The Frobenius relations meet linear distributivity

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Abstract

The notion of Frobenius algebra originally arose in ring theory, but it is a fairly easy observation that this notion can be extended to arbitrary monoidal categories. But, is this really the correct level of generalisation?

For example, when studying Frobenius algebras in the $*$-autonomous category **Sup**, the standard concept using only the usual tensor product is less interesting than a similar one in which both "tensor" and its de Morgan dual ("par") are used.

Thus we maintain that the notion of linear-distributive category (which has both a tensor and a par, but is nevertheless more general than the notion of monoidal category) provides the correct framework in which to interpret the concept of Frobenius algebra.

1 Introduction

We recall that a *linearly distributive category* is defined to be a category equipped with two tensor products, here denoted $\owedge$ and $\ovee$, which are related by a pair of (generally non-invertible) natural transformations

$$x \owedge (y \ovee z) \xrightarrow{\kappa^{(\ell\ell)}} (x \owedge y) \ovee z$$

$$(x \ovee y) \owedge z \xrightarrow{\kappa^{(rr)}} x \ovee (y \owedge z)$$

which are required to satisfy a large number of coherence conditions.

The reader is referred to [2] for the full definition, as well as for proofs of all statements in this section. The definition of *linear functor* between linearly distributive categories can be found in [3].

Example 1.1

Every (non-commutative) $*$-autonomous category $(\mathcal{K}, \owedge, e, \multimap, \multimapinv, d)$ has an underlying linearly distributive category, in which the second tensor product is defined as the "de Morgan dual" of the first.

$$x \ovee y := {}^*(y^* \owedge x^*)$$

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Here, as usual, x^* is an abbreviation for $x \multimap d$, and ${}^*x$ is an abbreviation for $d \multimapinv x$.

[Note that ${}^*(y^* \owedge x^*) \cong ({}^*y \owedge {}^*x)^*$ holds in any (non-commutative) $*$-autonomous category, so the asymmetry present in the definition above is only one of appearance.]

In particular, a boolean algebra may be viewed as a linearly distributive category with $\owedge = \wedge$ and $\ovee = \vee$.

One of the profoundest and most useful observations contained in [2] is that what one might call “duality theorems” in a (non-commutative) $*$-autonomous category can be characterised in terms of its underlying linearly distributive structure.

Theorem 1.2

Let $(\mathcal{K}, \owedge, e, \multimap, \multimapinv, d)$ be a (non-commutative) $*$-autonomous category, and suppose that we are given a pair of arrows

$$\begin{tikzcd} x \arrow[r, shift left, "\vartheta"] & {}^*y. \arrow[l, shift left, "\varphi"] \end{tikzcd}$$

Let $x \owedge y \xrightarrow{\gamma} d$ denote the transpose of ϑ, and τ the composite

$$e \xrightarrow{\ulcorner \varphi \urcorner} {}^*y \multimap x \xrightarrow{\sim} y \ovee x.$$

Then ϑ and φ are inverse to each other if and only if the following diagrams commute.

$$\begin{tikzcd} x \owedge e \arrow[r, "\iota \owedge \tau"] \arrow[dd, "\upsilon^{(r)}"'] & x \owedge (y \ovee x) \arrow[dr, bend left, "\kappa^{(\ell\ell)}"] & \\ & (\mathrm{LTI}_1) & (x \owedge y) \ovee x \arrow[d, "\gamma \ovee \iota"] \\ x \arrow[rr, "\upsilon^{(\ell)-1}"'] & & d \ovee x \end{tikzcd}$$

$$\begin{tikzcd} e \owedge y \arrow[r, "\tau \owedge \iota"] \arrow[dd, "\upsilon^{(\ell)}"'] & (y \ovee x) \owedge y \arrow[dr, bend left, "\kappa^{(rr)}"] & \\ & (\mathrm{LTI}_2) & y \ovee (x \owedge y) \arrow[d, "\iota \ovee \gamma"] \\ y \arrow[rr, "\upsilon^{(r)-1}"'] & & y \ovee d \end{tikzcd}$$

[Here we abuse notation by using the same symbol to denote both of the left-unit isomorphisms; similarly, the right-unit isomorphisms.]

Definitions 1.3

1. A pair of arrows in a linearly distributive category $(\mathcal{K}, \owedge, e, \ovee, d)$

$$x \owedge y \xrightarrow{\quad\gamma\quad} d \qquad\qquad e \xrightarrow{\quad\tau\quad} y \ovee x$$

   which satisfies (LTI$_1$) and (LTI$_2$) is called a *linear adjoint.*

2. A pair of linear adjoints

$$x \owedge y \xrightarrow{\quad\gamma_1\quad} d \qquad\qquad e \xrightarrow{\quad\tau_1\quad} y \ovee x$$

$$y \owedge x \xrightarrow{\quad\gamma_2\quad} d \qquad\qquad e \xrightarrow{\quad\tau_2\quad} x \ovee y$$

   is called a *cyclic linear adjoint.*

Remark 1.4

It is to be emphasised that any monoidal category $(\mathcal{K}, \otimes, i)$ can be regarded as a linearly distributive category by choosing $\owedge = \otimes = \ovee$, $e = i = d$, $\kappa^{(rr)} = \alpha$, and $\kappa^{(\ell\ell)} = \alpha^{-1}$.

