MEALY MORPHISMS OF ENRICHED CATEGORIES

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BACKGROUND

A MONOIDAL CATEGORY IS A CATEGORY $\underline{V}$ EQUIPPED WITH A "TENSOR PRODUCT" $\otimes : \underline{V} \times \underline{V} \to \underline{V}$ AND A UNIT OBJECT I WHICH IS ASSOCIATIVE

$$\alpha_{ABC} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$$

AND UNITARY

$$\rho_A : A \otimes I \xrightarrow{\cong} A \quad \& \quad \lambda_A : I \otimes A \xrightarrow{\cong} A$$

THE α, ρ, λ SATISFYING SOME "COHERENCE CONDITIONS".

Ex: $\underline{V} = \underline{Ab}$, $\otimes$ THE USUAL, $I = \mathbb{Z}$.

Ex: $\underline{V} = \underline{SET}$, $\otimes$ = CARTESIAN PRODUCT, $I = 1$.

Ex: $\underline{V} = \underline{CAT}$, " " $I = \mathbb{1}$.

A $\underline{V}$-CATEGORY $\underline{A}$ CONSISTS OF A CLASS OF OBJECTS $\mathrm{Ob}\,\underline{A}$, FOR EACH PAIR OF OBJECTS A, B AN OBJECT $\underline{A}(A,B) \in \underline{V}$, FOR EACH OBJECT A AN "IDENTITY MORPHISM" $\mathrm{ID}_A : I \to \underline{A}(A,A)$ AND A "COMPOSITION MORPHISM" FOR EACH A, B, C

$$\circ_{A,B,C} : \underline{A}(B,C) \otimes \underline{A}(A,B) \longrightarrow \underline{A}(A,C)$$

SATISFYING ASSOCIATIVITY AND UNIT LAWS.

EXAMPLE: $\underline{V} = \underline{Ab}$ A $\underline{V}$-CATEGORY IS AN ADDITIVE CATEGORY. E.G. R-$\underline{MOD}$.

A RING R IS AN $\underline{Ab}$-CATEGORY WITH ONE OBJECT.

EXAMPLE: A $\underline{SET}$-CATEGORY IS AN ORDINARY CATEGORY.

EXAMPLE: A CAT-CATEGORY IS CALLED A 2-CATEGORY. IT HAS OBJECTS, ARROWS AND 2-CELLS

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Ex: CAT ITSELF IS A 2-CATEGORY - THE 2-CELLS ARE NATURAL TRANSFORMATIONS.

A BICATEGORY IS A WEAK 2-CATEGORY: COMPOSITION OF ARROWS IS ONLY ASSOCIATIVE AND UNITARY UP TO COHERENT ISOMORPHISMS.

EXAMPLE: OBJ - RINGS, ARROWS $R \to S$ S-R BIMODULES, 2-CELLS LINEAR FUNCTIONS

COMPOSITION

$$\begin{tikzcd} & S \arrow[dr, "{}_T Q_S"] & \\ R \arrow[ur, "{}_S P_R"] \arrow[rr, "Q \otimes_S P"'] & & T \end{tikzcd}$$

<u>EXAMPLE</u>: A MONOIDAL CATEGORY CAN BE CONSIDERED AS A ONE OBJECT BICATEGORY.

<u>MORPHISMS OF $\underline{V}$-CATEGORIES</u>

A $\underline{V}$-FUNCTOR $F : \underline{A} \to \underline{B}$ IS GIVEN BY A FUNCTION $F : \mathrm{Ob}\,\underline{A} \longrightarrow \mathrm{Ob}\,\underline{B}$, AND FOR EACH PAIR OF OBJECTS $A, A' \in \underline{A}$, A $\underline{V}$-MORPHISM $F : \underline{A}(A, A') \longrightarrow \underline{B}(FA, FA')$ SATISFYING THE COMMUTATIVITY OF 2 DIAGRAMS EXPRESSING THAT F PRESERVES COMPOSITION AND UNITS, IN A WAY INTERNAL TO $\underline{V}$.

<u>EXAMPLE</u>: $\underline{V} = \underline{SET}$ WE GET ORDINARY FUNCTORS

<u>EXAMPLE</u>: $\underline{V} = \underline{Ab}$ " " ADDITIVE "

<u>EXAMPLE</u>: $\underline{V} = \underline{CAT}$ " " 2-FUNCTORS.

