GENERAL ASSOCIATIVITY & GENERAL COMPOSITION IN DOUBLE CATEGORIES

BY

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DOUBLE CATEGORY = CATEGORY IN $\underline{\text{CAT}}$

$$\underline{D}_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \underline{D}_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \underline{D}_0$$

$\underline{D}_1$: Obj = HOR MORPH, MORPH = DOUBLE CELLS

$\underline{D}_0$: Obj = Obj, MORPH = VERT MORPHISMS

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, dashed, "w"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \qquad \text{OR} \qquad \boxed{\;\alpha\;}$$

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline\end{array} \xmapsto{\text{HOR}} \begin{array}{|c|}\hline \alpha\beta \\ \hline\end{array}$$

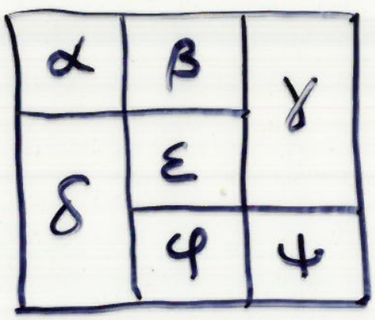
ASSOC

$$\begin{array}{|c|}\hline \alpha \\ \hline \gamma \\ \hline\end{array} \xmapsto{\text{VERT}} \begin{array}{|c|}\hline \alpha\cdot\gamma \\ \hline\end{array}$$

ASSOC

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline\end{array} \longmapsto (\alpha\beta)\cdot(\gamma\delta) = (\alpha\cdot\gamma)(\beta\cdot\delta)$$

<u>EXAMPLE</u>:



$$((\alpha\beta)\cdot(\delta(\varepsilon\cdot\varphi)))(\gamma\cdot\psi)$$
$$= (\alpha\cdot\delta)((\beta\cdot\varepsilon)\gamma)\cdot(\varphi\psi)$$

?

<u>IN DIM 1</u>: $\mu_1(a_1 \cdots a_n) = \mu_2(a_1 \cdots a_n)$?

INDUCTION ON n : ASSUME TRUE FOR $k < n$.

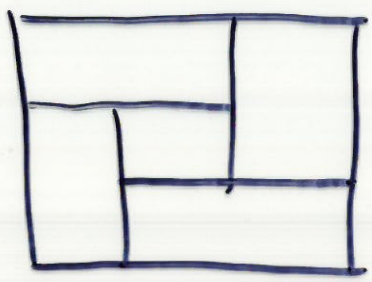
$$\mu_1(a_1 \cdots a_n) = (a_1 \cdots a_\ell)(a_{\ell+1} \cdots a_n)$$

$(\ell < m)$

$$\begin{aligned}\mu_2(a_1 \cdots a_n) &= (a_1 \cdots a_m)(a_{m+1} \cdots a_n) \\ &= ((a_1 \cdots a_\ell)(a_{\ell+1} \cdots a_m))(a_{m+1} \cdots a_n) \\ &= (a_1 \cdots a_\ell)((a_{\ell+1} \cdots a_m)(a_{m+1} \cdots a_n)) = \mu_1(a_1 \cdots a_n)\end{aligned}$$

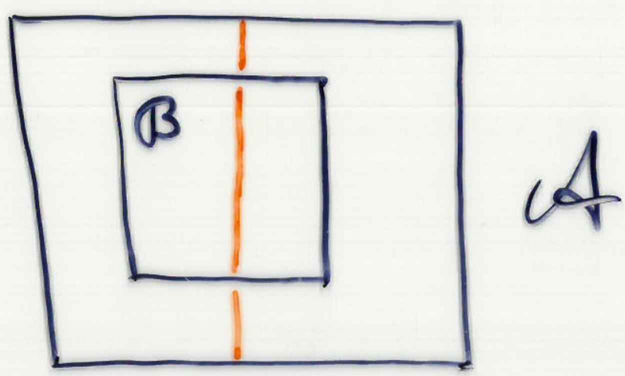
PROBLEM IS $\exists$ ARRANGEMENTS WHICH ARE NOT COMPOSABLE

E.G. THE PINWHEEL



<u>LEMMA</u>: IF A COMPATIBLE ARRANGEMENT $\mathcal{A}$ OF DOUBLE CELLS IS COMPOSABLE & $\mathcal{B}$ IS A COMPATIBLE SUB-ARRANGEMENT, THEN $\mathcal{B}$ is COMPOSABLE.