In this sense, linearly distributive categories are actually more general than monoidal categories and when, as here, duality theory is at the centre of one's attention, this is often a useful point of view: that monoidal categories are merely *degenerate* linearly distributive categories.

Note that in the strict degenerate case, the diagrams defining linear adjoint reduce to the more usual triangle identities.

Lemma 1.5

In an arbitrary linearly distributive category, each half of a linear adjoint determines the other in the sense that, if

$$x \owedge y \xrightarrow{\quad\gamma_1\quad} d \qquad\qquad e \xrightarrow{\quad\tau_1\quad} y \ovee x$$

$$x \owedge y \xrightarrow{\quad\gamma_2\quad} d \qquad\qquad e \xrightarrow{\quad\tau_2\quad} y \ovee x$$

are linear adjoints, then $(\gamma_1 = \gamma_2) \iff (\tau_1 = \tau_2)$.

2 Linear points and Frobenius monoids

Throughout the next two sections $(\mathcal{K}, \owedge, e, \ovee, d)$ shall denote an arbitrary linearly distributive category. In particular, we shall not be considering *symmetries*, *braidings*, or even *non-planar linear distributions* on $(\mathcal{K}, \owedge, e, \ovee, d)$, until section 5.

Also, it shall be our convention that: the term *monoid* refers to a monoid in $(\mathcal{K}, \owedge, e)$; the term *comonoid* refers to a comonoid in $(\mathcal{K}, \ovee, d)$; and, the term *monoid/comonoid pair* refers to a monoid and a comonoid (in the senses above) which have the same underlying object.

Lemma 2.1

Let $M = (m, \mu, \eta)$ be a monoid, $(x, \sigma^{(\ell)})$ a left M-object, and $(y, \sigma^{(r)})$ a right M-object; then the maps

$$m \boxtimes (x \uplus y) \xrightarrow{\kappa^{(\ell\ell)}} (m \boxtimes x) \uplus y \xrightarrow{\sigma^{(\ell)} \uplus \iota} x \uplus y$$

$$(x \uplus y) \boxtimes m \xrightarrow{\kappa^{(rr)}} x \uplus (y \boxtimes m) \xrightarrow{\iota \uplus \sigma^{(r)}} x \uplus y$$

define a two-sided action of M on $x \uplus y$.

Proof

We shall prove only the left/right-compatibility diagram, sometimes also known as "middle associativity", as an illustration.

$$\begin{tikzcd}
(m \boxtimes (x \uplus y)) \boxtimes m \arrow[rrrr, "\alpha"] \arrow[d, "\kappa^{(\ell\ell)} \boxtimes \iota"'] & & & & m \boxtimes ((x \uplus y) \boxtimes m) \arrow[d, "\iota \boxtimes \kappa^{(rr)}"] \\
((m \boxtimes x) \uplus y) \boxtimes m \arrow[d, "(\sigma^{(\ell)} \uplus \iota) \boxtimes \iota"'] \arrow[drr, "\kappa^{(rr)}"] & & (\textsc{coh}) & & m \boxtimes (x \uplus (y \boxtimes m)) \arrow[d, "\iota \boxtimes (\iota \uplus \sigma^{(r)})"] \arrow[dll, "\kappa^{(\ell\ell)}"'] \\
(x \uplus y) \boxtimes m \arrow[d, "\kappa^{(rr)}"'] & (\textsc{nat}) & (m \boxtimes x) \uplus (y \boxtimes m) \arrow[dll, "\sigma^{(\ell)} \uplus \iota"] \arrow[dd, "\sigma^{(\ell)} \uplus \sigma^{(r)}"] \arrow[drr, "\iota \uplus \sigma^{(r)}"'] & (\textsc{nat}) & m \boxtimes (x \uplus y) \arrow[d, "\kappa^{(\ell\ell)}"] \\
x \uplus (y \boxtimes m) \arrow[drr, "\iota \uplus \sigma^{(r)}"'] & & & & (m \boxtimes x) \uplus y \arrow[dll, "\sigma^{(\ell)} \uplus \iota"] \\
& & x \boxtimes y & &
\end{tikzcd}$$

The remainder is left as an exercise. Q.E.D.

Remark 2.2

In the case where $(\mathcal{K}, \boxtimes, e, \uplus, d)$ arises from a (non-commutative) $*$-autonomous category $(\mathcal{K}, \boxtimes, e, \multimap, \multimapinv, d)$, we have canonical isomorphisms $x^* \multimap y \cong x \uplus y \cong x \multimapinv {}^*y$.

Now the left (right) m-action we have defined on $x \uplus y$ is the same as the "obvious" one on $x \multimapinv {}^*y$ (respectively, $x^* \multimap y$).

But in an arbitrary (left- and right-) closed monoidal category, there would seem to be no way of expressing the idea that these left- and right- actions combine to form a two-sided action.

