ENRICHED DINATURAL NUMBERS

FEB. 7, 1995

FOR $F, G : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{B}$

DINATURAL TRANSFS :

$$
\begin{tikzcd}
 & F(A,A) \arrow[r, "t_A"] & G(A,A) \arrow[dr, "{G(A,f)}"] & \\
F(A',A) \arrow[ur, "{F(f,A)}"] \arrow[dr, "{F(A',f)}"'] & & & G(A,A') \\
 & F(A',A') \arrow[r, "t_{A'}"'] & G(A',A') \arrow[ur, "{G(f,A')}"'] &
\end{tikzcd}
$$

BARR DINATURALS :

$$
\begin{tikzcd}
 & F(A,A) \arrow[dr] \arrow[rr, "t_A"] & & G(A,A) \arrow[dr, "{G(A,f)}"] & \\
PB \arrow[ur] \arrow[dr] & & F(A,A') & & G(A,A') \\
 & F(A',A') \arrow[ur] \arrow[rr, "t_{A'}"'] & & G(A',A') \arrow[ur] &
\end{tikzcd}
$$

THEOREM:

$$
\frac{\text{STRONG BARR DINATS } \mathrm{Hom}_{\underline{A}} \longrightarrow \mathrm{Hom}_{\underline{A}}^{L}}{L \longrightarrow \mathbb{N}} \quad \cong
$$

$\mathrm{Hom}_{\underline{A}}^{L} : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{\mathrm{Set}}$; $(A, A') \mapsto \mathrm{Hom}(A \times L, A')$

$\mathbb{N}$ = NNO in $\underline{A}$

WHAT IS STRONG ?

STRONG BDN $t : \mathrm{Hom}_{\underline{A}} \Rightarrow \mathrm{Ham}_{\underline{A}}^{L}$

$$A \xrightarrow{\alpha} A \quad \xmapsto{t} \quad A \times L \xrightarrow{t(\alpha)} A$$

$$\begin{tikzcd}
X \times A \arrow[r, "X \times \alpha"] & X \times A \arrow[r, mapsto] & X \times A \times L \arrow[r, "t(X \times \alpha)"] \arrow[d, equal] & X \times A \arrow[d, equal] \\
 & & X \times A \times L \arrow[r, "X \times t(\alpha)"'] & X \times A
\end{tikzcd}$$

$$\boxed{t(X \times \alpha) = X \times t(\alpha)}$$

Something special about $\mathrm{Ham}_{\underline{A}}, \mathrm{Ham}_{\underline{A}}^{L}$ allows this.

CLUE: $E(\underline{A}) = \underline{A}^{\cdot 2}$ = Cat of endos of $\underline{A}$

$$\frac{\text{BDNs} \quad t : \mathrm{Ham}_{\underline{A}} \longrightarrow \mathrm{Ham}_{\underline{A}}}{\begin{tikzcd} E(\underline{A}) \arrow[rr, "T"] \arrow[dr, "U"'] & & E(\underline{A}) \arrow[dl, "U"] \\ & \underline{A} & \end{tikzcd}} \cong$$

$\underline{A}$ cart. closed with equalizers $\Rightarrow E(\underline{A})$ enriched in $\underline{A}$ & U is an $\underline{A}$-funct.

$$\boxed{t \text{ strong} \iff T \ \underline{A}\text{-functor}}$$

$E(\underline{A})$ is <u>tensored</u>, i.e.

$$E(\underline{A})\big[(A,\alpha),-\big] : E(\underline{A}) \longrightarrow \underline{A}$$

has a ladj $(\;)\otimes(A,\alpha)$.

$$\frac{X\otimes(A,\alpha) \longrightarrow (B,\beta)}{X \longrightarrow E(\underline{A})[(A,\alpha),(B,\beta)]}$$

$$X\otimes(A,\alpha) = (X\times A,\ X\times\alpha).$$

$$\boxed{\begin{array}{l} t \text{ strong} \iff T \text{ preserves } \otimes \text{, i.e.} \\ \qquad\qquad T(X\otimes(A,\alpha)) = X\otimes T(A,\alpha) \end{array}}$$

Each functor of an adjoint pair determines the other. For $E(\underline{A})$ $X\otimes(A,\alpha)$ always exists (regardless of whether $\underline{A}$ is cartesian closed) So we reformulate everything in terms of $\otimes$ rather than Hom.