LAX MORPHISMS

FOR 2-CATEGORIES AND BICATEGORIES WE MIGHT WANT TO RELAX THE PRESERVATION OF COMPOSITION AND IDENTITIES

$$F : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$$

OBJ $\mapsto$ OBJ , ARR $\mapsto$ ARR , 2-CELLS $\mapsto$ 2-CELLS

PRESERVING ALL DOMAINS & CODOMAINS BUT WITH ADDED STRUCTURE

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(f'f)"'] & FA' \arrow[r, "Ff'"] \arrow[d, phantom, "\Downarrow \mu_{f',f}"] & FA'' \\ & {} & \end{tikzcd}$$

AND

$$\begin{tikzcd} FA \arrow[r, bend left, "1_{FA}"] \arrow[r, bend right, "F(1_A)"'] \arrow[r, phantom, "\Downarrow \eta_A"] & FA \end{tikzcd}$$

SATISFYING ASSOCIATIVITY AND UNIT EQUATIONS.

IF μ, η ARE IDENTITIES WE GET <u>STRICT</u> MORPHISMS

IF μ, η ARE ISOS WE GET <u>PSEUDO</u> MORPHISMS

IF μ, η ARE NOT ISOS WE GET <u>LAX</u> MORPHISMS

$\underline{V}$ A $\otimes$ CATEGORY

IN "BICATEGORIES I", BÉNABOU REMARKED THAT A $\underline{V}$-CATEGORY IS THE SAME AS A CLASS $\mathrm{Ob}\,\underline{A}$ AND A LAX MORPHISM OF BICATEGORIES $\underline{A} : \underline{\underline{\mathrm{Ob}\,A}} \longrightarrow \underline{\underline{V}}$.

$\underline{\underline{\mathrm{Ob}}}\,\underline{A}$: OBJECTS ARE $A, A', A'' \in \mathrm{Ob}\,\underline{A}$

UNIQUE 1-CELL $A \to A'$ $\forall\, A, A'$

ONLY IDENTITY 2-CELLS

$\underline{\underline{V}}$: ONE OBJECT $*$

1-CELLS OBJECTS OF $\underline{V}$

COMPOSITION $* \xrightarrow{V} * \xrightarrow{V'} * \;=\; * \xrightarrow{V' \otimes V} *$

$\underline{A}$: $A \longmapsto *$

$(A, A') : A \to A' \longmapsto \underline{A}(A, A') \in \underline{V}$

$$\begin{tikzcd}[column sep=small]
 & A' \arrow[dr, "{(A',A'')}"] & \\
A \arrow[ur, "{(A,A')}"] \arrow[rr, "{(A,A'')}"'] & & A''
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=small]
 & * \arrow[dr, "{\underline{A}(A',A'')}"] \arrow[d, phantom, "\Downarrow\circ"] & \\
* \arrow[ur, "{\underline{A}(A,A')}"] \arrow[rr, "{\underline{A}(A,A'')}"'] & \, & *
\end{tikzcd}$$

$\underline{A}(A', A'') \otimes \underline{A}(A, A') \longrightarrow \underline{A}(A, A'')$

LAX AND OPLAX TRANSFORMATIONS OF LAX FUNCTORS WERE DEFINED IN "BICATEGORIES I" BUT $\underline{V}$-FUNCTORS WERE NOT CONSIDERED IN THIS CONTEXT.

A $\underline{V}$-FUNCTOR $\underline{A} \longrightarrow \underline{B}$ IS A PAIR (F, φ)

$$\begin{tikzcd} \underline{\underline{Ob}}\,\underline{A} \arrow[rr, "F"] \arrow[dr, "\underline{A}"'] & \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & \underline{\underline{Ob}}\,\underline{B} \arrow[dl, "\underline{B}"] \\ & \underline{\underline{V}} & \end{tikzcd}$$

WHERE F IS A FUNCTION ON OBJECTS AND φ IS A LAX TRANSFORMATION $\underline{B} \circ F \longrightarrow \underline{A}$ WHICH IS THE IDENTITY ON OBJECTS (C.F. ICON).