For the most part, we shall be interested in only the following cases of Lemma 2.1, and their duals: if $M = (m, \mu, \eta)$ is a monoid, then of course m carries a canonical two-sided M-action; and so any two-sided action on x induces canonical two-sided actions on both $x \uplus m$ and $m \uplus x$.

This observation is necessary to make the following definition type correctly.

Definitions 2.3

1. A *cyclic nuclear monoid* in $(\mathcal{K}, \boxtimes, e, \boxplus, d)$ consists of:

   (a) a monoid $M = (m, \mu, \eta)$;

   (b) a comonoid $G = (g, \delta, \varepsilon)$;

   (c) a two-sided M-action $(\sigma^{(\ell)}, \sigma^{(r)})$ on g; and

   (d) a two-sided G-coaction $(\vartheta^{(\ell)}, \vartheta^{(r)})$ on m;

   such that:

   (e) $m \xrightarrow{\vartheta^{(\ell)}} g \boxplus m$ and $m \xrightarrow{\vartheta^{(r)}} m \boxplus g$ are M-equivariant; and

   (f) $m \boxtimes g \xrightarrow{\sigma^{(\ell)}} g$ and $g \boxtimes m \xrightarrow{\sigma^{(r)}} g$ are G-coequivariant.

2. A *Frobenius monoid* in $(\mathcal{K}, \boxtimes, e, \boxplus, d)$ is a cyclic nuclear monoid with $m = g$, $\sigma^{(\ell)} = \mu = \sigma^{(r)}$, and $\vartheta^{(\ell)} = \delta = \vartheta^{(r)}$.

Theorem 2.4

A cyclic nuclear monoid in $(\mathcal{K}, \boxtimes, e, \boxplus, d)$ is the same thing as a linear point of $(\mathcal{K}, \boxtimes, e, \boxplus, d)$—*i.e.*, a linear functor $1 \longrightarrow (\mathcal{K}, \boxtimes, e, \boxplus, d)$.

A Frobenius monoid is the same thing as a monoid/comonoid pair $(m, \mu, \eta, \delta, \varepsilon)$ which satisfy the *linear Frobenius relations* depicted below.

$$\begin{tikzcd}
m \boxtimes (m \boxplus m) \arrow[rr, "\kappa^{(\ell\ell)}"] & & (m \boxtimes m) \boxplus m \arrow[d, "\mu \boxplus \iota"] \\
m \boxtimes m \arrow[u, "\iota \boxtimes \delta"] \arrow[d, "\delta \boxtimes \iota"'] \arrow[r, "\mu"] & m \arrow[r, "\delta"] \arrow[u, phantom, "(\mathrm{LFR}_1)"] \arrow[d, phantom, "(\mathrm{LFR}_2)"] & m \boxplus m \\
(m \boxplus m) \boxtimes m \arrow[rr, "\kappa^{(rr)}"'] & & m \boxplus (m \boxtimes m) \arrow[u, "\iota \boxplus \mu"']
\end{tikzcd}$$

Proof

It follows directly from the definition of linear functor that a linear point consists of: a monoid $M = (m, \mu, \eta)$, a comonoid $G = (g, \delta, \varepsilon)$, and maps $\sigma^{(\ell)}, \sigma^{(r)}, \vartheta^{(\ell)}, \vartheta^{(r)}$ whose sources and targets match those given in the definition of cyclic nuclear monoid. Therefore all that remains to show is that the eighteen coherence conditions required of $\sigma^{(\ell)}, \sigma^{(r)}, \vartheta^{(\ell)}, \vartheta^{(r)}$ by the definition of linear functor are (collectively) equivalent to the conditions placed on them by the definition of cyclic nuclear monoid.

In fact, Definition 2.3 also requires that $\sigma^{(\ell)}$, $\sigma^{(r)}$, $\vartheta^{(\ell)}$ and $\vartheta^{(r)}$ satisfy a total of eighteen diagrams:

1. five diagrams for $\sigma^{(\ell)}, \sigma^{(r)}$ being an M-action;

2. five diagrams for $\vartheta^{(\ell)}, \vartheta^{(r)}$ being a G-coaction;

3. four diagrams for $\vartheta^{(\ell)}, \vartheta^{(r)}$ being M-equivariant; and

4. four diagrams for $\sigma^{(\ell)}, \sigma^{(r)}$ being G-coequivariant.

We claim that the two sets of conditions are not merely equivalent, but actually equal.

To demonstrate this claim, we need merely identify the five symmetry-based groupings which appear in [3, Definition 1]. These are as follows:

1. the four (co)unit diagrams;

2. the four (co)associativity diagrams;

3. the two left/right-compatibility diagrams;

4. the four "same-parity" (co)equivariance diagrams—*i.e.*, the left equivariance of $\vartheta^{(\ell)}$, the right equivariance of $\vartheta^{(r)}$, the left coequivariance of $\sigma^{(\ell)}$ and the right coequivariance of $\sigma^{(r)}$;

5. the remaining four (co)equivariance diagrams.