Let $\underline{A} = (\underline{V}, \otimes, I, \alpha, \lambda, \rho)$ Monoidal Cat.

A category $\underline{B}$ is $\underline{V}$-tensored if there is a left $\underline{V}$-action on it, i.e.

$$\underline{V} \times \underline{B} \xrightarrow{\otimes} \underline{B}$$

such that

$$\lambda : I \otimes B \xrightarrow{\cong} B$$

$$\alpha : V_1 \otimes (V_2 \otimes B) \xrightarrow{\cong} (V_1 \otimes V_2) \otimes B$$

with obvious coherence conditions w.r.t. α, λ, ρ of $\underline{V}$.

<u>Ex</u>: $\underline{V}$, $\underline{V}^{\mathbb{C}}$, $E(\underline{V}) = \underline{V}^{?}$

If $(\ ) \otimes B : \underline{V} \longrightarrow \underline{B}$ has r.adj $\underline{B}[B, -] : \underline{B} \longrightarrow \underline{V}$, then $\underline{B}$ becomes a $\underline{V}$-category.

If $\underline{B}$ and $\underline{C}$ are $\underline{V}$-tensored

A functor $F : \underline{B} \to \underline{C}$ is

<u>strong</u> if it comes with "strength" morphisms

$$s_{V,B} : V \otimes F(B) \longrightarrow F(V \otimes B)$$

- Natural in V & B
- Compatible with λ, α.

If $\underline{B}$ and $\underline{C}$ are also $\underline{V}$-cats

$$\frac{F : \underline{B} \longrightarrow \underline{C} \quad \text{strong}}{F : \underline{B} \longrightarrow \underline{C} \quad \underline{V}\text{-functor}}$$

(c.f. RJ Wood – thesis)

A nat. trans. $t : F \to G$ is <u>strong</u> if

$$\begin{tikzcd} V \otimes F(B) \arrow[r, "s"] \arrow[d, "V \otimes t(B)"'] \arrow[dr, phantom, "///"] & F(V \otimes B) \arrow[d, "t(V \otimes B)"] \\ V \otimes G(B) \arrow[r, "s"'] & G(V \otimes B) \end{tikzcd}$$

$$\frac{t : F \longrightarrow G \quad \text{strong}}{t : F \longrightarrow G \quad \underline{V}\text{-nat}}$$

Have a 2-cat $\underline{V}$-$\underline{\underline{\text{Tens}}}$

Have strong adjoints

$$\underline{B} \underset{U}{\overset{F}{\rightleftarrows}} \underline{C}$$

$F \dashv_{\underline{V}} U \iff F \dashv U$ & F, U strong

& $V \otimes FB \xrightarrow[s]{\cong} F(V \otimes B)$

equiv:

$$\frac{V \otimes FB \longrightarrow C}{V \otimes B \longrightarrow UC} \cong \text{Nat in } V, B, C.$$

<u>Ex</u>: $U : E(\underline{V}) \longrightarrow \underline{V}$ is strong.

U has strong laj $\iff$

$\underline{V}$ has a NNO (in $\otimes$ sense)

$$
\begin{tikzcd}
V\otimes I \arrow[r, "V\otimes o"] \arrow[d, "\rho"', "\cong"] & V\otimes N \arrow[r, "V\otimes s"] \arrow[d, dashed, "\exists!\, h"'] \arrow[d, dashed, "h"] & V\otimes N \arrow[d, dashed, "h"] \\
V \arrow[r, "g"'] & A \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

$$
\frac{h : V\otimes F(I) \longrightarrow (A,\alpha)}{V\otimes I \longrightarrow A} \cong
$$

<u>Thm</u>:

$$
\frac{\text{Strong BDN } : \mathrm{Ham}_{\underline{V}} \longrightarrow \mathrm{Ham}^{L}_{\underline{V}}}{L \longrightarrow N}
$$

For $F, G : \underline{B}^{op} \times \underline{B} \longrightarrow \underline{C}$

