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Monoidal Categories With Natural Numbers Object*

Abstract. The notion of a natural numbers object in a monoidal category is defined and it is shown that the theory of primitive recursive functions can be developed. This is done by considering the category of cocommutative comonoids which is cartesian, and where the theory of natural numbers objects is well developed. A number of examples illustrate the usefulness of the concept.

Introduction

The purpose of this paper is to show that one can define the concept of a natural numbers object in a monoidal category so as to include a number of naturally occurring examples and then to construct the usual primitive recursive functions in this setting. It is somewhat surprising that we can obtain the primitive recursive functions since one of the basic ingredients in their construction are projections, and these are not available in a monoidal category. Our method is first of all to reduce the problem to the case of a *symmetric* monoidal category by constructing a symmetry for an appropriate full subcategory. We then show that our natural numbers object is a cocommutative comonoid, and is in fact a natural numbers object in the category of all such, a category well-known to be cartesian. Then standard results yield all primitive recursive functions. The last section shows that the initial monoidal category with natural numbers object is isomorphic to the initial cartesian category with natural numbers object.

1. Natural numbers objects in a monoidal category

1.1. Definition. Let $V=(V, \otimes, I, \alpha, \lambda, \varrho)$ be a (not necessarily symmetric) monoidal category (see [6, p. 21]). By a *left natural numbers object* (*LNNO*) in V we mean an object N and two morphisms $0\colon I \to N$, $S\colon N \to N$ such that for any pair of morphisms $f\colon A \to B$, $g\colon B \to B$ in V, there exists a unique $h\colon N \otimes A \to B$ such that

$$\begin{tikzcd} I \otimes A \arrow[r, "0 \otimes A"] \arrow[d, "\lambda"'] & N \otimes A \arrow[r, "S \otimes A"] \arrow[d, "h"] & N \otimes A \arrow[d, "h"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & B \end{tikzcd}$$

commutes.

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A *right natural numbers object* is defined similarly except that we require a unique h: $A \otimes N \to B$ in this case.

If they exist, left (right) natural numbers objects are unique up to canonical isomorphism. Furthermore, if V has a left natural numbers object and a right one, they are isomorphic; either one will be called a *(2-sided) natural numbers object* (NNO). In fact, if $(N_l, 0_l, S_l)$ is a left natural numbers object and $(N_r, 0_r, S_r)$ a right one, then for any A we get unique morphisms σ_A and τ_A filling in the diagram

$$\begin{tikzcd}
I \otimes A \arrow[r, "0_l \otimes A"] \arrow[d, "\varrho^{-1}\lambda"'] & N_l \otimes A \arrow[r, "S_l \otimes A"] \arrow[d, "\sigma_A"] & N_l \otimes A \arrow[d, "\sigma_A"] \\
A \otimes I \arrow[r, "A \otimes 0_r"] \arrow[d, "\lambda^{-1}\varrho"'] & A \otimes N_r \arrow[r, "A \otimes S_r"] \arrow[d, "\tau_A"] & A \otimes N_r \arrow[d, "\tau_A"] \\
I \otimes A \arrow[r, "0_l \otimes A"'] & N_l \otimes A \arrow[r, "S_l \otimes A"'] & N_l \otimes A
\end{tikzcd}$$

and uniqueness gives $\tau_A \sigma_A = 1_{N_l \otimes A}$ and $\sigma_A \tau_A = 1_{A \otimes N_r}$. Thus $N_l \otimes A \cong A \otimes N_r$ and, specializing to the case $A = I$, we get $N_l \cong N_r$. Thus $N_l \otimes A \cong A \otimes N_l$, i.e. N_l (and N_r too) "commutes" with any object A.

If V is symmetric, then the concepts of left, right, and two-sided NNOs coincide and σ_A and τ_A are instances of the symmetry for V.

In this paper we shall be dealing mainly with two-sided natural numbers objects but there is one important example of a left one which is not two-sided, and that is given by Burroni's concept of Peano-Lawvere category discussed below.

