

MEALY MORPHISMS
OF
ENRICHED CATEGORIES

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FREDERICTON
JUNE 2010

BACKGROUND

A MONOIDAL CATEGORY IS A CATEGORY V EQUIPPED WITH A "TENSOR PRODUCT"

$$\otimes : \underline{V} \times \underline{V} \rightarrow \underline{V} \quad \text{AND A UNIT OBJECT } I$$

WHICH IS ASSOCIATIVE

$$\alpha_{ABC} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$$

AND UNITARY

$$\rho_A : A \otimes I \xrightarrow{\cong} A \quad \& \quad \lambda_A : I \otimes A \xrightarrow{\cong} A$$

THE α, ρ, λ SATISFYING SOME "COHERENCE CONDITIONS".

Ex: V = Ab, \otimes THE USUAL, $I = \mathbb{Z}$.

Ex: V = SET, \otimes = CARTESIAN PRODUCT, $I = 1$.

Ex: V = CAT, " " " " $I = 1$.

A V-CATEGORY A CONSISTS OF A CLASS OF OBJECTS Ob_A , FOR EACH PAIR OF OBJECTS A, B AN OBJECT $\underline{A}(A, B) \in \underline{V}$, FOR EACH OBJECT A AN "IDENTITY MORPHISM" $ID_A : I \rightarrow \underline{A}(A, A)$ AND A "COMPOSITION MORPHISM" FOR EACH A, B, C

$$\circ_{A,B,C} : \underline{A}(B, C) \otimes \underline{A}(A, B) \rightarrow \underline{A}(A, C)$$

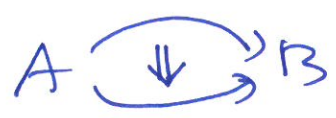
SATISFYING ASSOCIATIVITY AND UNIT LAWS.

EXAMPLE : V = Ab A V-CATEGORY IS AN ADDITIVE CATEGORY. E.G. R-MOD.

A RING R IS AN Ab-CATEGORY WITH ONE OBJECT.

EXAMPLE : A SET-CATEGORY IS AN ORDINARY CATEGORY.

EXAMPLE: A CAT-CATEGORY IS CALLED A 2-CATEGORY. IT HAS OBJECTS, ARROWS AND 2-CELLS



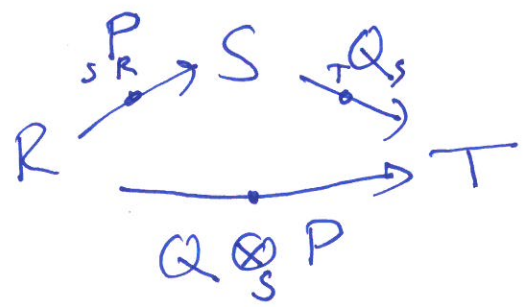
Ex: CAT ITSELF IS A 2-CATEGORY - THE 2-CELLS ARE NATURAL TRANSFORMATIONS.

A BICATEGORY IS A WEAK 2-CATEGORY:

COMPOSITION OF ARROWS IS ONLY ASSOCIATIVE AND UNITARY UP TO COHERENT ISOMORPHISMS.

EXAMPLE: OBJ - RINGS, ARROWS $R \rightarrow S$ S - R BIMODULES, 2-CELLS LINEAR FUNCTIONS

COMPOSITION



EXAMPLE: A MONOIDAL CATEGORY CAN BE CONSIDERED AS A ONE OBJECT BICATEGORY.

MORPHISMS OF \underline{V} -CATEGORIES

A \underline{V} -FUNCTOR $F: \underline{A} \rightarrow \underline{B}$ IS GIVEN BY

A FUNCTION $F: \text{Ob } \underline{A} \rightarrow \text{Ob } \underline{B}$, AND FOR

EACH PAIR OF OBJECTS $A, A' \in \underline{A}$, A

\underline{V} -MORPHISM $F: \underline{A}(A, A') \rightarrow \underline{B}(FA, FA')$

SATISFYING THE COMMUTATIVITY OF 2 DIAGRAMS

EXPRESSING THAT F PRESERVES COMPOSITION

AND UNITS, IN A WAY INTERNAL TO \underline{V} .

EXAMPLE: $\underline{V} = \underline{\text{SET}}$ WE GET ORDINARY FUNCTORS

EXAMPLE: $\underline{V} = \underline{\text{AB}}$ " " ADDITIVE "

EXAMPLE: $\underline{V} = \underline{\text{CAT}}$ " " 2-FUNCTORS.

