DOUBLE CATEGORIES OF MODELS

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DOUBLE CATEGORIES

$\mathbb{A}$ HAS OBJECTS $A, B, C, \cdots$

HORIZONTAL ARROWS $f : A \to B$

VERTICAL ARROWS $\begin{array}{c} A \\ \downarrow v \\ C \end{array}$

CELLS
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

HORIZONTAL COMPOSITION OF ARROWS AND CELLS

GIVE CATEGORIES.

VERTICAL COMPOSITION TOO.

INTERCHANGE
$$\begin{array}{|c|c|} \hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array} \qquad (\delta\gamma)\cdot(\beta\alpha) = (\delta\cdot\beta)(\gamma\cdot\alpha)$$

TODAY ONLY CONSIDER FLAT DOUBLE CATEGORIES

I.E. FOR ANY BOUNDARY $f, g ; v, w$ THERE IS

AT MOST ONE α.

BASIC EXAMPLES

1. $\mathbb{S}$ET   OBJECTS ARE SETS, HORIZ ARR FNS

VERTICAL ARROWS ARE RELATIONS

CELLS

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "S"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \Leftrightarrow \left(a \underset{R}{\sim} c \Rightarrow fa \underset{S}{\sim} gc \right)$$

2. REGULAR THEORY (ONE SORTED, FINITARY)

GIVEN SET OF <u>OPERATION</u> <u>SYMBOLS</u> WITH ARITIES

SET OF <u>PREDICATE</u> <u>SYMBOLS</u> " "

CAN CONSTRUCT <u>TERMS</u>, REGULAR <u>FORMULAS</u>

$(=, \top, \wedge, \exists)$

GIVEN SET OF <u>AXIOMS</u>, SEQUENTS $\Phi \vdash \Psi$

WHERE Φ, Ψ ARE REGULAR FORMULAS

CONSTRUCT A <u>DOUBLE</u> <u>LAWVERE</u> <u>THEORY</u> $\mathbb{T}$

<u>OBJECTS</u> $[0], [1], [2], \ldots$

<u>HORIZONTAL</u> <u>ARROWS</u> $[m] \to [n]$ ARE n-TUPLES OF (EQUIVALENCE CLASSES OF) TERMS IN VARIABLES $x_1, x_2, \ldots x_m$.

<u>VERTICAL</u> <u>ARROWS</u> $[m] \xrightarrow{\bullet} [p]$ ARE FORMULAS $\Phi(x_1, \ldots x_m; y_1, \ldots y_p)$.

<u>CELLS</u>

$$\begin{tikzcd}
{[m]} \arrow[r, "t"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\vdash"] & {[n]} \arrow[d, "\Psi"] \\
{[p]} \arrow[r, "u"'] & {[q]}
\end{tikzcd}$$

IFF $\Phi(\bar{x};\bar{y}) \Rightarrow \Psi(t(\bar{x}); u(\bar{y}))$ FOLLOWS FROM THE AXIOMS

<u>VERTICAL</u> <u>COMPOSITION</u> $[m] \xrightarrow{\Phi} [p] \xrightarrow{\Theta} [r]$

$$\Theta \cdot \Phi(\bar{x};\bar{z}) \equiv \exists \bar{y}\,(\Phi(\bar{x};\bar{y}) \wedge \Theta(\bar{y};\bar{z}))$$

GET A <u>WEAK</u> DOUBLE CATEGORY.

DOUBLE FUNCTORS

$F : \mathbb{A} \to \mathbb{B}$

OBJ $\mapsto$ OBJ ; $A \mapsto FA$

HORIZ $\mapsto$ HORIZ ; $(A \xrightarrow{f} A') \mapsto (FA \xrightarrow{Ff} FA')$

VERT $\mapsto$ VERT ;

$$
\begin{tikzcd}
A \arrow[d, "v"'] \\ \bar{A}
\end{tikzcd}
\mapsto
\begin{tikzcd}
FA \arrow[d, "Fv"] \\ F\bar{A}
\end{tikzcd}
$$

CELLS $\mapsto$ CELLS

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{A}'
\end{tikzcd}
\mapsto
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "Fv"'] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fv'"] \\
F\bar{A} \arrow[r, "F\bar{f}"'] & F\bar{A}'
\end{tikzcd}
$$

HORIZONTAL COMPOSITION PRESERVED.