$t : F \longrightarrow G$ BDN strong if

$$
V\otimes t_B = t_{V\otimes B} \quad ?
$$

$$
\begin{tikzcd}
F(B,B) \arrow[r, "t_B"] \arrow[d, dashed, "V\otimes -"'] & G(B,B) \arrow[d, dashed, "V\otimes -"] \\
F(V\otimes B, V\otimes B) \arrow[r, "t_{V\otimes B}"'] & G(V\otimes B, V\otimes B)
\end{tikzcd}
\quad ?
$$

A <u>strong bifunctor</u> $F : \underline{B}^{op} \times \underline{B} \longrightarrow \underline{C}$

$$\sigma_{V,B,B'} : F(B, B') \longrightarrow F(V \otimes B, V \otimes B')$$

- Nat in B, B' ; Dinat in V
- Compatible with λ, α

$$\begin{tikzcd}
F(B,B') \arrow[r, "\sigma"] \arrow[dd, "\sigma"'] & F(V\otimes B, V\otimes B') \arrow[d, "\sigma"] \\
 & F(V'\otimes(V\otimes B), V'\otimes(V\otimes B')) \arrow[d, "{F(*,\alpha)}"] \\
F((V'\otimes V)\otimes B, (V'\otimes V)\otimes B') \arrow[r, "{F(\alpha,*)}"'] & F(V'\otimes(V\otimes B), (V'\otimes V)\otimes B')
\end{tikzcd}$$

$\underline{B}$ tensored, $\underline{C}$ no conditions.

<u>Ex:</u> $\mathrm{Hom}_{\underline{A}} : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{Set}$

$$\mathrm{Hom}_{\underline{A}}(A, A') \longrightarrow \mathrm{Hom}_{\underline{A}}(V\otimes A, V\otimes A')$$

$$(A \xrightarrow{f} A') \longmapsto (V\otimes A \xrightarrow{V\otimes f} V\otimes A')$$

But $\underline{\text{not}}$ $\underline{A}[-,-] : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{V}$ if $\exists$.

$$\underline{A}[A,A'] \xrightarrow{\ ?\ } \underline{A}[V\otimes A, V\otimes A']$$

$$(W \to \underline{A}[A,A']) \overset{?}{\rightsquigarrow} (W \to \underline{A}[V\otimes A, V\otimes A'])$$

$$(W\otimes A \to A') \overset{??}{\rightsquigarrow} (W\otimes V\otimes A \to V\otimes A')$$

$$Hom^L : \underline{V}^{op} \times \underline{V} \longrightarrow \underline{Set}$$

$$Hom^L(A,B) = \{A\otimes L \to B\} = Hom(A\otimes L, B).$$

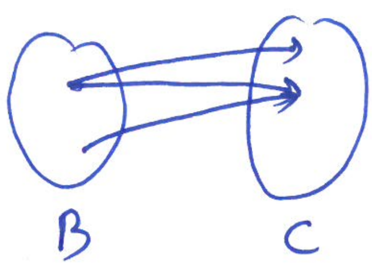
<u>Examples</u>: ? <u>Profunctors</u> !

<u>SET</u>		<u>CAT</u>
function $B \to C$	———	functor $\underline{B} \to \underline{C}$
relation $B \nrightarrow C$	———	profunctor $\underline{B} \nrightarrow \underline{C}$
$B \longrightarrow 2^C$	———	$\bar{P} : \underline{B} \longrightarrow \underline{Set}^{\underline{C}^{op}}$
$B\times C \longrightarrow 2$	———	$P : \underline{C}^{op} \times \underline{B} \longrightarrow \underline{Set}$
$R \subseteq B\times C$ (graph)		$\underline{C} \xleftarrow{\text{disc. fib.}} El(P) \longrightarrow \underline{B}$

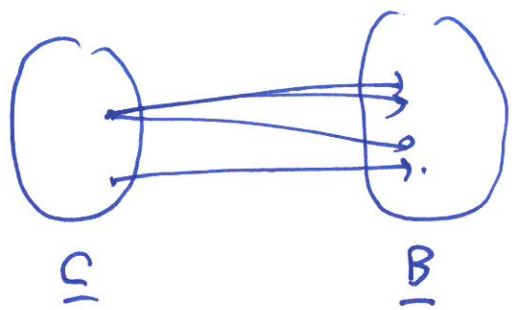
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Incidence matrix

$$\begin{array}{cc} \begin{array}{c} \uparrow \\ \underline{C} \\ \downarrow \end{array} & \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & & P(C,B) & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \\ & \longleftarrow \underline{B} \longrightarrow \end{array}$$



cograph



(disc. cofib.)