LAX MORPHISMS

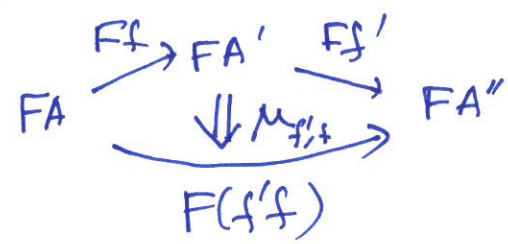
FOR 2-CATEGORIES AND BICATEGORIES WE MIGHT WANT TO RELAX THE PRESERVATION OF COMPOSITION AND IDENTITIES

$$F: \underline{A} \longrightarrow \underline{B}$$

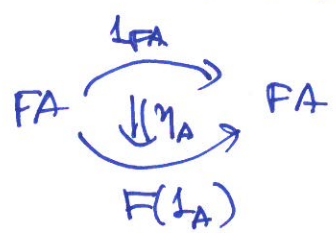
OBJ \mapsto OBJ , ARR \mapsto ARR , 2-CELLS \mapsto 2-CELLS

PRESERVING ALL DOMAINS & CODOMAINS BUT WITH

ADDED STRUCTURE



AND



SATISFYING ASSOCIATIVITY AND UNIT EQUATIONS.

IF μ, η ARE IDENTITIES WE GET STRICT MORPHISMS

IF μ, η ARE ISOS WE GET PSEUDO MORPHISMS

IF μ, η ARE NOT ISOS WE GET LAX MORPHISMS

V A ⊗ CATEGORY

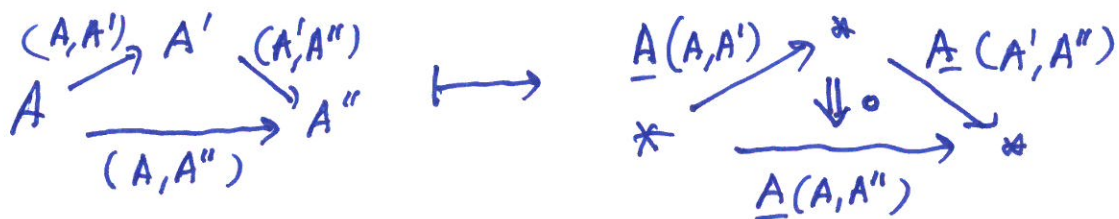
IN "BICATEGORIES I", BÉNABOU REMARKED THAT A V-CATEGORY IS THE SAME AS A CLASS ObA AND A LAX MORPHISM OF BICATEGORIES $\underline{A}: \underline{\underline{ObA}} \longrightarrow \underline{V}$.

ObA : OBJECTS ARE $A, A', A'' \in \underline{\underline{ObA}}$
 UNIQUE 1-CELL $A \rightarrow A' \quad \forall A, A'$
 ONLY IDENTITY 2-CELLS

V : ONE OBJECT $*$
 1-CELLS OBJECTS OF V
 COMPOSITION $* \xrightarrow{V} * \xrightarrow{V'} * = * \xrightarrow{V' \otimes V} *$

A : $A \longmapsto *$

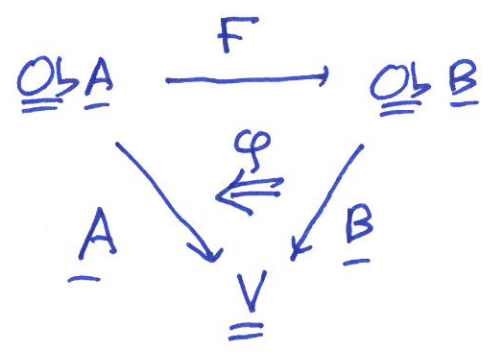
$(A, A') : A \rightarrow A' \longmapsto \underline{A}(A, A') \in \underline{V}$