$$\begin{tikzcd} \underline{B}(FA) \arrow[r, "\varphi A"] \arrow[d, "{\underline{B}(FA,FA')}"'] \arrow[dr, phantom, "\overset{\varphi(A,A')}{\Leftarrow}"] & \underline{A}(A) \arrow[d, "{\underline{A}(A,A')}"] \\ \underline{B}(FA') \arrow[r, "\varphi A'"'] & \underline{A}(A') \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r, "I"] \arrow[d, "{\underline{B}(FA,FA')}"'] \arrow[dr, phantom, "\overset{\varphi(A,A')}{\Leftarrow}"] & * \arrow[d, "{\underline{A}(A,A')}"] \\ * \arrow[r, "I"'] & * \end{tikzcd}$$

I.E. $\varphi(A,A') : \underline{A}(A,A') \to \underline{B}(FA, FA')$ + CONDITIONS.

DEFINITION LET $\underline{A}$ AND $\underline{B}$ BE $\underline{V}$-CATEGORIES.

A MEALY MORPHISM $(F,\varphi): \underline{A} \longrightarrow \underline{B}$ CONSISTS OF:

OBJECT FUNCTIONS $F: \mathrm{ob}\underline{A} \to \mathrm{ob}\underline{B}$, $\varphi: \mathrm{ob}\underline{A} \longrightarrow \mathrm{ob}\underline{V}$;

FOR EACH PAIR $A, A' \in \underline{A}$ A MORPHISM

$$\varphi(A,A'): \underline{A}(A,A') \otimes \varphi A \longrightarrow \varphi A' \otimes \underline{B}(FA, FA') ;$$

SATISFYING

(Mm1)

$$\begin{tikzcd}
I \otimes \varphi A \arrow[r, "\mathrm{id} \otimes \varphi A"] \arrow[d, "\mathrm{CAN}", "\cong"'] & \underline{A}(A,A) \otimes \varphi A \arrow[d, "\varphi(A,A)"] \\
\varphi A \otimes I \arrow[r, "\varphi A \otimes \mathrm{id}"'] & \varphi A \otimes \underline{B}(FA, FA)
\end{tikzcd}$$

(Mm2)

$$\begin{tikzcd}
\underline{A}(A',A'') \otimes \underline{A}(A,A') \otimes \varphi A \arrow[rr, "\circ \otimes \varphi A"] \arrow[d, "\underline{A}(A',A'') \otimes \varphi(A,A')"'] & & \underline{A}(A,A'') \otimes \varphi A \arrow[dd, "\varphi(A,A'')"] \\
\underline{A}(A',A'') \otimes \varphi A' \otimes \underline{B}(FA,FA') \arrow[d, "\varphi(A',A'') \otimes \underline{B}(FA,FA')"'] & & \\
\varphi A'' \otimes \underline{B}(FA',FA'') \otimes \underline{B}(FA,FA') \arrow[rr, "\varphi A'' \otimes \circ"'] & & \varphi A'' \otimes \underline{B}(FA, FA'')
\end{tikzcd}$$

MEALY MORPHISMS ARE EASILY COMPOSED

COMPOSITION IS NOT STRICTLY ASSOCIATIVE AS IT USES $\otimes$. NEED 2-CELLS.

DEFINITION LET $(F,\varphi), (G,\gamma) : \underline{A} \longrightarrow \underline{B}$ BE MEALY MORPHISMS. A MEALY CELL $t : (F,\varphi) \longrightarrow (G,\gamma)$ IS GIVEN BY MORPHISMS

$$tA : \varphi A \longrightarrow \gamma A \otimes \underline{B}(FA, GA)$$

SATISFYING

(Mc)
$$\begin{tikzcd}
\underline{A}(A,A') \otimes \varphi A \arrow[r, "\varphi(A,A')"] \arrow[d, "\underline{A}(A,A') \otimes tA"'] & \varphi A' \otimes \underline{B}(FA,FA') \arrow[d, "tA' \otimes \underline{B}(FA,FA')"] \\
\underline{A}(A,A') \otimes \gamma A \otimes \underline{B}(FA,GA) \arrow[d, "\gamma(A,A') \otimes \underline{B}(FA,GA)"'] & \gamma A' \otimes \underline{B}(FA',GA') \otimes \underline{B}(FA,FA') \arrow[d, "\gamma A' \otimes \circ"] \\
\gamma A' \otimes \underline{B}(FA,FA') \otimes \underline{B}(FA,GA) \arrow[r, "\gamma A' \otimes \circ"'] & \gamma A' \otimes \underline{B}(FA,GA')
\end{tikzcd}$$