[What distinguishes the same-parity (co)equivariance diagrams is that, in each case, they contain an alternating ternary string: either mgm or gmg—for an example, see below.]

Turning to the case of Frobenius monoids, we see that the first ten diagrams (according to either grouping) are trivial; the remaining eight diagrams collapse to two.

Consider, for example, the M-equivariance of $\vartheta^{(\ell)}$. In general, this states that the diagrams

$$
\begin{tikzcd}
m \boxtimes m \arrow[r, "\iota \boxtimes \vartheta^{(\ell)}"] \arrow[dd, "\mu"'] & m \boxtimes (g \uplus m) \arrow[d, "\kappa^{(\ell\ell)}"] \\
& (m \boxtimes g) \uplus m \arrow[d, "\sigma^{(\ell)} \uplus \iota"] \\
m \arrow[r, "\vartheta^{(\ell)}"'] & g \uplus m
\end{tikzcd}
\qquad
\begin{tikzcd}
m \boxtimes m \arrow[r, "\vartheta^{(\ell)} \boxtimes \iota"] \arrow[dd, "\mu"'] & (g \uplus m) \boxtimes m \arrow[d, "\kappa^{(rr)}"] \\
& g \uplus (m \boxtimes m) \arrow[d, "\iota \uplus \mu"] \\
m \arrow[r, "\vartheta^{(\ell)}"'] & g \uplus m
\end{tikzcd}
$$

commute—but, if $g = m$, $\sigma^{(\ell)} = \mu$ and $\vartheta^{(\ell)} = \delta$, then these reduce to (LFR$_1$) and (LFR$_2$), respectively.

Each of the three other pairs of diagrams—the G-coequivariance of $\sigma^{(\ell)}$, that of $\sigma^{(r)}$, and the M-equivariance of $\vartheta^{(r)}$—do the same.

[Note that each pair contains one diagram with a $\kappa^{(\ell\ell)}$ and one with a $\kappa^{(rr)}$—the former reduces to (LFR$_1$) and the latter to (LFR$_2$).] Q.E.D.

Scholium 2.5

For a monoid/comonoid pair $(m, \mu, \eta, \delta, \varepsilon)$ in $(\mathcal{K}, \boxtimes, e, \boxslash, d)$, the following are equivalent:

1. $m \xrightarrow{\delta} m \boxslash m$ is left (m, μ, η)-equivariant;
2. $m \xrightarrow{\delta} m \boxslash m$ is right (m, μ, η)-equivariant;
3. $m \boxtimes m \xrightarrow{\mu} m$ is left (m, δ, ε)-coequivariant;
4. $m \boxtimes m \xrightarrow{\mu} m$ is right (m, δ, ε)-coequivariant;
5. $(m, \mu, \eta, \delta, \varepsilon)$ is a Frobenius monoid.

In the degenerate case, this fact is well-known and appears, for example, in [5].

3 Duality

In the degenerate case, Frobenius monoids are usually studied in connection with duality theory, so there should be little surprise that the same is true in the general case. Indeed, it follows from general results in [3] and [1] that cyclic nuclear monoids and Frobenius monoids must be self-dual. Here we formulate and prove the same result with somewhat more precision.

Theorem 3.1

If $((m, \mu, \eta), (g, \delta, \varepsilon), (\sigma^{(\ell)}, \sigma^{(r)}), (\vartheta^{(\ell)}, \vartheta^{(r)}))$ form a cyclic nuclear monoid, then the maps

$$e \xrightarrow{\quad\eta\quad} m \xrightarrow{\quad\vartheta^{(\ell)}\quad} g \boxslash m \qquad\qquad m \boxtimes g \xrightarrow{\quad\sigma^{(\ell)}\quad} g \xrightarrow{\quad\varepsilon\quad} d$$

$$e \xrightarrow{\quad\eta\quad} m \xrightarrow{\quad\vartheta^{(r)}\quad} m \boxslash g \qquad\qquad g \boxtimes m \xrightarrow{\quad\sigma^{(r)}\quad} g \xrightarrow{\quad\varepsilon\quad} d$$

form a cyclic linear adjoint.

In particular, if $(m, \mu, \eta, \delta, \varepsilon)$ is a Frobenius monoid, then

$$e \xrightarrow{\quad\eta\quad} m \xrightarrow{\quad\delta\quad} m \boxslash m \qquad\qquad m \boxtimes m \xrightarrow{\quad\mu\quad} m \xrightarrow{\quad\varepsilon\quad} d$$

form a linear adjoint.

Proof

Each of the four "same-parity" (co)equivariance diagrams is used once, together with appropriate (co)unit axioms, to obtain one of the four necessary diagrams.