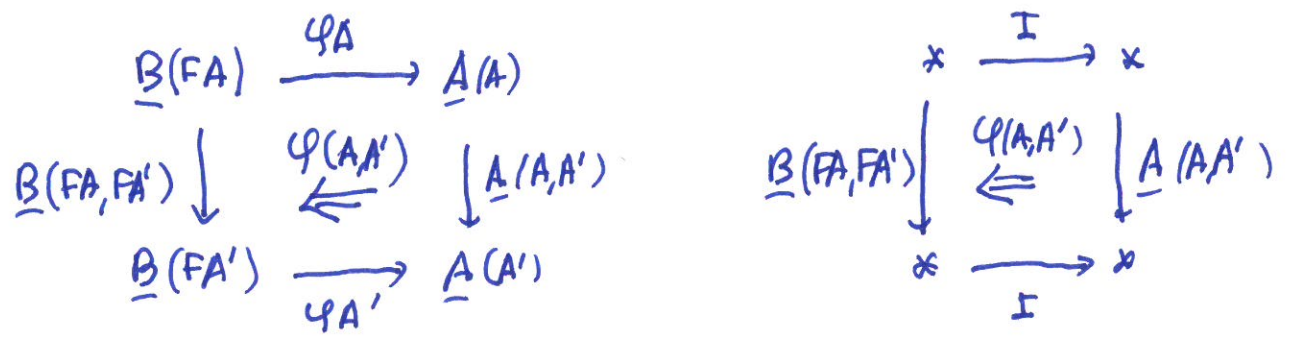
$$\underline{A}(A', A'') \otimes \underline{A}(A, A') \longrightarrow \underline{A}(A, A'')$$

LAX AND OPLAX TRANSFORMATIONS OF LAX FUNCTORS WERE DEFINED IN "BICATEGORIES I" BUT V-FUNCTORS WERE NOT CONSIDERED IN THIS CONTEXT.

A V-FUNCTOR A → B IS A PAIR (F, φ)



WHERE F IS A FUNCTION ON OBJECTS AND φ IS A LAX TRANSFORMATION B ∘ F → A WHICH IS THE IDENTITY ON OBJECTS (C.F. ICON).



I.E. φ(A, A') : A(A, A') → B(FA, FA') + CONDITIONS.

DEFINITION LET \underline{A} AND \underline{B} BE \underline{V} -CATEGORIES.

A MEALY MORPHISM $(F, \varphi): \underline{A} \rightarrow \underline{B}$ CONSISTS OF:

OBJECT FUNCTIONS $F: \text{Ob } \underline{A} \rightarrow \text{Ob } \underline{B}$, $\varphi: \text{Ob } \underline{A} \rightarrow \text{Ob } \underline{V}$;

FOR EACH PAIR $A, A' \in \underline{A}$ A MORPHISM

$$\varphi(A, A'): \underline{A}(A, A') \otimes \varphi_A \longrightarrow \varphi_{A'} \otimes \underline{B}(FA, FA');$$

SATISFYING

(Mm1)

$$\begin{array}{ccc} \Gamma \otimes \varphi_A & \xrightarrow{\text{id} \otimes \varphi_A} & \underline{A}(A, A) \otimes \varphi_A \\ \cong \downarrow \text{CAN} & & \downarrow \varphi(A, A) \\ \varphi_A \otimes \Gamma & \xrightarrow{\varphi_A \otimes \text{id}} & \varphi_A \otimes \underline{B}(FA, FA) \end{array}$$

(Mm2)

$$\begin{array}{ccc} \underline{A}(A', A'') \otimes \underline{A}(A, A') \otimes \varphi_A & \xrightarrow{\circ \otimes \varphi_A} & \underline{A}(A, A'') \otimes \varphi_A \\ \downarrow \underline{A}(A', A'') \otimes \varphi(A, A') & & \downarrow \varphi(A, A'') \\ \underline{A}(A', A'') \otimes \varphi_{A'} \otimes \underline{B}(FA, FA') & & \\ \downarrow \varphi(A', A'') \otimes \underline{B}(FA, FA') & & \\ \varphi_{A''} \otimes \underline{B}(FA', FA'') \otimes \underline{B}(FA, FA') & \xrightarrow{\varphi_{A''} \otimes \circ} & \varphi_{A''} \otimes \underline{B}(FA, FA'') \end{array}$$

MEALY MORPHISMS ARE EASILY COMPOSED

COMPOSITION IS NOT STRICTLY ASSOCIATIVE

AS IT USES \otimes . NEED 2-CELLS.