GIVEN MEALY MORPHISMS $(F,\varphi): \underline{A} \to \underline{B}$ AND $(H,\eta): \underline{B} \to \underline{C}$ THE *COMPOSITE* IS $(HF, \varphi \otimes \eta F)$ WHERE $(\varphi \otimes \eta F)A = \varphi A \otimes \eta FA$ AND $(\varphi \otimes \eta F)(A, A')$ IS

$$
\begin{tikzcd}
\underline{A}(A,A') \otimes \varphi A \otimes \eta FA \arrow[r, "\varphi(A,A') \otimes 1"'] & \varphi A' \otimes \underline{B}(FA, FA') \otimes \eta FA \arrow[d, "1 \otimes \eta(FA, FA')"] \\
& \varphi A' \otimes \eta FA' \otimes \underline{C}(HFA, HFA').
\end{tikzcd}
$$

THEOREM $\underline{V}$-CATEGORIES, MEALY MORPHISMS AND MEALY CELLS FORM A BICATEGORY $\underline{V}$-$\underline{\underline{\text{MEALY}}}$.

EXAMPLE LET $\underline{I}$ BE THE $\underline{V}$-CATEGORY WITH ONE OBJECT $*$ AND $\underline{I}(*,*) = I$. FOR ANY $\underline{V}$-CATEGORY $\underline{B}$, A MEALY MORPHISM $\underline{I} \longrightarrow \underline{B}$ IS DETERMINED BY A PAIR (V, B) V IN $\underline{V}$, B IN $\underline{B}$.

A MEALY CELL $b : (V, B) \longrightarrow (V', B')$ IS A MORPHISM

$$b : V \longrightarrow V' \otimes \underline{B}(B, B').$$

THIS DESCRIBES THE CATEGORY $\underline{\tilde{B}}_0 := \underline{V}\text{-}\underline{\underline{\text{MEALY}}}(\underline{I}, \underline{B})$

IT IS THE UNDERLYING CATEGORY OF A $\underline{V}$-CATEGORY (ASSUMING SOME CLOSEDNESS CONDITIONS ON $\underline{V}$) WHICH WILL PLAY AN IMPORTANT ROLE LATER.

EXAMPLE $\underline{V}\text{-}\underline{\underline{\text{MEALY}}}(\underline{B}, \underline{I}) \cong \underline{V}\text{-}\underline{\underline{\text{CAT}}}(\underline{B}, \underline{V})$

EXAMPLE FOR $\underline{V} = \underline{Set}$, THE MORPHISMS

$$\varphi(A,A') : \underline{A}(A,A') \times \varphi A \longrightarrow \varphi A' \times \underline{B}(FA, FA')$$

HAVE TWO COMPONENTS:

$$\varphi_1(A,A') : \underline{A}(A,A') \times \varphi A \longrightarrow \varphi A'$$

WHICH MAKE φ INTO A FUNCTOR $\underline{A} \longrightarrow \underline{Set}$, AND

$$\varphi_2(A,A') : \underline{A}(A,A') \times \varphi A \longrightarrow \underline{B}(FA, FA')$$

WHICH GIVE A LIFTING OF F TO A FUNCTOR

$$\begin{tikzcd} & El(\Phi) \arrow[dl, "P"'] \arrow[dr, "\bar{F}"] & \\ \underline{A} & & \underline{B} \end{tikzcd}$$

IN THIS WAY A MEALY MORPHISM $(F, \varphi) : \underline{A} \longrightarrow \underline{B}$ CAN BE VIEWED AS A SPAN IN $\underline{CAT}$ WHERE THE LEFT LEG IS A DISCRETE OPFIBRATION AND THE RIGHT LEG IS OBJECT-WISE CONSTANT ON THE FIBRES.