For instance, the left-equivariance of $\vartheta^{(\ell)}$ (see proof of Theorem 2.4 above) can be combined with its left counit axiom and the right unit axiom for μ to produce (LTI_1) for $\eta \mathbin{;} \vartheta^{(\ell)}$

and $\sigma^{(\ell)}$; ε.

$$
\begin{tikzcd}
m \owedge e \arrow[r, "\iota \owedge \eta"] \arrow[ddr, bend right, "\upsilon^{(r)}"'] & m \owedge m \arrow[r, "\iota \owedge \vartheta^{(\ell)}"] \arrow[dd, "\mu"] & m \owedge (g \ovee m) \arrow[dr, bend left, "\kappa^{(\ell\ell)}"] & \\
& & & (m \owedge g) \ovee m \arrow[d, "\sigma^{(\ell)} \ovee \iota"] \\
& m \arrow[rr, "\vartheta^{(\ell)}"] \arrow[drr, bend right, "\upsilon^{(\ell)-1}"'] & & g \ovee m \arrow[d, "\varepsilon \ovee \iota"] \\
& & & d \ovee m
\end{tikzcd}
$$

Q.E.D.

Scholium 3.2

By examining the proof of Theorem 3.1, we can derive the following formula for $\vartheta^{(\ell)}$:

$$
\begin{tikzcd}
m \owedge e \arrow[r, "\iota \owedge (\eta\, ;\, \vartheta)"] & m \owedge (g \ovee m) \arrow[d, "\kappa^{(\ell\ell)}"] \\
& (m \owedge g) \ovee m \arrow[d, "\sigma^{(\ell)} \ovee \iota"] \\
m \arrow[uu, "\upsilon^{(r)^{-1}}"] \arrow[r, "\vartheta^{(\ell)}"'] & g \ovee m
\end{tikzcd}
$$

—but recall from Lemma 1.5, that the composite η ; $\vartheta^{(\ell)}$ is uniquely determined by $\sigma^{(\ell)}$; ε.

Thus, the structure of a cyclic nuclear monoid is overdetermined. In particular, $\vartheta^{(\ell)}$ is determined, indirectly, by $\sigma^{(\ell)}$ and ε.

In the case of a Frobenius monoid in a degenerate linearly distributive category, this is a celebrated result: the map $\delta = \vartheta^{(\ell)}$ is determined by $\mu = \sigma^{(\ell)}$ and ε—see, for example, [5].

Remark 3.3

It is interesting to note that only four of the eight (co)equivariance conditions are actually used in the proof of Theorem 3.1.

Also interesting is the fact that each linear adjoint is constructed using only one half of the action/coaction structure. This suggests that there exists a more general notion than that of cyclic nuclear monoid, for which non-cyclic linear adjoints can be constructed.

The latter idea will be more fully explored in a subsequent paper.

4 Frobenius quantales

Throughout the next two sections: $\mathcal{E}$ shall denote an arbitrary elementary topos; $1 \xrightarrow{\top} 2$ its subobject classifier; $\mathbb{P}$ the power-$\mathcal{E}$-object monad, $(2^{(\ )}, \bigcup, \{\})$; and $\check{\mathcal{E}} = \mathbf{Sup}(\mathcal{E})$ the category of $\mathbb{P}$-algebras—equivalently, the category of (internally) cocomplete ordered $\mathcal{E}$-objects and (internal) sup-homomorphisms. Henceforth, we shall speak of the objects of $\mathcal{E}$ as if they were sets, and therefore also drop the modifiers *internal* and *internally*.

As demonstrated in [4], $\check{\mathcal{E}}$ can be given a symmetric $*$-autonomous structure, and hence also a (symmetric) linearly distributive structure. [We shall postpone the definition of symmetric linearly distributive category until the next section, as we will not need this extra structure until then.] Following the convention laid out in Example 1.1, x^* does not denote x^{op} but rather $x \multimap 2^{\mathsf{op}}$; we write $\llcorner - \lrcorner$ for the canonical isomorphism $x^{\mathsf{op}} \longrightarrow x^* = {}^*x$.

We shall continue to write $\boxtimes$ and $\veebar$ for the two tensor products on $\check{\mathcal{E}}$, but we shall abuse notation, slightly, by denoting their units by 2 and 2^{op} respectively. [Each of these carries a canonical $\mathbb{P}$-algebra structure.]

We write $|-|$ for the forgetful functor $\check{\mathcal{E}} \longrightarrow \mathbf{Ord}(\mathcal{E})$, $\boxtimes$ also for the canonical map

$$|x| \times |y| \xrightarrow{\ \boxtimes\ } |x \boxtimes y|$$

—hence $(|-|, \boxtimes, \top)$ is the forgetful monoidal functor $(\check{\mathcal{E}}, \boxtimes, 2) \longrightarrow (\mathbf{Ord}(\mathcal{E}), \times, 1)$.

A *quantale* is a monoid in $(\check{\mathcal{E}}, \boxtimes, 2)$. [We shall only consider unital quantales in the current paper.] Given a quantale (q, μ, η), we write $(|q|, \&, \eta)$ for the corresponding monoid in $(\mathbf{Ord}(\mathcal{E}), \times, 1)$; in particular, $\&$ denotes the composite

$$|q| \times |q| \xrightarrow{\ \boxtimes\ } |q \boxtimes q| \xrightarrow{\ |\mu|\ } |q| \, .$$

We write $\rightarrow$ and $\leftarrow$ for the left- and right-closed structures on $(|q|, \&, \eta)$, respectively.