DEFINITION LET $(F, \varphi), (G, \gamma) : \underline{A} \rightarrow \underline{B}$ BE

MEALY MORPHISMS. A MEALY CELL $t : (F, \varphi) \rightarrow (G, \gamma)$

IS GIVEN BY MORPHISMS

$$t_A : \varphi_A \longrightarrow \gamma_A \otimes \underline{B}(F_A, G_A)$$

SATISFYING

(Mc)

$$\begin{array}{ccc}
 \underline{A}(A, A') \otimes \varphi_A & \xrightarrow{\varphi(A, A')} & \varphi_{A'} \otimes \underline{B}(F_A, F_{A'}) \\
 \downarrow \underline{A}(A, A') \otimes t_A & & \downarrow t_{A'} \otimes \underline{B}(F_A, F_{A'}) \\
 \underline{A}(A, A') \otimes \gamma_A \otimes \underline{B}(F_A, G_A) & & \gamma_{A'} \otimes \underline{B}(F_{A'}, G_{A'}) \otimes \underline{B}(F_A, F_{A'}) \\
 \downarrow \gamma(A, A') \otimes \underline{B}(F_A, G_A) & & \downarrow \gamma_{A'} \otimes \circ \\
 \gamma_{A'} \otimes \underline{B}(F_A, F_{A'}) \otimes \underline{B}(F_A, G_A) & \xrightarrow{\gamma_{A'} \otimes \circ} & \gamma_{A'} \otimes \underline{B}(F_A, G_{A'})
 \end{array}$$

GIVEN MEALY MORPHISMS $(F, \varphi): \underline{A} \rightarrow \underline{B}$ AND

$(H, \gamma): \underline{B} \rightarrow \underline{C}$ THE COMPOSITE IS $(HF, \varphi \otimes \gamma F)$

WHERE $(\varphi \otimes \gamma F)A = \varphi A \otimes \gamma FA$ AND $(\varphi \otimes \gamma F)(A, A')$ IS

$$\begin{array}{ccc}
 \underline{A}(A, A') \otimes \varphi A \otimes \gamma FA & \xrightarrow{\varphi(A, A') \otimes 1} & \varphi A' \otimes \underline{B}(FA, FA') \otimes \gamma FA \\
 & & \searrow 1 \otimes \gamma(FA, FA') \\
 & & \varphi A' \otimes \gamma FA' \otimes \underline{C}(HFA, HFA') .
 \end{array}$$

THEOREM \underline{V} -CATEGORIES, MEALY MORPHISMS AND MEALY CELLS FORM A BICATEGORY \underline{V} -MEALY.

EXAMPLE LET \underline{I} BE THE \underline{V} -CATEGORY WITH ONE OBJECT $*$ AND $\underline{I}(*, *) = \underline{I}$. FOR ANY \underline{V} -CATEGORY \underline{B} , A MEALY MORPHISM $\underline{I} \rightarrow \underline{B}$ IS DETERMINED BY A PAIR (V, B) V IN \underline{V} , B IN \underline{B} .

A MEALY CELL $b: (V, B) \rightarrow (V', B')$ IS A MORPHISM $b: V \rightarrow V' \otimes \underline{B}(B, B')$.

THIS DESCRIBES THE CATEGORY $\tilde{\underline{B}}_0 := \underline{V}$ -MEALY($\underline{I}, \underline{B}$)

IT IS THE UNDERLYING CATEGORY OF A \underline{V} -CATEGORY (ASSUMING SOME CLOSEDNESS CONDITIONS ON \underline{V}) WHICH WILL PLAY AN IMPORTANT ROLE LATER.

EXAMPLE \underline{V} -MEALY($\underline{B}, \underline{I}$) \cong \underline{V} -CAT($\underline{B}, \underline{V}$)

EXAMPLE FOR $\underline{V} = \underline{SET}$, THE MORPHISMS

$$\varphi(A, A'): \underline{A}(A, A') \times \varphi_A \longrightarrow \varphi_{A'} \times \underline{B}(FA, FA')$$

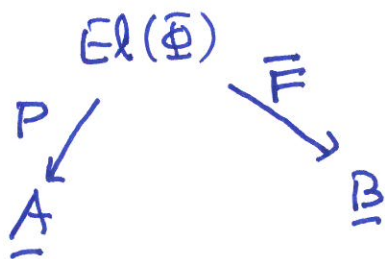
HAVE TWO COMPONENTS:

$$\varphi_1(A, A'): \underline{A}(A, A') \times \varphi_A \longrightarrow \varphi_{A'}$$

WHICH MAKE φ INTO A FUNCTOR $\underline{A} \longrightarrow \underline{SET}$, AND

$$\varphi_2(A, A'): \underline{A}(A, A') \times \varphi_A \longrightarrow \underline{B}(FA, FA')$$

WHICH GIVE A LIFTING OF F TO A FUNCTOR



IN THIS WAY A MEALY MORPHISM $(F, \varphi): \underline{A} \longrightarrow \underline{B}$

CAN BE VIEWED AS A SPAN IN \underline{CAT} WHERE

THE LEFT LEG IS A DISCRETE OPFIBRATION

AND THE RIGHT LEG IS OBJECT-WISE CONSTANT

ON THE FIBRES.