VERTICAL COMPOSITION

(I) LAX FUNCTORS

$$
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "id_{FA}"'] \arrow[dr, phantom, "\leq"] & FA \arrow[d, "F(id_A)"] \\
FA \arrow[r, equal] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[dd, "F(\bar{v} \cdot v)"] \\
F\bar{A} \arrow[d, "F\bar{v}"'] \arrow[r, phantom, "\leq"] & \\
F\bar{\bar{A}} \arrow[r, equal] & F\hat{A}
\end{tikzcd}
$$

(II) OPLAX FUNCTORS DUAL

(III) STRONG FUNCTORS - ISOS

(IV) STRICT FUNCTORS - EQUALITIES

EXAMPLES

$\mathbb{G}r$ THE DOUBLE CATEGORY OF GROUPS, HOMOMORPHISMS AND HOMOMORPHIC RELATIONS

$U : \mathbb{G}r \to \mathbb{S}et$ IS A STRICT DOUBLE FUNCTOR

DEFINE $F : \mathbb{S}et \to \mathbb{G}r$ TO BE THE FREE GROUP FUNCTOR ON SETS AND FUNCTIONS.

FOR A RELATION $R : A \to B$, DEFINE $F(R)$ TO BE THE LEAST HOMOMORPHIC RELATION S.T

$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[d, "R"'] \arrow[dr, phantom, "\leq"] & FA \arrow[d, "F(R)"] \\ B \arrow[r, "\eta B"'] & FB \end{tikzcd}$$

MAKES F INTO OPLAX (NORMAL) DOUBLE FUNCTOR $\mathbb{S}et \to \mathbb{G}r$.

F IS <u>LEFT</u> <u>ADJOINT</u> TO U IN APPROPRIATE DOUBLE CATEGORY SENSE (OPLAX $\dashv$ LAX).

LET $\mathcal{T}$ BE A REGULAR THEORY

$\mathbb{T}$ THE CORRESPONDING DOUBLE THEORY

A MODEL A OF $\mathcal{T}$ GIVES A STRONG

DOUBLE FUNCTOR $M : \mathbb{T} \to \mathbb{S}\mathrm{et}$

$M[n] = A^n \quad , \quad M(t) = [\![t]\!]_A : A^m \to A^n$

$M(\Phi) = [\![\Phi]\!]_A \subseteq A^m \times A^p .$

<u>HORIZONTAL</u> <u>TRANSFORMATIONS</u>

$F, G : \mathbb{A} \to \mathbb{B} \quad , \quad h : F \to G$

$\forall A$ GIVE $hA : FA \to GA$

$\forall v \; A \xrightarrow{} \bar{A}$ GIVE

$$\begin{array}{ccc} FA & \xrightarrow{hA} & GA \\ {\scriptstyle Fv}\downarrow & hv & \downarrow{\scriptstyle Gv} \\ F\bar{A} & \xrightarrow[h\bar{A}]{} & G\bar{A} \end{array}$$

HORIZONTALLY NATURAL

VERTICALLY FUNCTORIAL

(IN THE FLAT CASE - MUCH EASIER)

EXAMPLES

FOR $U : \mathbb{G}r \to \mathbb{S}et$ AND $F : \mathbb{S}et \to \mathbb{G}r$

$\eta : 1_{\mathbb{S}et} \to UF$ IS A HORIZONTAL TRANSFORMATION

GIVEN MODELS A, B OF A REGULAR THEORY $\mathcal{T}$

$M, N : \mathbb{T} \to \mathbb{S}et$ THE CORRESP. DOUB FUNCTORS

HORIZONTAL TRANSFS $M \to N$ ARE THE SAME

AS MORPHISMS OF MODELS $A \to B$.