We recall that a quantale equipped with a cyclic dualising element is commonly called a *Girard quantale*, [7]. [By a (cyclic) dualising element for (q, μ, η), one actually means a (cyclic) dualising element for $(|q|, \&, \eta, \rightarrow, \leftarrow)$.]

Surprisingly, there does not seem to be a standard (short) name for a quantale equipped with an arbitrary dualising element; but this proves to be fortunate, in light of the following theorem.

Theorem 4.1

A Frobenius monoid in $(\check{\mathcal{E}}, \boxtimes, 2, \veebar, 2^{\mathsf{op}})$ amounts to a quantale equipped with a dualising element; equivalently, a $*$-autonomous cocomplete poset.

Proof

Suppose that $(q, \mu, \eta, \delta, \varepsilon)$ is a Frobenius monoid in $(\check{\mathcal{E}}, \boxtimes, 2, \veebar, 2^{\mathsf{op}})$, and let $\ni = \bigvee \ker \varepsilon$ (so that $\varepsilon = \llcorner \ni \lrcorner$). We will abbreviate $(\ ) \rightarrow \ni$ and $\ni \leftarrow (\ )$ to $(\ )^{\perp}$ and ${}^{\perp}(\ )$ respectively.

By Theorem 3.1

$$2 \xrightarrow{\ \eta\ } q \xrightarrow{\ \delta\ } q \veebar q \qquad\qquad q \boxtimes q \xrightarrow{\ \mu\ } q \xrightarrow{\ \varepsilon\ } 2^{\mathsf{op}}$$

form a linear adjoint. Hence, by Theorem 1.2, the transpose of the latter composite

$$q \xrightarrow{\ (\mu\, ;\, \varepsilon)^r\ } 2^{\mathsf{op}} \multimapinv q = {}^*q$$

is invertible.

But it is easy to check that this transpose equals the map

$$\alpha \mapsto {\llcorner}\alpha^{\perp}{\lrcorner}$$

since we have

$$\begin{aligned} {\llcorner}\alpha^{\perp}{\lrcorner}(\beta) = \bot \quad &\iff \quad \beta \leq \alpha^{\perp} = \alpha \to \ni \\ &\iff \quad \mu(\alpha \otimes \beta) = \alpha \ \&\ \beta \leq \ni \\ &\iff \quad \varepsilon(\mu(\alpha \otimes \beta)) = \bot. \end{aligned}$$

[This may not look contructive, but it is: we're just using the universal properties of $\otimes$ and of 2. Here, $\bot$ means $\bot_{2^{\mathsf{op}}}$—*i.e.*, $\top_{2}$.]

Moreover, this suffices to demonstrate that $\ni$ is a dualising element, since for posets we always have $({}^{\perp}((\ )^{\perp}))^{\perp} = (\ )^{\perp}$.

The previous arguments are reversible in the sense that if we start with a dualising element $\ni$, then we can define $\varepsilon = {\llcorner}\ni{\lrcorner}$, and the composite $q \otimes q \xrightarrow{\mu} q \xrightarrow{\varepsilon} 2^{\mathsf{op}}$ defines half of a linear adjoint. According to Scholium 3.2, there is then a unique δ making $(q, \mu, \eta, \delta, \varepsilon)$ into a Frobenius monoid in $(\check{\mathcal{E}}, \otimes, 2, \oplus, 2^{\mathsf{op}})$. Q.E.D.

We therefore propose that a quantale equipped with a (not necessarily) dualising element should be called a *Frobenius quantale*. We shall also use the term *Frobenius locale* for a Frobenius quantale whose underlying quantale is a locale; this amounts to a complete boolean algebra.

By way of comparison, note that a Frobenius monoid in $(\check{\mathcal{E}}, \otimes, 2, \otimes, 2)$ whose underlying quantale is a locale amounts to a power-object, [6]. Thus, in a boolean topos, $\mathcal{E}$, Frobenius quantales are more general than Frobenius monoids in $(\check{\mathcal{E}}, \otimes, 2, \otimes, 2)$; but, in non-boolean toposes, neither concept is contained in the other.

5 Canonical cyclicity

We recall that, by definition, a linearly distributive category $(\mathcal{K}, \otimes, e, \oplus, d)$ is called *symmetric* if both $\otimes$ and $\oplus$ are equipped with a symmetry, and if these symmetries (which shall both be denoted χ) satisfy one extra coherence condition:

$$\begin{tikzcd} x \otimes (y \oplus z) \arrow[r, "\iota \otimes \chi"] \arrow[d, "\kappa^{(\ell\ell)}"'] & x \otimes (z \oplus y) \arrow[r, "\chi"] & (z \oplus y) \otimes x \arrow[d, "\kappa^{(rr)}"] \\ (x \otimes y) \oplus z \arrow[r, "\chi"'] & z \oplus (x \otimes y) \arrow[r, "\iota \oplus \chi"'] & z \oplus (y \otimes x). \end{tikzcd}$$

The following lemma should be fairly obvious, but its proof does utilise this extra coherence condition, demonstrating the latter's worth.