"PARTIAL FUNCTORS"

EXAMPLE/DEFINITION A MEALY MACHINE [GEORGE MEALY 1955] HAS

- A (FINITE) SET S OF STATES
- AN INPUT ALPHABET Σ
- AN OUTPUT ALPHABET Λ
- A TRANSITION FUNCTION $t : \Sigma \times S \longrightarrow S$
- AN OUTPUT FUNCTION $g : \Sigma \times S \longrightarrow \Lambda$
- A START STATE $s_0 \in S$

RUN AS FOLLOWS:

TAKE A WORD IN INPUT ALPHABET $a_m a_{m-1} \cdots a_1$

START IN STATE s_0

NEW STATE $s_1 = t(a_1, s_0)$ OUTPUT $b_1 = g(a_1, s_0)$

THEN $s_2 = t(a_2, s_1)$ $b_2 = g(a_2, s_1)$

ETC. $\vdots$ $\vdots$

SO $a_m \cdots a_2 a_1 \rightsquigarrow b_m \cdots b_2 b_1$

EXTEND t AND g TO FREE MONOIDS

$$\bar{t} : \Sigma^* \times S \longrightarrow S$$
$$\bar{g} : \Sigma^* \times S \longrightarrow \Lambda^*$$

OR $\Sigma^* \times S \longrightarrow S \times \Lambda^*$

EXAMPLE A $\underline{V}$-FUNCTOR $F : \underline{A} \to \underline{B}$ CORRESPONDS EXACTLY TO A MEALY MORPHISM $F_\bullet = (F, I) : \underline{A} \longrightarrow \underline{B}$, WHERE I REPRESENTS THE FUNCTION $\mathrm{OB}\,\underline{A} \longrightarrow \mathrm{OB}\,\underline{V}$ WITH CONSTANT VALUE I. COMPOSITION IS PRESERVED

$$G_\bullet F_\bullet \cong (GF)_\bullet$$

$\underline{V}$-NATURAL TRANSFORMATIONS $F \to K$ ARE IN BISECTION WITH MEALY CELLS $F_\bullet \to K$.

THUS WE GET A LOCALLY FULLY FAITHFUL EMBEDDING

$$(\ )_\bullet : \underline{V}\text{-}\underline{\underline{\mathrm{CAT}}} \longrightarrow \underline{V}\text{-}\underline{\underline{\mathrm{MEALY}}}$$

DEFINITION $\underline{B}$ IS LEFT TENSORED IF FOR EVERY V IN $\underline{V}$ AND B IN $\underline{B}$ THERE IS AN OBJECT $V \otimes B$ IN $\underline{B}$ AND A MORPHISM

$$k : V \longrightarrow \underline{B}(B, V \otimes B)$$

WHICH MEDIATES A BIJECTION

$$\frac{X \longrightarrow \underline{B}(V \otimes B, B')}{X \otimes V \longrightarrow \underline{B}(B, B')}$$

FOR EACH X IN $\underline{V}$ AND B' IN $\underline{B}$.

THEOREM LET $\underline{A}$ AND $\underline{B}$ BE $\underline{V}$-CATEGORIES WITH $\underline{B}$ TENSORED. THEN THE EMBEDDING

$$(\ )_\bullet : \underline{V}\text{-}\underline{\underline{CAT}}(\underline{A}, \underline{B}) \longrightarrow \underline{V}\text{-}\underline{\underline{MEALY}}(\underline{A}, \underline{B})$$

HAS A LEFT ADJOINT.

"PROOF" IF $(F, \varphi) : \underline{A} \longrightarrow \underline{B}$ IS A MEALY MORPHISM, THEN $\hat{F}A = \varphi A \otimes FA$ IS THE OBJECT PART OF A $\underline{V}$-FUNCTOR WHICH WILL BE THE REFLECTION OF (F, φ). $\square$

PROFUNCTORS = THE CATEGORICIAN'S RELATIONS

A RELATION $R : A \to B$ IS A FUNCTION

$A \times B \to \{T, F\}$.

THE COMPOSITE OF $A \xrightarrow{R} B \xrightarrow{S} C$ IS

$$a(S \circ R)c \iff \exists b\,(aRb \wedge bSc).$$

THE IDENTITY $I_A : A \to A$ IS EQUALITY

$$a\, I_A\, a' \iff a = a'$$

THIS GIVES THE CATEGORY REL

REL IS A 2-CATEGORY: THERE IS A 2-CELL

(UNIQUE) $R \Rightarrow R'$ IFF $\forall_{ab}(aRb \Rightarrow aR'b)$.

