Math 2135: Linear Algebra, Reading Week Stuff

1. (a) Show that

$$\lim_{t\to\pm\infty} t^n e^{-t^2/2} = 0$$

for all $n \in \mathbb{N}$. [Hint: Use L'Hospital's rule and (strong) induction.]

(b) Use integration by parts to show that

$$\int t^{n+2} e^{-t^2/2}\, dt = t^{n+1} e^{-t^2/2} + (n+1) \int t^n e^{-t^2/2}\, dt$$

for all $n \in \mathbb{N}$.

(c) Conclude that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2/2}\, dt = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n!}{2^{n/2}(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

[Hint: if you don't know what $\int_{-\infty}^{\infty} e^{-t^2/2}\, dt$ equals, ask someone who does.]

(d) If n is even, what does $\dfrac{n!}{2^{n/2}(n/2)!}$ really represent?

2. Let $\mathbb{P}_{normal}$ denote the real inner product space of polynomials with the inner product defined by

$$\langle p(t), q(t) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(t)q(t)e^{-t^2/2}\, dt$$

(a) Show that $\{1, t, t^2 - 1\}$ is an orthogonal set in $\mathbb{P}_{normal}$.

(b) Find the magnitudes of $1, t, t^2 - 1$ in $\mathbb{P}_{normal}$.

(c) Find a cubic polynomial $q(t)$ such that $\{1, t, t^2 - 1, q(t)\}$ is an orthogonal set in $\mathbb{P}_{normal}$, and such that $\|q(t)\| = \sqrt{6}$.

(d) Do you think it is possible to find an orthogonal basis for $\mathbb{P}_{normal}$?

(e) Google *Hermite polynomial.*

3. Let $\mathbb{P}_{uniform}$ denote the real inner product space of polynomials with the inner product defined by

$$\langle p(t), q(t) \rangle = \frac{1}{2} \int_{-1}^{1} p(t)q(t)\, dt$$

(a) Show that $\{1, t, \frac{1}{2}(3t^2 - 1)\}$ is an orthogonal set in $\mathbb{P}_{uniform}$.

(b) Find the magnitudes of $1, t, \frac{1}{2}(3t^2 - 1)$ in $\mathbb{P}_{uniform}$.

(c) Find a cubic polynomial $q(t)$ such that $\{1, t, \frac{1}{2}(3t^2 - 1), q(t)\}$ is an orthogonal set in $\mathbb{P}_{uniform}$, and such that $\|q(t)\| = \frac{1}{\sqrt{7}}$.

(d) Do you think it is possible to find an orthogonal basis for $\mathbb{P}_{uniform}$?

(e) Google *Legendre polynomial.*