<u>PROOF</u>: INDUCTION ON # OF CELLS IN $\mathcal{A}$.



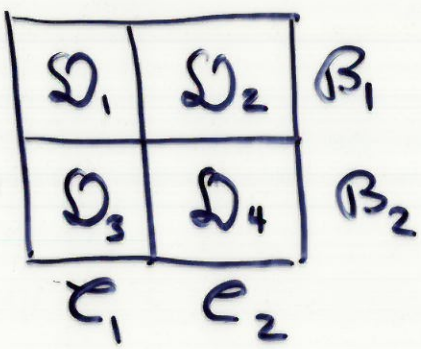
THEOREM 1: IF $\mathcal{A}$ IS COMPOSABLE IN TWO WAYS, THE RESULT IS THE SAME.

PROOF: IND. ON # CELLS OF $\mathcal{A}$.

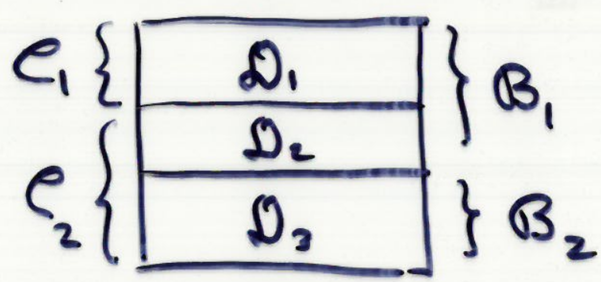
$$\mu_1(\mathcal{A}) = \mu(\mathcal{B}_1) * \mu(\mathcal{B}_2) \qquad * = \text{HOR, VERT}$$

$$\mu_2(\mathcal{A}) = \mu(\mathcal{C}_1) \,\square\, \mu(\mathcal{C}_2) \qquad \square = \text{HOR, VERT}$$

CASES → (1)



(2)



(1)
$$\begin{aligned}
\mu_1(\mathcal{A}) &= \mu(\mathcal{B}_1) \cdot \mu(\mathcal{B}_2) \\
&= \big(\mu(\mathcal{D}_1)\mu(\mathcal{D}_2)\big) \cdot \big(\mu(\mathcal{D}_3)\mu(\mathcal{D}_4)\big) \\
&= \big(\mu(\mathcal{D}_1) \cdot \mu(\mathcal{D}_3)\big)\big(\mu(\mathcal{D}_2) \cdot \mu(\mathcal{D}_4)\big) \\
&= \mu(\mathcal{C}_1)\mu(\mathcal{C}_2) \\
&= \mu_2(\mathcal{A}).
\end{aligned}$$

(2) & OTHER CASES SIMILAR. ∎

EXAMPLES:

- A CAT $\mathbb{C}$ GIVES DOUBLE CATEGORIES WITH DOUBLE CELLS

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \text{OR} \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

- A 2-CAT $\underline{\underline{A}}$ GIVES DOUBLE CATEGORIES WITH

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \text{OR} \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

- ~~A TOP SPACE X GIVES A DOUBLE CAT $\pi_2(X)$ WITH DOUBLE CELLS HOMOTOPY CLASSES OF CONTINUOUS FNS $\varphi : [0,1]^2 \longrightarrow X$.~~