A FUNCTION $f : A \to B$ GIVES TWO RELATIONS

$f_* : A \to B$, $a f_* b \iff (fa = b)$

$f^* : B \to A$, $b f^* a \iff (b = fa)$

f_* IS LEFT ADJOINT TO f^*

FOR $R: A \nrightarrow B$ AND $S: B \nrightarrow A$, R LEFT ADJOINT TO S

$R \dashv S$ if

$$R \circ S \Rightarrow I_B \quad \text{AND} \quad I_A \Rightarrow S \circ R$$

R IS (COMES FROM) A FUNCTION $A \to B$ IFF

IT HAS A RIGHT ADJOINT.

A <u>PROFUNCTOR</u> $P: \underline{A} \nrightarrow \underline{B}$ IS A FUNCTOR

$P: \underline{B}^{op} \times \underline{A} \longrightarrow \underline{SET}$.

THE <u>COMPOSITE</u> OF $\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{C} \quad = Q \otimes P$

IS DEFINED BY

$$Q \otimes P(C,A) = \sum_B P(B,A) \times Q(C,B) \Big/ \sim \qquad \text{(}\sim\text{MATRIX MULT)}$$

$$= \int^B P(B,A) \times Q(C,B)$$

$$\sum_{B,B'} P(B,A) \times \underline{B}(B',B) \times Q(C,B') \rightrightarrows \sum_B P(B,A) \times Q(C,B) \twoheadrightarrow Q \otimes P(C,A) \qquad (\sim M \otimes_R N)$$

THE <u>IDENTITY</u> $Id_{\underline{A}} : \underline{A} \nrightarrow \underline{A}$ IS THE HOM FUNCTOR

$$\underline{A}(-,-) : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{SET} .$$

A <u>2-CELL</u> $P \Rightarrow P' : \underline{A} \nrightarrow \underline{B}$ IS A NATURAL TRANSFORMATION $t : P \to P' : \underline{B}^{op} \times \underline{A} \longrightarrow \underline{SET}$

WE GET A <u>BICATEGORY</u> $\underline{\underline{PROF}}$.

<u>NOTATION</u> WRITE $x \in P(B,A)$ AS $B \xrightarrow[P]{x} A$

FUNCTORIALITY SAYS WE CAN "COMPOSE" x WITH MORPHISMS $A \xrightarrow{a} A'$, $ax : B \nrightarrow A'$ AND MORPHISMS $B' \xrightarrow{b} B$, $xb : B' \nrightarrow A$. THESE COMPOSITS ARE ASSOCIATIVE AND UNITARY.

AN ELEMENT OF $Q \otimes P(C,A)$ IS AN EQUIVALENCE CLASS OF PAIRS $[C \xrightarrow[Q]{y} B \xrightarrow[P]{x} A]$. THE EQUIVALENCE IS GENERATED BY

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, equal] & B \arrow[r, "x"] \arrow[d, "b"'] & A \arrow[d, equal] \\
C \arrow[r, "y'"'] & B' \arrow[r, "x'"'] & A
\end{tikzcd}$$

IF WE WRITE THE EQUIVALENCE CLASS AS $x \otimes y$

THEN $x'b \otimes y = x' \otimes by$.

EVERY FUNCTOR $F : \underline{A} \longrightarrow \underline{B}$ DETERMINES TWO PROFUNCTORS

$F_* : \underline{A} \nrightarrow \underline{B}$, $\quad F_*(B, A) = \underline{B}(B, FA)$ AND

$F^* : \underline{B} \nrightarrow \underline{A}$, $\quad F^*(A, B) = \underline{B}(FA, B)$.

WE HAVE $F_* \dashv F^*$

NOT QUITE TRUE THAT AN ADJOINT PAIR OF PROFUNCTORS $P \dashv Q$ COME FROM A FUNCTOR.

$\underline{B}$ IS <u>CAUCHY COMPLETE</u> IF THIS IS TRUE FOR EVERY $P : \underline{A} \nrightarrow \underline{B}$ WITH A RIGHT ADJOINT.

FOR <u>SET</u> BASED PROFUNCTORS, $\underline{B}$ IS CAUCHY COMPLETE IFF $\underline{B}$ HAS SPLIT IDEMPOTENTS.

THE <u>CAUCHY COMPLETION</u> OF $\underline{B}$ IS THE CATEGORY OF PROFUNCTORS $\mathbb{1} \nrightarrow \underline{B}$ WITH A RIGHT ADJOINT.