PRODUCTS IN DOUBLE CATEGORIES (BINARY)

(1) $\forall A, B$

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & B \end{tikzcd}$$

WITH UNIVERSAL PROPERTY WITH RESPECT TO HORIZONTAL MORPHISMS

(2) $\forall$

$$\begin{tikzcd} A \arrow[d, "v"'] & B \arrow[d, "w"] \\ \bar{A} & \bar{B} \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[r, "p_1"] \arrow[d, "v \times w"'] \arrow[dr, phantom, "\pi_1"] & A \arrow[d, "v"] \\ \bar{A} \times \bar{B} \arrow[r, "q_1"'] & \bar{A} \end{tikzcd} , \quad \begin{tikzcd} A \times B \arrow[r, "p_2"] \arrow[d, "v \times w"'] \arrow[dr, phantom, "\pi_2"] & B \arrow[d, "w"] \\ \bar{A} \times \bar{B} \arrow[r, "q_2"'] & \bar{B} \end{tikzcd}$$

WITH UNIV PROP WRT CELLS

MAY BE LAX

$$\begin{tikzcd} A \times B \arrow[r, equal] \arrow[d, "v \times w"'] & A \times B \arrow[dd, "(\bar{v} \cdot v) \times (\bar{w} \cdot w)"] \\ \bar{A} \times \bar{B} \arrow[d, "\bar{v} \times \bar{w}"'] \arrow[r, phantom, "\leq"] & \\ \bar{\bar{A}} \times \bar{\bar{B}} \arrow[r, equal] & \bar{\bar{A}} \times \bar{\bar{B}} \end{tikzcd}$$

OR STRONG

$(=)$

WE ALWAYS REQUIRE NORMAL $\quad id_A \times id_B = id_{A \times B}$

ALL OUR EXAMPLES WILL BE STRONG.

EXAMPLES

IN $\mathbb{S}$ET : $\quad R : A \longrightarrow \bar{A}$, $S : B \longrightarrow \bar{B}$

$R \times S : A \times B \longrightarrow \bar{A} \times \bar{B}$

$$(a,b) \underset{R\times S}{\sim} (\bar{a},\bar{b}) \iff \left(a \underset{R}{\sim} \bar{a} \wedge b \underset{S}{\sim} \bar{b} \right)$$

IN $\mathbb{T}$: $\quad \Phi : [m] \longrightarrow [p]$, $\Psi : [n] \longrightarrow [q]$

$$\Phi \times \Psi (\bar{x}, \bar{x}' ; \bar{y}, \bar{y}') \equiv \Phi(\bar{x}; \bar{y}) \wedge \Psi(\bar{x}'; \bar{y}')$$

A MODEL A OF $\mathcal{T}$ CORRESPONDS TO A FINITE PRODUCT PRESERVING DOUBLE FUNCTOR $M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{et}$

THEOREM: THERE IS AN EQUIVALENCE OF CATEGORIES

$$\underline{\mathrm{Mod}}(\mathcal{T}) \simeq \underline{\mathrm{Prod}}(\mathbb{T}, \mathbb{S}\mathrm{ET}) .$$

PROBLEM: RECOVER $\mathbb{T}$ FROM THE CATEGORY OF MODELS.

IN LAWVERE'S FUNCTORIAL SEMANTICS, HE RECOVERS $\mathbb{T}$ BY TAKING NATURAL TRANSFORMATIONS $U^m \to U^n$ ($U : \underline{Alg} \to \underline{Set}$) AS MORPHISMS $[m] \to [n]$ IN $\mathbb{T}$.

IN GENERAL THE CATEGORIES $\underline{Mod}(\mathbb{T})$ HAVE POOR CATEGORICAL PROPERTIES AND MORE STRUCTURE MIGHT BE WELCOME.

SINCE OUR FORMALISM IS IN TERMS OF DOUBLE CATEGORIES, WE MAY BE ABLE TO EXPLOIT THE EXTRA FREEDOM THIS AFFORDS.