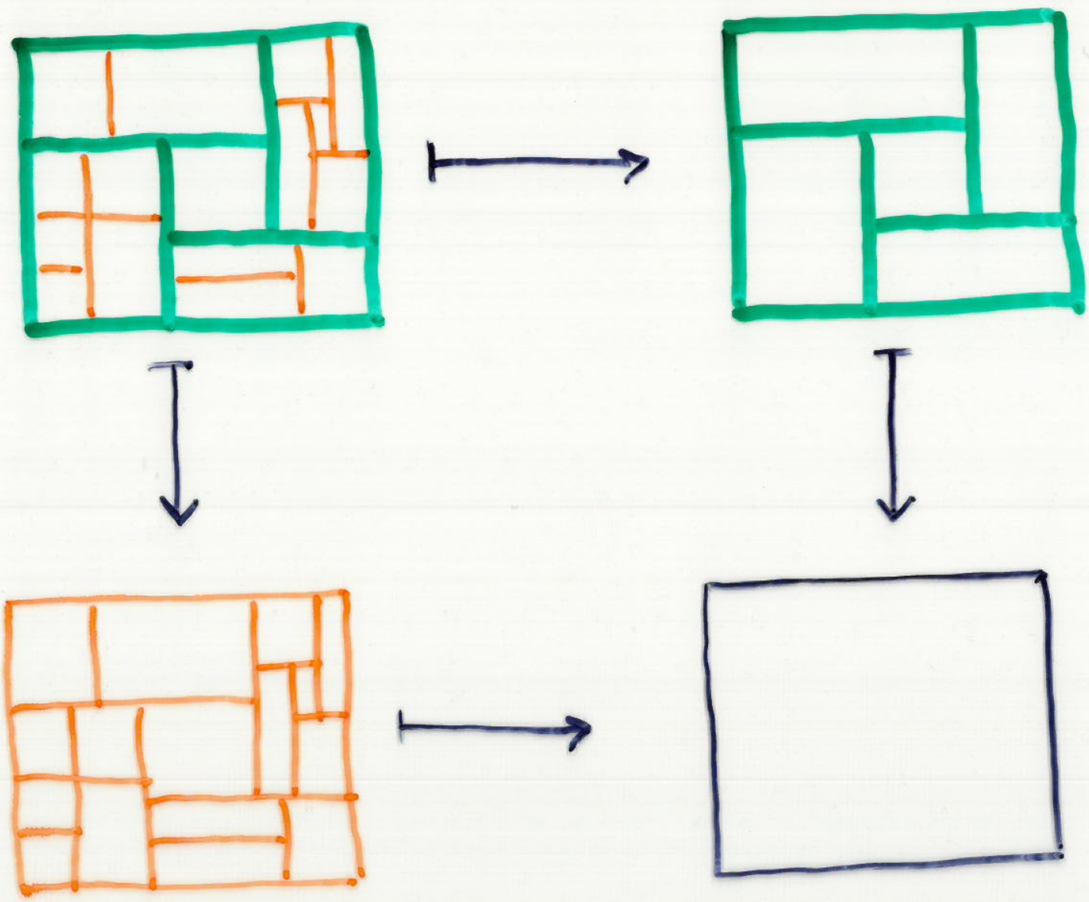
EACH OF THESE HAS <u>GENERAL COMPOSITION</u> FOR ALL ARRANGEMENTS:

$$\mu : \mathrm{ARR}(\mathbb{D}) \longrightarrow \mathbb{D}$$

WHY? WHAT'S INVOLVED?

SUPPOSE WE HAVE **GENERAL ASSOCIATIVITY**

$$
\begin{tikzcd}
\mathrm{ARR}\,\mathrm{ARR}(\mathbb{D}) \arrow[r, "\mathrm{ARR}(\mu)"'] \arrow[d, "\mu\,\mathrm{ARR}"'] & \mathrm{ARR}(\mathbb{D}) \arrow[d, "\mu"] \\
\mathrm{ARR}(\mathbb{D}) \arrow[r, "\mu"'] & \mathbb{D}
\end{tikzcd}
$$

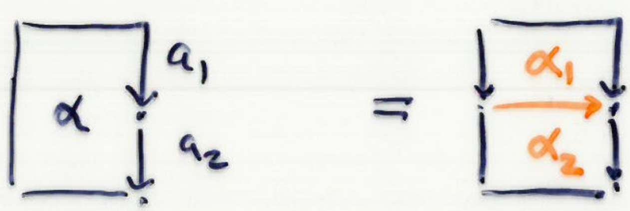


THEN IF $\mathcal{A} \mapsto \mathcal{B}$ BY COMPOSING 2 CELLS, $\mu(\mathcal{A}) = \mu(\mathcal{B})$.

REVERSELY IF $\mathcal{A} \mapsto \mathcal{B}$ BY FACTORING A CELL, $\mu(\mathcal{A}) = \mu(\mathcal{B})$ TOO.