"PARTIAL FUNCTORS"

EXAMPLE/DEFINITION A MEALY MACHINE

[GEORGE MEALY 1955] HAS

- A (FINITE) SET S OF STATES
- AN INPUT ALPHABET Σ
- AN OUTPUT ALPHABET Λ
- A TRANSITION FUNCTION $t: \Sigma \times S \rightarrow S$
- AN OUTPUT FUNCTION $g: \Sigma \times S \rightarrow \Lambda$
- A START STATE $s_0 \in S$

RUN AS FOLLOWS :

TAKE A WORD IN INPUT ALPHABET $a_m a_{m-1} \dots a_1$

START IN STATE s_0

NEW STATE	$s_1 = t(a_1, s_0)$	OUTPUT	$b_1 = g(a_1, s_0)$
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THEN	$s_2 = t(a_2, s_1)$		$b_2 = g(a_2, s_1)$
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ETC.	\vdots		\vdots
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So $a_m \dots a_2 a_1 \rightsquigarrow b_m \dots b_2 b_1$

EXTEND t AND g TO FREE MONOIDS

$$\bar{t}: \Sigma^* \times S \rightarrow S$$

$$\text{OR } \Sigma^* \times S \rightarrow S \times \Lambda^*$$

$$\bar{g}: \Sigma^* \times S \rightarrow \Lambda^*$$

EXAMPLE A \underline{V} -FUNCTOR $F: \underline{A} \rightarrow \underline{B}$

CORRESPONDS EXACTLY TO A MEALY MORPHISM

$F_\bullet = (F, I): \underline{A} \rightarrow \underline{B}$, WHERE I REPRESENTS

THE FUNCTION $OB \underline{A} \rightarrow OB \underline{V}$ WITH CONSTANT

VALUE I . COMPOSITION IS PRESERVED

$G \circ F \cong (GF)$.

\underline{V} -NATURAL TRANSFORMATIONS $F \rightarrow K$ ARE

IN BIJECTION WITH MEALY CELLS $F_\bullet \rightarrow K_\bullet$.

THUS WE GET A LOCALLY FULLY FAITHFUL

EMBEDDING

$$(\)_\bullet : \underline{V}\text{-}\underline{\underline{\text{CAT}}} \longrightarrow \underline{V}\text{-}\underline{\underline{\text{MEALY}}}$$

DEFINITION \underline{B} IS LEFT TENSORED IF FOR EVERY V IN \underline{V} AND B IN \underline{B} THERE IS AN OBJECT $V \otimes B$ IN \underline{B} AND A MORPHISM $k: V \rightarrow \underline{B}(B, V \otimes B)$

WHICH MEDIATES A BIJECTION

$$\frac{X \rightarrow \underline{B}(V \otimes B, B')}{X \otimes V \rightarrow \underline{B}(B, B')}$$

FOR EACH X IN \underline{V} AND B' IN \underline{B} .

THEOREM LET \underline{A} AND \underline{B} BE \underline{V} -CATEGORIES WITH \underline{B} TENSORED. THEN THE EMBEDDING

$$(\cdot)_\bullet : \underline{V}\text{-CAT}(\underline{A}, \underline{B}) \longrightarrow \underline{V}\text{-MEALY}(\underline{A}, \underline{B})$$

HAS A LEFT ADJOINT.

"PROOF" IF $(F, \varphi): \underline{A} \rightarrow \underline{B}$ IS A MEALY MORPHISM, THEN $\hat{F}A = \varphi_A \otimes FA$ IS THE OBJECT PART OF A \underline{V} -FUNCTOR WHICH WILL BE THE REFLECTION OF (F, φ) . □

PROFUNCTORS = THE CATEGORICIAN'S RELATIONS

A RELATION $R: A \rightarrow B$ IS A FUNCTION

$$A \times B \rightarrow \{T, F\}.$$

THE COMPOSITE OF $A \xrightarrow{R} B \xrightarrow{S} C$ IS

$$a (S \circ R) c \iff \exists b (a R b \wedge b S c).$$

THE IDENTITY $I_A: A \rightarrow A$ IS EQUALITY

$$a I_A a' \iff a = a'$$

THIS GIVES THE CATEGORY REL

REL IS A 2-CATEGORY: THERE IS A 2-CELL

$$(UNIQUE) \quad R \Rightarrow R' \quad \text{IFF} \forall_{a,b} (a R b \iff a R' b).$$