A RELATION $R : A \nrightarrow B$ CAN EQUALLY WELL BE GIVEN BY ITS GRAPH $\Gamma R \subseteq A \times B$.

ANALAGOUSLY, GIVEN A PROFUNCTOR $P : \underline{A} \nrightarrow \underline{B}$ WE CONSTRUCT ITS CATEGORY OF ELEMENTS $\underline{El}(P)$

AN OBJECT OF $\underline{El}(P)$ is $(B, A, x \in P(B,A))$

A MORPHISM $(B, A, x) \to (B', A', x')$ IS A PAIR $b : B \to B'$, $a : A \to A'$ SUCH THAT

$$P(B,a)(x) = P(b,A')(x)$$

IE

$$\begin{tikzcd} B \arrow[r, "x", "\bullet"' marking] \arrow[d, "b"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "a"] \\ B' \arrow[r, "x'"', "\bullet" marking] & A' \end{tikzcd}.$$

WE GET

$$\begin{tikzcd} & \underline{El}(P) \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \underline{A} & & \underline{B} \end{tikzcd}$$

A DISCRETE BIFIBRATION

$P \cong \pi'_* \otimes \pi^*$.

MEALY MORPHISMS CAN BE VIEWED AS PROFUNCTORS

RECALL THAT A PROFUNCTOR $P : \underline{A} \to\!\!\!\!\!\bullet\; \underline{B}$ IS GIVEN BY AN OBJECT FUNCTION $P : \mathrm{Ob}\,\underline{B} \times \mathrm{Ob}\,\underline{A} \longrightarrow \mathrm{Ob}\,\underline{V}$ AND LEFT AND RIGHT ACTIONS

$$\lambda : \underline{A}(A, A') \otimes P(B, A) \longrightarrow P(B, A')$$
$$\rho : P(B, A) \otimes \underline{B}(B', B) \longrightarrow P(B', A)$$

SATISFYING UNIT LAWS (2) AND ASSOCIATIVITY (3). (THINK $\underline{V}$-FUNCTOR $\underline{B}^{op} \otimes \underline{A} \longrightarrow \underline{V}$.)

A MORPHISM OF PROFUNCTORS $t : P \longrightarrow Q$ IS AN EQUIVARIANT FAMILY OF MORPHISMS

$$t(B, A) : P(B, A) \longrightarrow Q(B, A)$$

COMPOSITION OF PROFUNCTORS REQUIRES CERTAIN WELL-BEHAVED COLIMITS IN $\underline{V}$.

WHEN $\underline{V} = \underline{Ab}$, A $\underline{V}$-CATEGORY IS AN ADDITIVE CATEGORY, AND AN ADDITIVE CATEGORY WITH ONE OBJECT "IS" A RING.

GIVEN RINGS R, S, A PROFUNCTOR

$P : \underline{R} \nrightarrow \underline{S}$ IS AN ADDITIVE FUNCTOR

$P : \underline{S}^{op} \times \underline{R} \to \underline{Ab}$ WHICH IS AN S-R BIMODULE

PROFUNCTOR COMPOSITION

$$\underline{R} \overset{P}{\nrightarrow} S \overset{Q}{\nrightarrow} T \;=\; {}_TQ_S \otimes_S {}_SP_R$$

THE IDENTITY IS ${}_RR_R$.

AN ADDITIVE CATEGORY IS CAUCHY COMPLETE IF IT HAS FINITE SUMS AND SPLIT IDEMPOTENTS.

LET $(F,\varphi) : \underline{A} \longrightarrow \underline{B}$ BE A MEALY MORPHISM

DEFINE THE ASSOCIATED PROFUNCTOR $(F,\varphi)_*$ BY

$$(F,\varphi)_*(B,A) = \varphi A \otimes \underline{B}(B,FA)$$

$$\lambda : \underline{A}(A,A') \otimes \varphi A \otimes \underline{B}(B,FA) \xrightarrow{\varphi \otimes 1} \varphi A' \otimes \underline{B}(FA,FA') \otimes \underline{B}(B,FA) \xrightarrow{1 \otimes \circ} \varphi A' \otimes \underline{B}(B,FA)$$

$$\rho : \varphi A \otimes \underline{B}(B,FA) \otimes \underline{B}(B',B) \xrightarrow{1 \otimes \circ} \varphi A \otimes \underline{B}(B',FA)$$