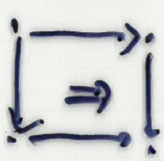
<u>IDEA</u>: USE COMPOSITIONS & FACTORIZATIONS TO REDUCE $\mathcal{A}$ ~~TO~~ A SINGLE CELL!

RL- FACTORIZATION (ALSO LR, TB, BT)

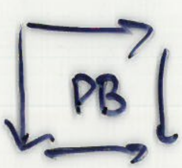


<u>EX</u>:

RL & LR

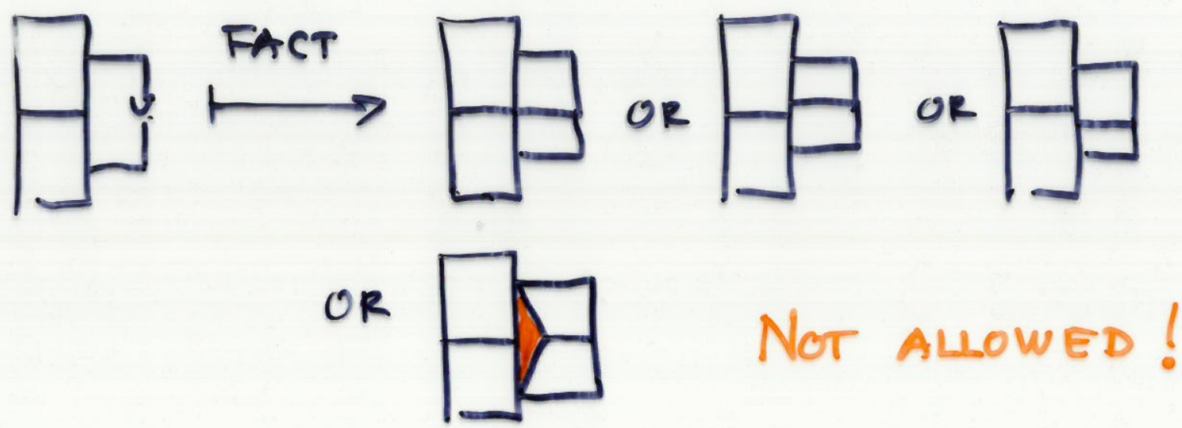


ALL 4 ... "$\pi_2(X)$" TOO.



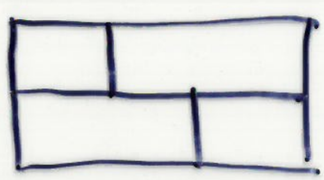
RL & BT

2



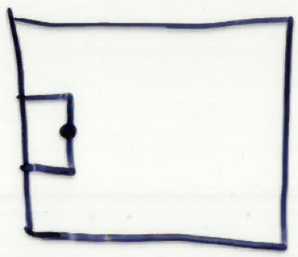
MUST BE CLEAR ABOUT WHAT A COMPATIBLE ARRANGEMENT IS (TILE ORDERS).

INDEXED BY PARTITIONS OF RECTANGLES ($\text{IN } \mathbb{R}^2$ OR $\mathbb{N}^2$). MAIN THING IS ALL SEGMENTS REPRESENTED AS ARROWS



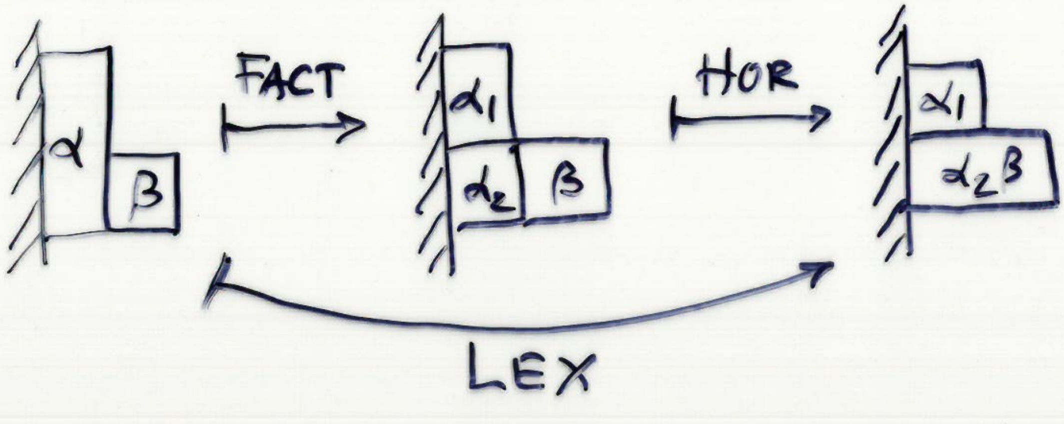
<u>FACT</u>: MEANS $\exists\, \mathcal{B} \xmapsto{\text{COMP}} \mathcal{A}$ (WHOLE ARRANGEMENT).