PLAN IS TO MAKE A DOUBLE CATEGORY OF MODELS WITH A DOUBLE FORGETFUL FUNCTOR $U : \mathbb{M}od \to \mathbb{S}et$. THEN CONSTRUCT A DOUBLE THEORY BY TAKING ITS HORIZONTAL ARROWS $[m] \to [n]$ TO BE HORIZONTAL TRANSFORMATIONS $U^m \to U^n$ AND TAKING ITS VERTICAL ARROWS TO BE VERTICAL TRANSFORMATIONS $U^m \to\!\!\!\!\bullet\;\, U^p$.

AS MODELS CORRESPOND TO CERTAIN DOUBLE FUNCTORS $\mathbb{T} \to \mathbb{S}et$ WE CAN TAKE VERTICAL MORPHISMS OF MODELS TO ALSO BE VERTICAL TRANSFORMATIONS.

BUT THERE ARE SEVERAL NOTIONS OF VERTICAL TRANSFORMATION - NOT CLEAR WHICH IS RIGHT.

THE CANDIDATES $\qquad t : F \to G : \mathbb{A} \to \mathbb{B}$

1. STRONG TRANSFORMATIONS

$\forall A$ GIVE A VERTICAL ARROW $tA : FA \to GA$

$\forall a : A \to A'$ GIVE A CELL

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "tA"'] \arrow[dr, phantom, "ta"] & FA' \arrow[d, "tA'"] \\
GA \arrow[r, "Ga"'] & GA'
\end{tikzcd}$$

WHICH ARE HORIZ FUNCTORIAL

AND VERTICALLY NATURAL

$$\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[d, "tA"] \\
F\bar{A} \arrow[r, phantom, "\cong"] \arrow[d, "t\bar{A}"'] \arrow[dr, phantom, "(*)"] & GA \arrow[d, "Gv"] \\
G\bar{A} \arrow[r, equal] & G\bar{A}
\end{tikzcd}$$

TOO STRONG

2. LAX TRANSFORMATIONS

SAME BUT WITH A CELL IN (*) + EQUATIONS.

3. MEALY MORPHISMS

$\forall\ v : A \nrightarrow \bar{A}$ GIVE A VERTICAL ARROW $tv : FA \nrightarrow G\bar{A}$

$\forall$
$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'"] \\ \bar{A} \arrow[r, "\bar{a}"'] & \bar{A}' \end{tikzcd}$$
GIVE A CELL
$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "tv"'] \arrow[dr, phantom, "t\alpha"] & FA' \arrow[d, "tv'"] \\ G\bar{A} \arrow[r, "G\bar{a}"'] & G\bar{A}' \end{tikzcd}$$

$\forall\ A \xrightarrow{v} \bar{A} \xrightarrow{\bar{v}} \bar{\bar{A}}$ GIVE A CELL

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[d, "tv"] \\ F\bar{A} \arrow[r, phantom, "\tau"] \arrow[d, "t\bar{v}"'] & G\bar{A} \arrow[d, "G\bar{v}"] \\ G\bar{\bar{A}} \arrow[r, equal] & G\bar{\bar{A}} \end{tikzcd}$$

SATISFYING "OBVIOUS" EQUATIONS

4. MODULAR MORPHISMS (COCKETT-KOSLOWSKI-SEELY-WOOD)

SIMILAR TO (3) BUT WITH τ REPLACED BY CELLS

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[dd, "t(\bar{v}\cdot v)"] \\ F\bar{A} \arrow[r, phantom, "\rho"] \arrow[d, "t\bar{v}"'] & {} \\ G\bar{\bar{A}} \arrow[r, equal] & G\bar{\bar{A}} \end{tikzcd}$$
AND
$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "tv"'] & FA \arrow[dd, "t(\bar{v}\cdot v)"] \\ G\bar{A} \arrow[r, phantom, "\lambda"] \arrow[d, "G\bar{v}"'] & {} \\ G\bar{\bar{A}} \arrow[r, equal] & G\bar{\bar{A}} \end{tikzcd}$$
SATISFYING EQUATIONS (3 ASSOC + 2 UNIT)