If V is right-closed, i.e. each of the functors $(\ )\otimes A$: $V \to V$ has a right adjoint $[A, -]$: $V \to V$, then $(N, 0, S)$ is a left natural numbers object iff for every b: $I \to B$, g: $B \to B$ there exists a unique h: $N \to B$ such that

$$\begin{tikzcd}
I \arrow[r, "0"] \arrow[d, equal] & N \arrow[r, "S"] \arrow[d, "h"] & N \arrow[d, "h"] \\
I \arrow[r, "b"'] & B \arrow[r, "g"'] & B
\end{tikzcd}$$

commutes. Indeed, given f: $A \to B$ and g: $B \to B$, the existence of the required h: $N \otimes A \to B$ satisfying the above condition is equivalent to the existence of $\bar{h}$: $N \to [A, B]$ such that

$$\begin{tikzcd}
I \arrow[r, "0"] \arrow[d, "\ulcorner 1_A \urcorner"'] & N \arrow[r, "S"] \arrow[d, "\bar{h}"] & N \arrow[d, "\bar{h}"] \\
{[A, A]} \arrow[r, "{[A, f]}"'] & {[A, B]} \arrow[r, "{[A, g]}"'] & {[A, B]}.
\end{tikzcd}$$

1.2. Examples. Every cartesian category (i.e. a category with finite products) with natural numbers object (see [12]) is a monoidal category with NNO, in particular every topos with NNO is one.

In the monoidal category $\boldsymbol{Ab}$ with the usual tensor, the polynomial ring $\boldsymbol{Z}[x]$ is an NNO with $0\colon \boldsymbol{Z} \to \boldsymbol{Z}[x]$ the "inclusion of scalars" and $S\colon \boldsymbol{Z}[x] \to \boldsymbol{Z}[x]$ "multiplication by x". Or, more generally, if R is a commutative ring with 1, $\boldsymbol{R\text{-}mod}$ has an NNO, viz. $R[x]$.

The monoidal category, $\boldsymbol{Ban}$, of Banach spaces and linear contractions with its usual tensor product (i.e. the projective tensor $\hat{\otimes}$ of [4]) has l_1 (the space of absolutely summable sequences) as NNO with $0\colon C \to l_1$ the inclusion of the first coordinate and $S\colon l_1 \to l_1$ the unilateral shift operator.

An example which is essentially non-symmetric is given by Burroni's categories satisfying the Peano-Lawvere axiom [1]. He defines a category $\boldsymbol{E}$ to be PL if for every object X in $\boldsymbol{E}$ there is given a diagram

$$X \xrightarrow{Z(X)} N(X) \xrightarrow{S(X)} N(X)$$

such that for every $X \xrightarrow{f} Y \xrightarrow{g} Y$ there exists a unique $h\colon N(X) \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "Z(X)"] \arrow[d, equal] & N(X) \arrow[r, "S(X)"] \arrow[d, "h"] & N(X) \arrow[d, "h"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

commutes. N is easily made functorial by defining $N(x)$ to be the unique morphism such that

$$\begin{tikzcd}
X \arrow[r, "Z(X)"] \arrow[d, "x"'] & N(X) \arrow[r, "S(X)"] \arrow[d, "N(x)"] & N(X) \arrow[d, "N(x)"] \\
X' \arrow[r, "Z(X')"'] & N(X') \arrow[r, "S(X')"'] & N(X')
\end{tikzcd}$$

commutes. And we also see that $Z\colon 1_{\boldsymbol{E}} \to N$ and $S\colon N \to N$ are natural transformations. Consider the monoidal category $\mathrm{End}(\boldsymbol{E})$ of endofunctors of $\boldsymbol{E}$ with composition as tensor. Then

$$1_{\boldsymbol{E}} \xrightarrow{Z} N \xrightarrow{S} N$$

is an LNNO in $\mathrm{End}(\boldsymbol{E})$; in fact, it is easily seen that being an LNNO in $\mathrm{End}(\boldsymbol{E})$ is equivalent to being a Burroni natural numbers object. However, it is an RNNO only for the most trivial $\boldsymbol{E}$.