Lemma 5.1

If $(\mathcal{K}, \otimes, e, \oplus, d)$ is a symmetric linearly distributive category, and

$$x \otimes y \xrightarrow{\gamma} d \qquad\qquad e \xrightarrow{\tau} y \oplus x$$

is a linear adjoint in $(\mathcal{K}, \otimes, e, \oplus, d)$, then

$$y \otimes x \xrightarrow{\chi} x \otimes y \xrightarrow{\gamma} d \qquad\qquad e \xrightarrow{\tau} y \oplus x \xrightarrow{\chi} x \oplus y$$

is also a linear adjoint in $(\mathcal{K}, \otimes, e, \oplus, d)$.

Briefly, every linear adjoint in $(\mathcal{K}, \otimes, e, \oplus, d)$ gives rises to a cyclic linear adjoint.

Proof

We can obtain $(\textsc{lti}_1)$ as follows:

$$\begin{tikzcd}
y \otimes e \arrow[r, "\iota \otimes \tau"] \arrow[d, "\chi"] \arrow[ddd, bend right, "\upsilon^{(r)}"'] & y \otimes (y \oplus x) \arrow[r, "\iota \otimes \chi"] \arrow[d, "\chi"] & y \otimes (x \oplus y) \arrow[dr, bend left, "\kappa^{(\ell\ell)}"] & \\
e \otimes y \arrow[r, "\tau \otimes \iota"] \arrow[dd, "\upsilon^{(\ell)}"] & (y \oplus x) \otimes y \arrow[dr, bend right, "\kappa^{(rr)}"] & y \oplus (y \otimes x) \arrow[r, "\chi"'] \arrow[d, "\iota \oplus \chi"] & (y \otimes x) \oplus y \arrow[d, "\chi \oplus \iota"] \\
 & & y \oplus (x \otimes y) \arrow[d, "\iota \oplus \gamma"] & (x \otimes y) \oplus y \arrow[d, "\gamma \oplus \iota"] \\
y \arrow[rr, "\upsilon^{(r)-1}"] \arrow[rrr, bend right, "\upsilon^{(\ell)-1}"'] & & y \oplus d \arrow[r, "\chi"] & d \oplus y
\end{tikzcd}$$

and $(\textsc{lti}_2)$ by a symmetric argument. Q.E.D.

What is, perhaps, not obvious is whether every cyclic linear adjoint in $(\mathcal{K}, \otimes, e, \oplus, d)$ arises in this way. As we shall see later in this section, the answer turns out to be *no*.

Definitions 5.2

1. A cyclic linear adjoint in $(\mathcal{K}, \otimes, e, \oplus, d)$ is called *canonically cyclic* if the diagram

$$\begin{tikzcd}
x \otimes y \arrow[r, "\gamma_1"] \arrow[d, "\chi"'] & d \arrow[d, equal] \\
y \otimes x \arrow[r, "\gamma_2"] & d
\end{tikzcd} \qquad\qquad \begin{tikzcd}
e \arrow[r, "\tau_1"] \arrow[d, equal] & y \oplus x \arrow[d, "\chi"] \\
e \arrow[r, "\tau_2"] & x \oplus y.
\end{tikzcd}$$

commutes.

2. A cyclic nuclear monoid in $(\mathcal{K}, \otimes, e, \oplus, d)$ is called *canonically cyclic* if the cyclic linear adjoint described in Theorem 3.1 is so—*i.e.*, if the diagram

$$\begin{tikzcd} e \arrow[r, "\eta"] \arrow[d, equal] & m \arrow[r, "\vartheta^{(\ell)}"] & g \oplus m \arrow[d, "\chi"] \\ e \arrow[r, "\eta"] & m \arrow[r, "\vartheta^{(r)}"] & m \oplus g \end{tikzcd} \qquad \begin{tikzcd} m \otimes g \arrow[r, "\sigma^{(\ell)}"] \arrow[d, "\chi"'] & g \arrow[r, "\varepsilon"] & d \arrow[d, equal] \\ g \otimes m \arrow[r, "\sigma^{(r)}"] & g \arrow[r, "\varepsilon"] & d \end{tikzcd}$$

commutes.

3. A *Girard monoid* in $(\mathcal{K}, \otimes, e, \oplus, d)$ is a Frobenius monoid whose underlying cyclic nuclear monoid is canonically cyclic—*i.e.*, if it satisfies

$$\begin{tikzcd} e \arrow[r, "\eta"] \arrow[d, equal] & m \arrow[r, "\delta"] & m \oplus m \arrow[d, "\chi"] \\ e \arrow[r, "\eta"] & m \arrow[r, "\delta"] & m \oplus m \end{tikzcd} \qquad \begin{tikzcd} m \otimes m \arrow[r, "\mu"] \arrow[d, "\chi"'] & m \arrow[r, "\varepsilon"] & d \arrow[d, equal] \\ m \otimes m \arrow[r, "\mu"] & m \arrow[r, "\varepsilon"] & d \end{tikzcd}$$

Remark 5.3

As a result of Lemmata 1.5 and 5.1, it suffices to check only one of each pair of diagrams in the definitions above.