A FUNCTION $f: A \rightarrow B$ GIVES TWO RELATIONS

$$f_{\rightarrow}: A \rightarrow B, \quad a f_{\rightarrow} b \iff (f a = b)$$

$$f^{\times}: B \rightarrow A, \quad b f^{\times} a \iff (b = f a)$$

f_* IS LEFT ADJOINT TO f^*

FOR $R: A \rightarrow B$ AND $S: B \rightarrow A$, R LEFT ADJOINT TO S

$R \dashv S$ IF

$$R \circ S \Rightarrow I_B \quad \text{AND} \quad I_A \Rightarrow S \circ R$$

R IS (COMES FROM) A FUNCTION $A \rightarrow B$ IFF IT HAS A RIGHT ADJOINT.

A PROFUNCTOR $P: \underline{A} \rightarrow \underline{B}$ IS A FUNCTOR

$$P: \underline{B}^{op} \times \underline{A} \rightarrow \underline{SET}$$

THE COMPOSITE OF $\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{C} = Q \otimes P$

IS DEFINED BY

$$Q \otimes P(C, A) = \sum_B P(B, A) \times Q(C, B) \sim \text{(MATRIX MULTI)}$$

$$= \int^B P(B, A) \times Q(C, B)$$

$$\sum_{B, B'} P(B, A) \times B(B', B) \times Q(C, B') \Rightarrow \sum_B P(B, A) \times Q(C, B) \Rightarrow Q \otimes P(C, A) \quad (M \otimes_R N)$$

THE IDENTITY $\text{Id}_{\underline{A}} : \underline{A} \rightarrow \underline{A}$ IS THE HOM FUNCTOR

$$\underline{A}(-, -) : \underline{A}^{\text{op}} \times \underline{A} \rightarrow \underline{\text{SET}}.$$

A 2-CELL $P \Rightarrow P' : \underline{A} \rightarrow \underline{B}$ IS A NATURAL

TRANSFORMATION $\zeta : P \rightarrow P' : \underline{B}^{\text{op}} \times \underline{A} \rightarrow \underline{\text{SET}}$

WE GET A BICATEGORY PROF.

NOTATION WRITE $x \in P(B, A)$ AS $B \xrightarrow[x]{x} A$

FUNCTORIALITY SAYS WE CAN "COMPOSE" x WITH

MORPHISMS $A \xrightarrow{a} A'$, $ax : B \rightarrow A'$ AND

MORPHISMS $B' \xrightarrow{b} B$, $x'b : B' \rightarrow A$. THESE COMPOSITS

ARE ASSOCIATIVE AND UNITARY.

AN ELEMENT OF $\text{QOP}(C, A)$ IS AN EQUIVALENCE

CLASS OF PAIRS $[C \xrightarrow[a]{y} B \xrightarrow[p]{x} A]$. THE EQUIVALENCE

IS GENERATED BY

$$\begin{array}{ccccc} C & \xrightarrow{y} & B & \xrightarrow{x} & A \\ \parallel & & \downarrow b & & \parallel \\ C & \xrightarrow{y'} & B' & \xrightarrow{x'} & A \end{array}$$

IF WE WRITE THE EQUIVALENCE CLASS AS $x \otimes y$

THEN $x' \otimes y = x' \otimes y$.

EVERY FUNCTOR $F: \underline{A} \rightarrow \underline{B}$ DETERMINES TWO PROFUNCTORS

$F_*: \underline{A} \rightarrow \underline{B}$, $F_*(B, A) = \underline{B}(B, FA)$ AND

$F^*: \underline{B} \rightarrow \underline{A}$, $F^*(A, B) = \underline{B}(FA, B)$.

WE HAVE $F_* \dashv F^*$

NOT QUITE TRUE THAT AN ADJOINT PAIR OF PROFUNCTORS $P \dashv Q$ COME FROM A FUNCTOR.

B IS CAUCHY COMPLETE IF THIS IS TRUE FOR EVERY $P: \underline{A} \rightarrow \underline{B}$ WITH A RIGHT ADJOINT.

FOR SET BASED PROFUNCTORS, B IS CAUCHY COMPLETE IFF B HAS SPLIT IDEMPOTENTS.