On the other hand, if N is an LNNO, $\boldsymbol{V}$ is a PL category with $N(X) = N \otimes X$. So the two notions of category with natural numbers object (monoidal $\boldsymbol{V}$ with LNNO and PL category) are closely related. The main advantage of ours is that we have an actual object (and two morphisms) representing the natural numbers, and the theory more closely follows the usual theory of primitive recursive functions.

In [8], Lambek defines natural numbers objects in Gentzen multicategories. Gentzen multicategories are to multicategories what cartesian categories are to monoidal categories, so it would seem natural to generalize our concept of LNNO to multicategories. Such a generalization would not be difficult, and indeed there seem to be good reasons for it. For one thing, the theory would be smoother, avoiding much of the coherence questions. It would also place our theory in the logical context where it belongs. Also see [7].

The examples for ***Ab***, ***R-mod***, and ***Ban*** above all follow from the following easily established fact.

1.3. PROPOSITION. *If V has countable coproducts which are preserved by the functors $(\ )\otimes A$ for all A in V, then $\sum_N I$ (the coproduct of countably many copies of the unit for $\otimes$) is an LNNO.* ■

The reader may feel that we have been deliberately misleading by saying that the NNO in ***Ab*** is $\boldsymbol{Z}[x]$ when 1.3 says that it is merely $\oplus_N \boldsymbol{Z}$. Of course $\boldsymbol{Z}[x]$ and $\oplus_N \boldsymbol{Z}$ are isomorphic as abelian groups, but there is a real sense in which the NNO is $\boldsymbol{Z}[x]$ rather than $\oplus \boldsymbol{Z}$. The point is that on any natural numbers object we can define addition and thus get a commutative monoid structure. In ***Ab***, a commutative monoid is a commutative ring with 1, and the ring which is the NNO for ***Ab*** is precisely $\boldsymbol{Z}[x]$.

1.4. PROPOSITION. *If N is an LNNO in a monoidal category V, then we can define an addition $+\colon N\otimes N \to N$ which makes N into a monoid. If V is symmetric, then N is a commutative monoid.*

PROOF. Define $+$ to be the unique map making

$$\begin{tikzcd}
I\otimes N \arrow[r, "0\otimes N"] \arrow[d, "\lambda"'] & N\otimes N \arrow[r, "S\otimes N"] \arrow[d, "+"] & N\otimes N \arrow[d, "+"] \\
N \arrow[r, "1_N"'] & N \arrow[r, "S"'] & N
\end{tikzcd}$$

commute. The proofs of the claimed properties of $+$ are the same as in the cartesian case and are left to the reader. (In fact they will *follow* from the cartesian case once we have established our results of Section 3.) ■

In the case of ***Ab*** (or ***R-mod***) it is easily seen that this addition is multiplication of polynomials $\boldsymbol{Z}[x]\otimes \boldsymbol{Z}[x] \to \boldsymbol{Z}[x]$. As for ***Ban***, it corresponds to the "Cauchy product" of sequences

$$(a_n)_n\cdot(b_n)_n = \Big(\sum_{i+j=n} a_i b_j\Big)_n.$$

Multiplication $\cdot$: $N \otimes N \to N$, however, is more complicated. In the cartesian case it is often defined by noting that the morphism $\langle p_1, \cdot \rangle$: $N \times N \to N \times N$ is the unique φ such that

$$\begin{tikzcd} N \times 1 \arrow[r, "N \times 0"] \arrow[d, equal] & N \times N \arrow[r, "N \times S"] \arrow[d, "\varphi"] & N \times N \arrow[d, "\varphi"] \\ N \times 1 \arrow[r, "N \times 0"'] & N \times N \arrow[r, "(p_1, +)"'] & N \times N. \end{tikzcd}$$

Then $\cdot = p_2 \varphi$. Or, classically, multiplication is defined by primitive recursion

$$\begin{tikzcd} N \times 1 \arrow[r, "N \times 0"] \arrow[d] & N \times N \arrow[d, "\cdot"] & N \times N \arrow[l, "N \times S"'] \arrow[d, "\langle 1_{N \times N}, \cdot \rangle"] \\ 1 \arrow[r, "0"'] & N & (N \times N) \times N. \arrow[l, "p_1 + p_3"] \end{tikzcd}$$

(See [12] for more details.)