$$\frac{x \in P(C,B)}{C \underset{P}{\overset{x}{\rightsquigarrow}} B}$$ Denote

$$\frac{I_{\underline{B}} : \underline{B} \longrightarrow \underline{B}}{\mathrm{Hom}_{\underline{B}} : \underline{B}^{op} \times \underline{B} \longrightarrow \underline{Set}}$$

$$\underline{B} \overset{P}{\nrightarrow} \underline{C} \overset{Q}{\nrightarrow} \underline{D}$$

$$Q \otimes P(D,B) = \int^{C} Q(D,C) \times P(C,B)$$

$$= \sum_{C} Q(D,C) \times P(C,B) \Big/ \sim$$

$$x \underset{C}{\otimes} y = \left[D \overset{y}{\rightsquigarrow} C \overset{x}{\rightsquigarrow} B \right]_C$$

$$\begin{tikzcd} \underline{B} \arrow[r, bend left, "P", "\shortmid"{marking}] \arrow[r, bend right, "P'"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow"] & \underline{C} \end{tikzcd}$$ nat. transf.

Have adjointness for profunctors.

$F : \underline{B} \to \underline{C}$ functor

$F_* : \underline{B} \nrightarrow \underline{C}$ profunctor $F_* = \underline{C}(-, F-)$

$F^* : \underline{C} \nrightarrow \underline{B}$, $F^* = \underline{C}(F-, -)$.

Then $F_* \dashv F^*$ (The point of it!)

$\underline{B}, \underline{C}$ tensored.

$F_* : \underline{B} \nrightarrow \underline{C}$ strong $\Leftrightarrow$ $F : \underline{B} \to \underline{C}$ strong.

$F^* : \underline{B} \nrightarrow C$ strong $\Leftrightarrow$ $\exists\, \tau : F(V \otimes B) \longrightarrow V \otimes F(B)$

??

$F_* \dashv F^* \iff V \otimes F(B) \xrightarrow{\cong} F(V \otimes B)$!

Strong profunctors compose.

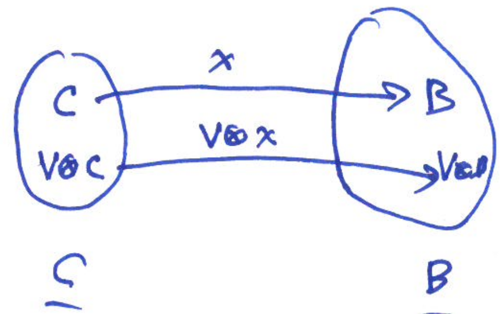
$$\underline{B} \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \underline{C} \qquad\qquad \sigma : P(C,B) \longrightarrow P(V\otimes C, V\otimes B)$$

$$\begin{tikzcd} & El(P) \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\ \underline{C} & & \underline{B} \end{tikzcd} \qquad x \in P(C,B) \longmapsto V\otimes x \in P(C,B)$$

$El(P)$ is tensored

P_1, P_2 strong

$$\begin{tikzcd} \underline{C} \arrow[dr] & & \underline{B} \arrow[dl] \\ & \underline{C} +_P \underline{B} & \end{tikzcd}$$



$\underline{C} +_P \underline{B}$ tensored

I_1, I_2 strong

$\underline{V}$ - profunctors $\quad P : \underline{C}^{op} \times \underline{B} \longrightarrow \underline{V}$

$\otimes$ requires colim in $\underline{V}$, pres by $\otimes$.

$\boxed{\text{Relationship ?}}$

For strong profunctors we have a notion of strong natural & dinatural transformations:

$$\begin{tikzcd}
P(C,B) \arrow[r, "t(C{,}B)"] \arrow[d, "\sigma"'] & Q(C,B) \arrow[d, "\sigma"] \\
P(V\otimes C, V\otimes B) \arrow[r, "t(V\otimes C{,}\, V\otimes B)"'] & Q(V\otimes C, V\otimes B)
\end{tikzcd}$$

or $x \in P(C,B)$: $\boxed{V\otimes t(x) = t(V\otimes x)}$