IF $t : (F,\varphi) \longrightarrow (G,\gamma)$ IS A MEALY CELL, DEFINE

$$t_*(B,A) : \varphi A \otimes \underline{B}(B,FA) \xrightarrow{tA \otimes 1} \gamma A \otimes \underline{B}(FA,GA) \otimes \underline{B}(B,FA) \xrightarrow{1 \otimes \circ} \gamma A \otimes \underline{B}(B,GA).$$

THEOREM (1) $(F,\varphi)_*$ IS A PROFUNCTOR $\underline{A} \looparrowright \underline{B}$

(2) t_* IS A MORPHISM OF PROFUNCTORS

(3) $(\ )_* : \underline{V}\text{-}\underline{\underline{\text{MEALY}}}(\underline{A},\underline{B}) \longrightarrow \underline{V}\text{-}\underline{\underline{\text{PROF}}}(\underline{A},\underline{B})$ IS AN EMBEDDING

(4) THE COMPOSITE $Q \otimes (F,\varphi)_*$ EXISTS FOR ALL PROFUNCTORS $Q : \underline{B} \looparrowright \underline{C}$

(5) $(G,\gamma)_* \otimes (F,\varphi)_* \cong ((G,\gamma)(F,\varphi))_*$

"PROOF" $Q \otimes (F,\varphi)_*(C,A) = \varphi A \otimes Q(C,FA)$

SUPPOSE $\underline{V}$ IS RIGHT CLOSED, I.E. $(\;)\otimes V$ HAS A RIGHT ADJOINT $[V,-]$ FOR EVERY $\underline{V}$.

WE WILL CONSTRUCT THE <u>MEALY MORPHISM CLASSIFIER</u>:

$$\frac{\text{MEALY MORPHISMS } \underline{A} \longrightarrow \underline{B}}{\underline{V}\text{-FUNCTORS } \underline{A} \longrightarrow \tilde{\underline{B}}}.$$

OBJECTS OF $\tilde{\underline{B}}$ ARE PAIRS (V, B), $V \in \underline{V}$, $B \in \underline{B}$

HOMS $\quad \tilde{\underline{B}}((V,B),(V',B')) = [V, V' \otimes \underline{B}(B,B')]$

PROPOSITION $\quad \tilde{\underline{B}}$ IS A $\underline{V}$-CATEGORY

THERE IS A MEALY MORPHISM $(K,\kappa) : \tilde{\underline{B}} \longrightarrow \underline{B}$

$$K(V,B) = B \quad , \quad \kappa(V,B) = V$$

$$\kappa((V,B),(V',B')) = ev : [V, V' \otimes \underline{B}(B,B')] \otimes V \longrightarrow V' \otimes \underline{B}(B,B')$$

THEOREM (K,κ) IS THE UNIVERSAL MEALY MORPHISM, I.E. WE GET AN EQUIVALENCE OF CATEGORIES

$$\underline{V}\text{-}\underline{\underline{CAT}}(\underline{A},\tilde{\underline{B}}) \xrightarrow{\sim} \underline{V}\text{-}\underline{\underline{MEALY}}(\underline{A},\underline{B})$$

BY COMPOSING WITH (K,κ).

$\tilde{(\ )}$ IS BI RIGHT ADJOINT TO THE INCLUSION

$$(\ )_{\bullet} : \underline{V}\text{-}\underline{\underline{CAT}} \longrightarrow \underline{V}\text{-}\underline{\underline{MEALY}}$$

$\tilde{\underline{B}}$ HAS ANOTHER UNIVERSAL PROPERTY: IT IS THE FREE LEFT TENSORED $\underline{V}$-CATEGORY GENERATED BY $\underline{B}$.

THERE IS A FULLY FAITHFUL $\underline{V}$-FUNCTOR $H : \underline{B} \longrightarrow \tilde{\underline{B}}$, $\quad HB = (I, B)$.

THEOREM $\tilde{\underline{B}}$ IS LEFT TENSORED AND COMPOSING WITH H GIVES AN EQUIVALENCE OF CATEGORIES

$$\underline{V}\text{-}\underline{\underline{TEN}}(\tilde{\underline{B}},\underline{C}) \xrightarrow{\sim} \underline{V}\text{-}\underline{\underline{CAT}}(\underline{B},\underline{C}).$$