In either case we use projections which are not available in monoidal categories. We shall define maps which act as projections but, before, we construct a twist map.

2. The twist map and symmetry axioms

Let N be an LNNO and define $N^k = ((N \otimes N) \otimes \ldots) \otimes N$ for $k \geqslant 1$, and $N^0 = I$. We wish to construct morphisms $\sigma_k(A)$: $N^k \otimes A \to A \otimes N^k$, meant to be the twist maps, and show that they satisfy

$$\begin{tikzcd} N^k \otimes I \arrow[rr, "\sigma_k(I)"] \arrow[dr, "\rho"'] & {} \arrow[d, phantom, "*"] & I \otimes N^k \arrow[dl, "\lambda"] \\ & N^k & \end{tikzcd}$$

$$\begin{tikzcd} (N^k \otimes A) \otimes B \arrow[r, "\alpha"] \arrow[d, "\sigma_k(A) \otimes B"'] & N^k \otimes (A \otimes B) \arrow[r, "\sigma_k(A \otimes B)"] \arrow[d, phantom, "**"] & (A \otimes B) \otimes N^k \arrow[d, "\alpha"] \\ (A \otimes N^k) \otimes B \arrow[r, "\alpha"'] & A \otimes (N^k \otimes B) \arrow[r, "A \otimes \sigma_k(B)"'] & A \otimes (B \otimes N^k), \end{tikzcd}$$

and $\sigma_k(N^l)\sigma_l(N^k) = 1_{N^l \otimes N^k}$ for all $k, l, \geqslant 0$. This will give us a symmetry on the full subcategory of V determined by all objects isomorphic to some N^k.

We first work out the details in the case $k = 1$.

2.1. Definition. For any A in V, define $\sigma(A)\colon N\otimes A\to A\otimes N$ to be the unique morphism filling in the diagram

$$\begin{tikzcd}
I\otimes A \arrow[r, "0\otimes A"] \arrow[d, "\varrho^{-1}\lambda"'] & N\otimes A \arrow[r, "S\otimes A"] \arrow[d, "\sigma(A)"] & N\otimes A \arrow[d, "\sigma(A)"] \\
A\otimes I \arrow[r, "A\otimes 0"'] & A\otimes N \arrow[r, "A\otimes S"'] & A\otimes N.
\end{tikzcd}$$

Note that if V is symmetric, with symmetry σ [6, p. 28], then uniqueness shows that $\sigma(A)=\sigma_{N,A}$ so there is no conflict of notation.

2.2. Proposition. *$\sigma\colon N\otimes(\ \ )\to(\ \ )\otimes N$ is a natural transformation.*

Proof. Let $f\colon A\to B$ in V. Then in

$$\begin{tikzcd}
I\otimes A \arrow[r, "0\otimes A"] \arrow[d, "I\otimes f"'] \arrow[dr, phantom, "(1)"] & N\otimes A \arrow[r, "S\otimes A"] \arrow[d, "N\otimes f"] \arrow[dr, phantom, "(2)"] & N\otimes A \arrow[d, "N\otimes f"] \\
I\otimes B \arrow[r, "0\otimes B"] \arrow[d, "\varrho^{-1}\lambda"'] \arrow[dr, phantom, "(3)"] & N\otimes B \arrow[r, "S\otimes B"] \arrow[d, "\sigma(B)"] \arrow[dr, phantom, "(4)"] & N\otimes B \arrow[d, "\sigma(B)"] \\
B\otimes I \arrow[r, "B\otimes 0"'] & B\otimes N \arrow[r, "B\otimes S"'] & B\otimes N
\end{tikzcd}$$