THE CAUCHY COMPLETION OF B IS THE CATEGORY OF PROFUNCTORS $\mathbb{1} \rightarrow \underline{B}$ WITH A RIGHT ADJOINT.

A RELATION $R: A \leftrightarrow B$ CAN EQUALLY WELL BE GIVEN BY ITS GRAPH $\Gamma R \subseteq A \times B$.

ANALAGOUSLY, GIVEN A PROFUNCTOR $P: \underline{A} \leftrightarrow \underline{B}$ WE CONSTRUCT ITS CATEGORY OF ELEMENTS $\underline{El}(P)$

AN OBJECT OF $\underline{El}(P)$ IS $(B, A, x \in P(B, A))$

A MORPHISM $(B, A, x) \rightarrow (B', A', x')$ IS A PAIR

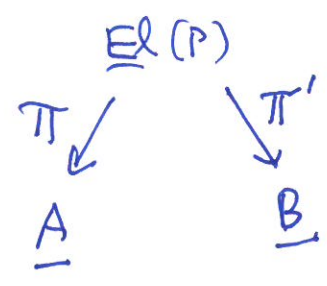
$b: B \rightarrow B', a: A \rightarrow A'$ SUCH THAT

$$P_{(B,a)}(x) = P_{(b,A')}(x')$$

IE

$$\begin{array}{ccc}
 B & \xrightarrow{x} & A \\
 b \downarrow & \cong & \downarrow a \\
 B' & \xrightarrow{x'} & A'
 \end{array}$$

WE GET



A DISCRETE BIFIBRATION

$$P \cong \pi'_* \otimes \pi^*$$

MEALY MORPHISMS CAN BE VIEWED AS PROFUNCTORS

RECALL THAT A PROFUNCTOR $P: \underline{A} \rightarrow \underline{B}$ IS GIVEN

BY AN OBJECT FUNCTION $P: \text{Ob } \underline{B} \times \text{Ob } \underline{A} \rightarrow \text{Ob } \underline{V}$

AND LEFT AND RIGHT ACTIONS

$$\lambda: \underline{A}(A, A') \otimes P(B, A) \rightarrow P(B, A')$$

$$\rho: P(B, A) \otimes \underline{B}(B', B) \rightarrow P(B', A)$$

SATISFYING UNIT LAWS (2) AND ASSOCIATIVITY (3).

(THINK \underline{V} -FUNCTOR $\underline{B}^{\text{op}} \otimes \underline{A} \rightarrow \underline{V}$.)

A MORPHISM OF PROFUNCTORS $t: P \rightarrow Q$ IS

AN EQUIVARIANT FAMILY OF MORPHISMS

$$t(B, A): P(B, A) \rightarrow Q(B, A)$$

COMPOSITION OF PROFUNCTORS REQUIRES

CERTAIN WELL-BEHAVED COLIMITS IN \underline{V} .

WHEN $\underline{V} = \underline{Ab}$, A \underline{V} -CATEGORY IS AN ADDITIVE CATEGORY, AND AN ADDITIVE CATEGORY WITH ONE OBJECT "IS" A RING.

GIVEN RINGS R, S , A PROFUNCTOR

$P: \underline{R} \rightarrow \underline{S}$ IS AN ADDITIVE FUNCTOR

$P: \underline{S}^{\text{op}} \times \underline{R} \rightarrow \underline{Ab}$ WHICH IS AN S - R BIMODULE

PROFUNCTOR COMPOSITION

$$\underline{R} \xrightarrow{P} \underline{S} \xrightarrow{Q} \underline{T} = \underline{T}^{\text{op}} \underset{S}{\otimes} \underset{R}{P}$$

THE IDENTITY IS $\underset{R}{R}$.

AN ADDITIVE CATEGORY IS CAUCHY COMPLETE IF

IT HAS FINITE SUMS AND SPLIT IDEMPOTENTS.