NO PROBLEM WITH RL-FACTORIZATION ALONG LEFT BORDER

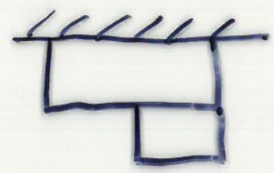


FACTORIZATION INTRODUCES MORE CELLS SO FURTHER FROM COMPOSITE

LEFT EXCHANGE



ALSO HAVE

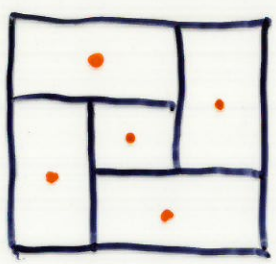
REX  TEX  (BEX)

RANK

REPRESENT THE ARRANGEMENT $\mathcal{A}$ AS RECTANGLES IN $\mathbb{N}^2$ (I.E. COORDINATIZE)

$$\rho(\mathcal{A}) = \sum \left(\begin{array}{l} \text{DIST FROM CENTRE OF EACH} \\ \text{RECT. TO TOP OF ARRANGEMENT} \end{array} \right)$$

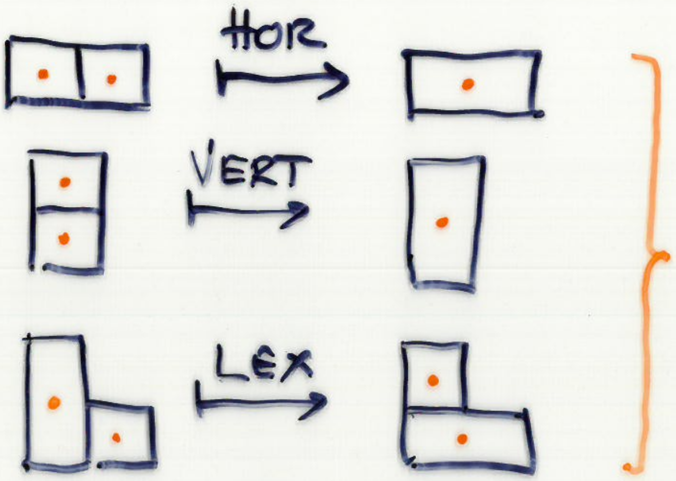
E.G.



3×3

$$\rho(\ \cdot\ ) = \tfrac{1}{2} + 1 + 2 + 1\tfrac{1}{2} + 2\tfrac{1}{2} = 7\tfrac{1}{2}$$

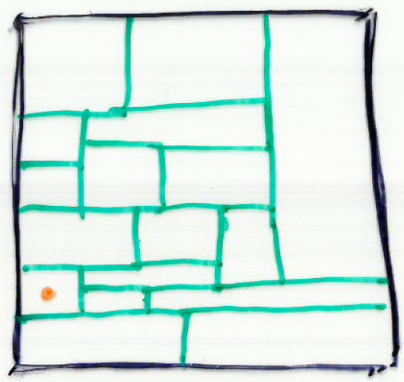
$\rho(\mathcal{A}) \in \frac{1}{2}\mathbb{N}$



ALL DECREASE RANK

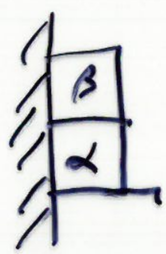
REX & TEX TOO

KEY TILE = THE LOWEST OF THOSE ON THE LEFT BORDER THAT HAS AT MOST ONE UPPER NEIGHBOUR.