From an abstract point of view, the significance of canonically cyclic linear adjoints has to do with the fact that in a symmetric $*$-autonomous category $(\mathcal{K}, \otimes, e, \multimap, \multimapinv, d)$, we have a canonical natural isomorphism ${}^*x \xrightarrow{\omega} x^*$ defined as the transpose of the composite

$$x \otimes {}^*x \xrightarrow{\chi} {}^*x \otimes x \xrightarrow{\varepsilon^{(r)}} d.$$

[Note that, for us, the term $*$-autonomous category denotes a (left- and right-) closed monoidal category equipped with a dualising object; and that the term symmetric $*$-autonomous category denotes a $*$-autonomous category equipped with a symmetry. Thus we do not assume in general that $x \multimap y = y \multimapinv x$, although this is what frequently occurs in practice.]

Theorem 5.4

Let $(\mathcal{K}, \otimes, e, \multimap, \multimapinv, d)$ be a symmetric $*$-autonomous category. Then a cyclic linear adjoint in $(\mathcal{K}, \otimes, e, \oplus, d)$

$$x \otimes y \xrightarrow{\gamma_1} d \qquad\qquad e \xrightarrow{\tau_1} y \oplus x$$

$$y \otimes x \xrightarrow{\gamma_2} d \qquad\qquad e \xrightarrow{\tau_2} x \oplus y$$

is canonically cyclic if and only if the diagram

$$\begin{tikzcd} x \arrow[r, "\gamma_1^r"] \arrow[d, equal] & {}^*y \arrow[d, "\omega"] \\ x \arrow[r, "\gamma_2^\ell"'] & y^* \end{tikzcd}$$

commutes.

Proof

It suffices to show that the composite

$$y \boxtimes x \xrightarrow{\iota \boxtimes \gamma_1^r} y \boxtimes {}^*y \xrightarrow{\iota \boxtimes \omega} y \boxtimes y^* \xrightarrow{\varepsilon^{(\ell)}} d$$

equals γ_2.

But the definition of ω amounts to

$$\begin{tikzcd}
y \boxtimes {}^*y \arrow[r, "\iota \boxtimes \omega"] \arrow[d, "\chi"'] & y \boxtimes y^* \arrow[d, "\varepsilon^{(\ell)}"] \\
{}^*y \boxtimes y \arrow[r, "\varepsilon^{(r)}"'] & d
\end{tikzcd}$$

and combining this with a naturality square, we obtain

$$\begin{tikzcd}
y \boxtimes x \arrow[r, "\iota \boxtimes \gamma_1^r"] \arrow[d, "\chi"'] & y \boxtimes {}^*y \arrow[r, "\iota \boxtimes \omega"] \arrow[d, "\chi"] & y \boxtimes y^* \arrow[d, "\varepsilon^{(\ell)}"] \\
x \boxtimes y \arrow[r, "\gamma_1^r \boxtimes \iota"] \arrow[rr, bend right, "\gamma_1"'] & {}^*y \boxtimes y \arrow[r, "\varepsilon^{(r)}"] & d.
\end{tikzcd}$$

Q.E.D.

Remark 5.5

The previous theorem is the main reason we have concentrated on symmetries rather than braidings.

In a braided $*$-autonomous category, we have at least two isomorphisms ${}^*x \xrightarrow{\omega} x^*$ which need not be equal; therefore neither is genuinely canonical.

Readers familiar with the classical theory of Frobenius algebras (*i.e.*, Frobenius monoids in the degenerate linearly distributive category $(\mathbf{Vec}, \otimes, k, \otimes, k)$) will have already recognised that not every Frobenius monoid is a Girard monoid, and hence that not every cyclic linear adjoint is canonically cyclic. [Girard monoids in $(\mathbf{Vec}, \otimes, k, \otimes, k)$ normally go by the uninspired, and potentially misleading, name *symmetric Frobenius algebras*.]

For everyone else we offer the following class of counter-examples.

Corollary 5.6

A Girard monoid in $(\check{\mathcal{E}}, \boxtimes, 2, \boxdot, 2^{\mathsf{op}})$ is the same thing as a Girard quantale.

Proof

By the previous theorem, a Girard monoid in $(\check{\mathcal{E}}, \boxtimes, 2, \boxtimes, 2^{\mathsf{op}})$ is the same thing as a Frobenius quantale $(q, \mu, \eta, \delta, \varepsilon)$ such that the diagram

$$\begin{tikzcd} q \arrow[r, "(\mu\ ;\ \varepsilon)^r"] \arrow[d, equal] & {}^*q \arrow[d, equal] \\ q \arrow[r, "(\mu\ ;\ \varepsilon)^\ell"'] & q^* \end{tikzcd}$$

commutes.

But we have already shown, in the proof of Theorem 4.1, that the top map equals $\alpha \mapsto \llcorner \alpha^{\perp} \lrcorner$; by a symmetric argument, the bottom map equals $\alpha \mapsto \llcorner {}^{\perp}\alpha \lrcorner$. Q.E.D.

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