LET $(F, \varphi): \underline{A} \rightarrow \underline{B}$ BE A MEALY MORPHISM

DEFINE THE ASSOCIATED PROFUNCTOR $(F, \varphi)_*$ BY

$$(F, \varphi)_*(B, A) = \varphi_A \otimes \underline{B}(B, FA)$$

$$\lambda: \underline{A}(\underline{A}, \underline{A}') \otimes \varphi_A \otimes \underline{B}(B, FA) \xrightarrow{\varphi \otimes 1} \varphi_{A'} \otimes \underline{B}(FA, FA') \otimes \underline{B}(B, FA) \xrightarrow{1 \otimes \varphi} \varphi_{A'} \otimes \underline{B}(B, FA)$$

$$\rho: \varphi_A \otimes \underline{B}(B, FA) \otimes \underline{B}(B', B) \xrightarrow{1 \otimes \varphi} \varphi_A \otimes \underline{B}(B', FA)$$

IF $t: (F, \varphi) \rightarrow (G, \gamma)$ IS A MEALY CELL, DEFINE

$$t_* (B, A): \varphi_A \otimes \underline{B}(B, FA) \xrightarrow{t_A \otimes 1} \gamma_A \otimes \underline{B}(FA, GA) \otimes \underline{B}(B, FA) \xrightarrow{1 \otimes \varphi} \gamma_A \otimes \underline{B}(B, GA)$$

THEOREM (1) $(F, \varphi)_*$ IS A PROFUNCTOR $\underline{A} \rightarrow \underline{B}$

(2) t_* IS A MORPHISM OF PROFUNCTORS

(3) $()_*: \underline{V}\text{-MEALY}(\underline{A}, \underline{B}) \rightarrow \underline{V}\text{-PROF}(\underline{A}, \underline{B})$ IS AN EMBEDDING

(4) THE COMPOSITE $Q \otimes (F, \varphi)_*$ EXISTS FOR ALL PROFUNCTORS $Q: \underline{B} \rightarrow \underline{C}$

$$(5) (G, \gamma)_* \otimes (F, \varphi)_* \cong ((G, \gamma)(F, \varphi))_*$$

"PROOF"

$$Q \otimes (F, \varphi)_*(C, A) = \varphi_A \otimes Q(C, FA)$$

SUPPOSE \underline{V} IS RIGHT CLOSED, I.E. $() \otimes V$ HAS A RIGHT ADJOINT $[V, -]$ FOR EVERY \underline{V} .

WE WILL CONSTRUCT THE MEALY MORPHISM CLASSIFIER:

$$\frac{\text{MEALY MORPHISMS } \underline{A} \rightarrow \underline{B}}{\underline{V}\text{-FUNCTORS } \underline{A} \rightarrow \underline{\tilde{B}}}$$

OBJECTS OF $\underline{\tilde{B}}$ ARE PAIRS (V, B) , $V \in \underline{V}$, $B \in \underline{B}$

$$\text{HOMs } \underline{\tilde{B}}((V, B), (V', B')) = [V, V' \otimes \underline{B}(B, B')]$$

PROPOSITION $\underline{\tilde{B}}$ IS A \underline{V} -CATEGORY

THERE IS A MEALY MORPHISM $(K, \kappa): \underline{\tilde{B}} \rightarrow \underline{B}$

$$K(V, B) = B, \quad \kappa(V, B) = V$$

$$\kappa((V, B), (V', B')) = \text{ev} : [V, V' \otimes \underline{B}(B, B')] \otimes V \rightarrow V' \otimes \underline{B}(B, B')$$

THEOREM (K, k) IS THE UNIVERSAL MEALY MORPHISM, I.E. WE GET AN EQUIVALENCE OF CATEGORIES

$$\underline{V}\text{-}\underline{\underline{CAT}}(\underline{A}, \underline{\tilde{B}}) \xrightarrow{\sim} \underline{V}\text{-}\underline{\underline{MEALY}}(\underline{A}, \underline{B})$$

BY COMPOSING WITH (K, k) .

$(\tilde{\quad})$ IS BI RIGHT ADJOINT TO THE INCLUSION

$$(\cdot) : \underline{V}\text{-}\underline{\underline{CAT}} \longrightarrow \underline{V}\text{-}\underline{\underline{MEALY}}$$

$\underline{\tilde{B}}$ HAS ANOTHER UNIVERSAL PROPERTY:

IT IS THE FREE LEFT TENSORED \underline{V} -CATEGORY GENERATED BY \underline{B} .

THERE IS A FULLY FAITHFUL \underline{V} -FUNCTOR

$$H : \underline{B} \longrightarrow \underline{\tilde{B}}, \quad H B = (I, B).$$

THEOREM $\underline{\tilde{B}}$ IS LEFT TENSORED AND COMPOSING WITH H GIVES AN EQUIVALENCE OF CATEGORIES

$$\underline{V}\text{-}\underline{\underline{TEN}}(\underline{\tilde{B}}, \underline{C}) \xrightarrow{\sim} \underline{V}\text{-}\underline{\underline{CAT}}(\underline{B}, \underline{C}).$$