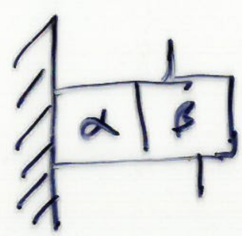
KEY OPERATION

(1) VERT OF KEY TILE WITH ONE ABOVE IT IF POSSIBLE



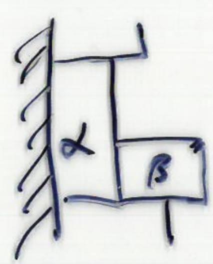
ELSE

(2) HOR OF KEY TILE WITH ONE TO RIGHT IF POSSIBLE



ELSE

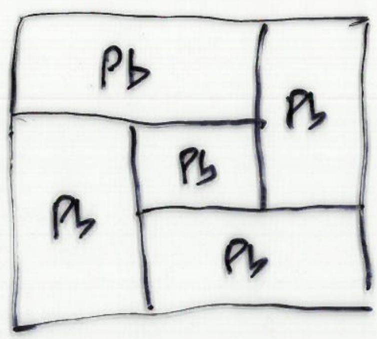
(3) LEX OF KEY TILE WITH LOWEST RIGHT NBR



THEOREM 2: IF $\mathbb{D}$ ADMITS RL-FACTORIZATIONS, ANY RECTANGULAR ARRANGEMENT CAN BE REDUCED TO A SINGLE CELL BY A FINITE NO OF COMPOSITIONS & FACTORIZATIONS (OR LEFT EXCHANGES).

PROOF: AS LONG AS THERE IS MORE THAN ONE CELL, THE KEY OPERATION CAN BE PERFORMED & THIS REDUCES THE RANK. $\square$

Ex:



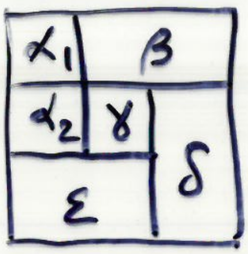
$\Rightarrow$ Pb

CHOOSE RL-FACTORIZATIONS IN $\mathbb{D}$.

GET $\mu(\mathcal{A})$ - KEY BASED COMPOSITE

(LIKE PUTTING ALL BRACKETS TO THE FRONT)

DO WE GET GENERAL ASSOC? NO

μ  $= (\alpha_1\beta)\cdot(((\alpha_2\gamma)\cdot\varepsilon)\delta)$

THEOREM 3: IF $\mathbb{D}$ HAS ANY 2 OF THE FACTORIZATIONS RL, LR, TB, BT, THEN ANY 2 WAYS OF REDUCING A RECT. ARRANG. TO A SINGLE CELL USING COMPOSITES AND FACTORIZATIONS YIELD THE SAME RESULT. CONSEQUENTLY μ SATISFIES GENERAL ASSOCIATIVITY.

"PROOF" ① WLOG HAVE RL & LR OR RL & BT.

② MAKE CHOICE OF RL-FACT & DEFINE μ THE KEY BASED COMPOSITE

WILL SHOW THAT IF $\mathcal{A} \overset{\lambda}{\longmapsto} \mathcal{B}$

$\lambda =$ (a) VERT or (b) HOR, THEN

$\mu(\mathcal{A}) = \mu(\mathcal{B})$.

③ COORDINATIZE $\mathcal{A}$ WITH INTEGERS

PROOF WILL BE BY INDUCTION ON RANK.

WILL SHOW THAT IF κ IS THE KEY OPERATION

$$\begin{tikzcd}
\mathcal{A} \arrow[r, mapsto, "\lambda"] \arrow[d, mapsto, "\kappa"'] & \mathcal{B} \\
\mathcal{C} &
\end{tikzcd}$$

$\exists$ CHAIN OF LESSER RANK JOINING $\mathcal{B}$ TO $\mathcal{C}$

$\mathcal{B} \mapsto \mathcal{D}_1 \leftarrow\!\shortmid \mathcal{D}_2 \mapsto \cdots \leftarrow\!\shortmid \mathcal{C}$. THEN

$\mu(\mathcal{B}) = \mu(\mathcal{D}_1) = \cdots = \mu(\mathcal{C}) = \mu(\mathcal{A})$.