$$\begin{tikzcd}
I\otimes A \arrow[r, "0\otimes A"] \arrow[d, "\varrho^{-1}\lambda"'] \arrow[dr, phantom, "(5)"] & N\otimes A \arrow[r, "S\otimes A"] \arrow[d, "\sigma(A)"] \arrow[dr, phantom, "(6)"] & N\otimes A \arrow[d, "\sigma(A)"] \\
A\otimes I \arrow[r, "A\otimes 0"] \arrow[d, "f\otimes I"'] \arrow[dr, phantom, "(7)"] & A\otimes N \arrow[r, "A\otimes S"] \arrow[d, "f\otimes N"] \arrow[dr, phantom, "(8)"] & A\otimes N \arrow[d, "f\otimes N"] \\
B\otimes I \arrow[r, "B\otimes 0"'] & B\otimes N \arrow[r, "B\otimes S"'] & B\otimes N
\end{tikzcd}$$

(1), (2), (7), (8) commute by functoriality of $\otimes$ and (3), (4), (5), (6) commute by definition of σ. Thus by uniqueness,

$$\sigma(B)\cdot N\otimes f=f\otimes N\cdot\sigma(A),$$

i.e. σ is natural. ■

2.3. Proposition. $\lambda\sigma(I)=\varrho$.

PROOF.

$$\begin{tikzcd}
I\otimes I \arrow[r, "0\otimes I"] \arrow[d, "\lambda^{-1}\varrho"'] & N\otimes I \arrow[r, "S\otimes I"] \arrow[d, "\lambda^{-1}\varrho"] & N\otimes I \arrow[d, "\lambda^{-1}\varrho"] \\
I\otimes I \arrow[r, "I\otimes 0"'] & I\otimes N \arrow[r, "I\otimes S"'] & I\otimes N
\end{tikzcd}$$

commutes by naturality of $\lambda^{-1}\varrho$ and since $\lambda^{-1}\varrho=\varrho^{-1}\lambda\colon I\otimes I\to I\otimes I$ (they are both equal to $1_{I\otimes I}$) uniqueness gives $\sigma(I)=\lambda^{-1}\varrho$. ■

This result gives commutativity of $*$ in the special case where $k=1$. We wish to prove a similar special case for $**$. In order to simplify notation, we shall omit the associativity isomorphisms α and the brackets in tensor products. The interested reader can easily supply them if he wishes, but Mac Lane's coherence theorem for monoidal categories [10, p. 162] says that it is not necessary. This theorem says (roughly) that any two morphisms with same domain and same codomain built out of instances of λ, ϱ, α and their inverses using composition and $\otimes$ are equal. These morphisms are called *canonical* (see *loc. cit.* p. 165). We often use the notation $f\bullet g$ for a composite in which an obvious canonical morphism has been omitted. See [11] where this notation is used systematically (*and* rigorously).

2.4. PROPOSITION. *For any objects A and B of V,*

$$\sigma(A\otimes B)=(N\otimes A\otimes B\xrightarrow{\sigma(A)\otimes B}A\otimes N\otimes B\xrightarrow{A\otimes\sigma(B)}A\otimes B\otimes N).$$

PROOF. In

$$\begin{tikzcd}
I\otimes A\otimes B \arrow[r, "0\otimes A\otimes B"] \arrow[d, "\varrho^{-1}\lambda\otimes B"'] & N\otimes A\otimes B \arrow[r, "S\otimes A\otimes B"] \arrow[d, "\sigma(A)\otimes B"] & N\otimes A\otimes B \arrow[d, "\sigma(A)\otimes B"] \\
A\otimes I\otimes B \arrow[r, "A\otimes 0\otimes B"] \arrow[d, "A\otimes\varrho^{-1}\lambda"'] & A\otimes N\otimes B \arrow[r, "A\otimes S\otimes B"] \arrow[d, "A\otimes\sigma(B)"] & A\otimes N\otimes B \arrow[d, "A\otimes\sigma(B)"] \\
A\otimes B\otimes I \arrow[r, "A\otimes B\otimes 0"'] & A\otimes B\otimes N \arrow[r, "A\otimes B\otimes S"'] & A\otimes B\otimes N
\end{tikzcd}$$

all squares commute by definition of $\sigma(A)$ and $\sigma(B)$. The coherence theorem says that the morphism on the left is equal to $\varrho^{-1}\lambda$ for $A\otimes B$, thus the result (follows by uniqueness. ■