3 POSSIBILITIES FOR κ : (1) VERT, (2) HOR, (3) LEX.

NEED MORE POSSIBILITIES FOR λ

(a) VERT, (b) HOR, (c) LEX, (d) REX, (e) TEX

15 CASES : (1a) – (3e)

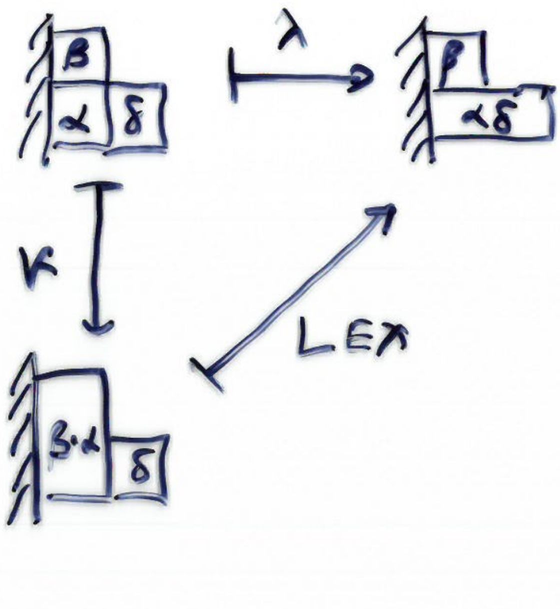
EX: (1b) $\quad \kappa =$ VERT

$\lambda =$ HOR

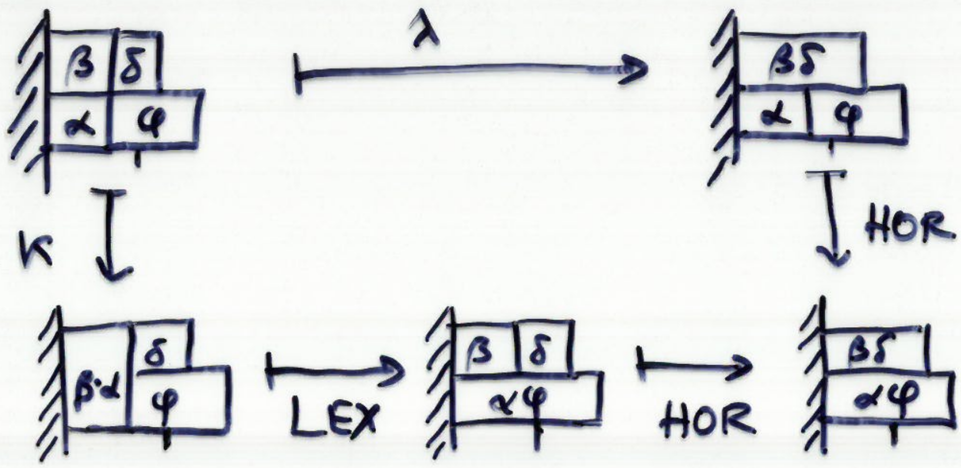
(0) NO OVERLAP : COMMUTE

$$\begin{tikzcd} \mathcal{A} \arrow[r, mapsto, "\text{HOR}"] \arrow[d, mapsto, "\text{VERT}"'] & \mathcal{B} \arrow[d, mapsto, "\text{VERT}"] \\ \mathcal{C} \arrow[r, mapsto, "\text{HOR}"'] & \mathcal{D} \end{tikzcd}$$

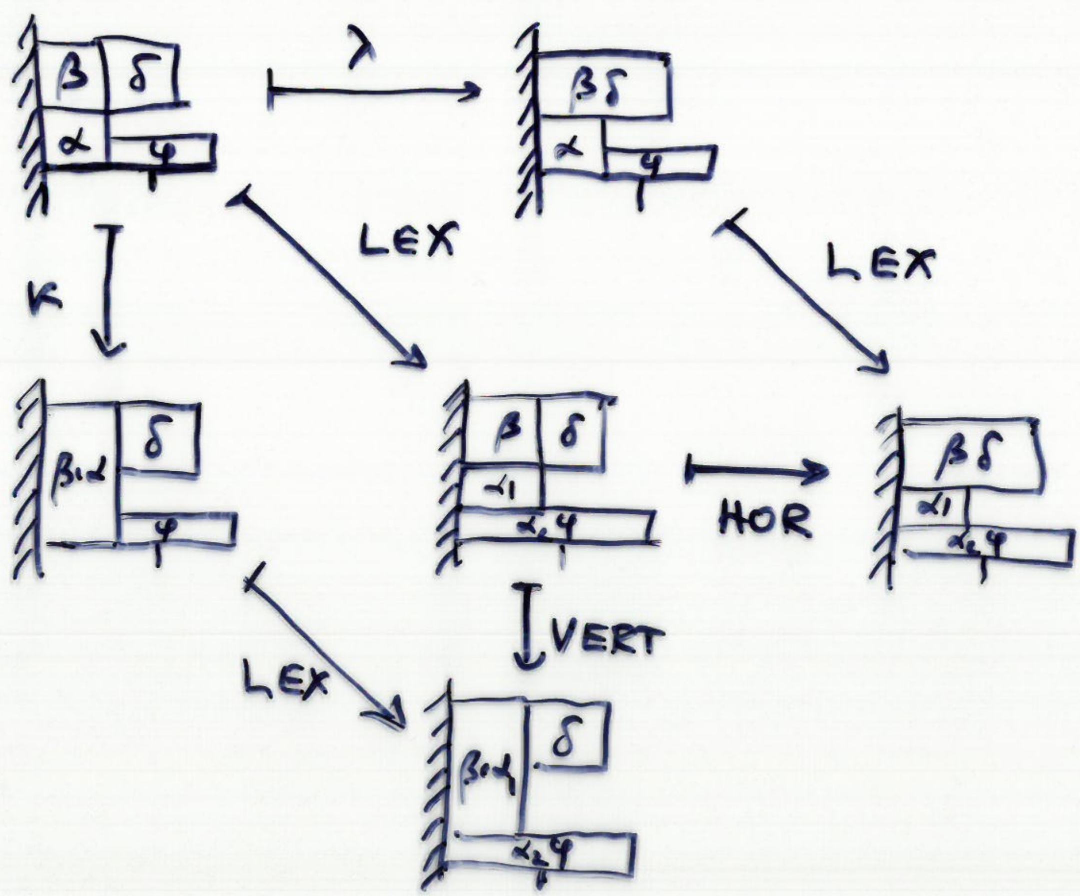
(i) $\gamma = \alpha$:



(ii) $\gamma = \beta$, α just 1 RIGHT NBR φ



(iii) $\gamma = \beta$, α SEVERAL RIGHT NBRS

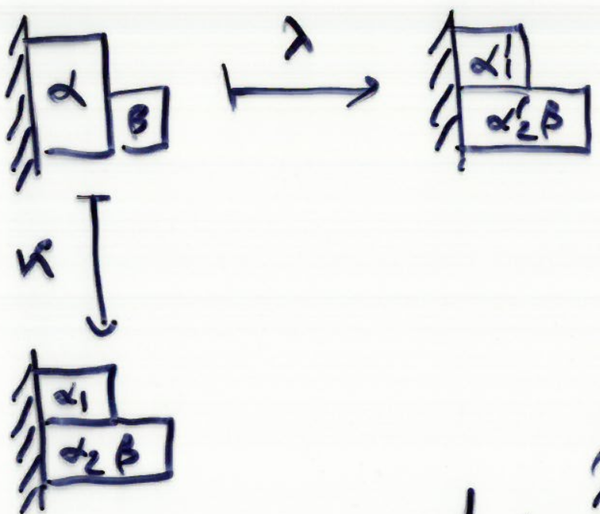


$14\frac{1}{2}$ CASES WORK THIS WAY — NO NEED FOR THE EXTRA FACTORIZATION.

ONLY (3c) NEEDS IT

κ = LEX

λ = LEX



LET ν BE TEX OR REX DUAL TO KEY OP.
(OR HOR OR VERT)
~~NOT~~ LEX

$$\begin{tikzcd} \mathcal{A} \arrow[r, mapsto, "\lambda"] \arrow[d, mapsto, "\kappa"'] \arrow[dr, mapsto, "\nu"] & \mathcal{B} \arrow[d, no head, squiggly] \\ \mathcal{C} \arrow[r, no head, squiggly] & \mathcal{D} \end{tikzcd}$$

SAME

BY (3a), (3b), (3d) OR (3e)