2.5. PROPOSITION. $\sigma(N)^2=1_{N\otimes N}$.

PROOF. Consider

$$\begin{tikzcd}
I\otimes N \arrow[r, "0\otimes N"] \arrow[d, "\varrho^{-1}\lambda"'] \arrow[dr, phantom, "(1)"] & N\otimes N \arrow[r, "S\otimes N"] \arrow[d, "\sigma(N)"] \arrow[dr, phantom, "(2)"] & N\otimes N \arrow[d, "\sigma(N)"] \\
N\otimes I \arrow[r, "N\otimes 0"] \arrow[d, "\lambda^{-1}\varrho"'] \arrow[dr, phantom, "(3)"] & N\otimes N \arrow[r, "N\otimes S"] \arrow[d, "\sigma(N)"] \arrow[dr, phantom, "(4)"] & N\otimes N \arrow[d, "\sigma(N)"] \\
I\otimes N \arrow[r, "0\otimes N"'] & N\otimes N \arrow[r, "S\otimes N"'] & N\otimes N.
\end{tikzcd}$$

Squares (1) and (2) commute by definition of $\sigma(N)$, whereas (3) and (4) commute by naturality of σ once we have noticed that $\lambda^{-1}\varrho = \sigma(I)$ by 2.3. Since the map on the left is $1_{I\otimes N}$, uniqueness gives $\sigma(N)^2 = 1_{N\otimes N}$. ■

2.6. DEFINITION. We define morphisms $\sigma_k(A)\colon N^k\otimes A\to A\otimes N^k$ for each natural number k by induction:

(1) $\sigma_0(A) = \varrho^{-1}\lambda\colon I\otimes A\to A\otimes I$

(2) $\sigma_1(A) = \sigma(A)\colon N\otimes A\to A\otimes N$

(3) $\sigma_{k+1}(A) = (\sigma_k(A)\otimes N)\bullet(N^k\otimes\sigma(A))$ for $k\geqslant 1$.

2.7. PROPOSITION. *For any $k\geqslant 0$, $\lambda\sigma_k(I) = \varrho$, i.e. $*$ holds.*

PROOF. For $k=0$, this follows since $\varrho^{-1}\lambda = \lambda^{-1}\varrho\colon I\otimes I\to I\otimes I$. Proposition 2.3 is the case $k=1$. For $k>1$, a simple induction shows that $\sigma_k(I)$ is a canonical morphism as described before 2.4. Thus $\lambda\sigma_k(I)$ and ϱ are two canonical morphisms $N^k\otimes I\to N^k$ and therefore are equal. ■

2.8. PROPOSITION. *For any $k\geqslant 0$, $\sigma_k(A\otimes B) = A\otimes\sigma_k(B)\bullet\sigma_k(A)\otimes B$, i.e. the diagram $**$ commutes.*

PROOF. For $k=0$, all morphisms involved are canonical so $**$ commutes by coherence. Proposition 2.4 is the case $k=1$. Now, assume we have $**$ for some $k\geqslant 1$ and we show that we have it for $k+1$.

$$\begin{aligned}
&A\otimes\sigma_{k+1}(B)\bullet\sigma_{k+1}(A)\otimes B\\
&\quad = A\otimes\sigma_k(B)\otimes N\bullet A\otimes N^k\otimes\sigma(B)\bullet\sigma_k(A)\otimes N\otimes B\bullet N^k\otimes\sigma(A)\otimes B\\
&\quad = A\otimes\sigma_k(B)\otimes N\bullet\sigma_k(A)\otimes B\otimes N\bullet N^k\otimes A\otimes\sigma(B)\bullet N^k\otimes\sigma(A)\otimes B\\
&\quad = \sigma_k(A\otimes B)\otimes N\bullet N^k\otimes\sigma(A\otimes B)\\
&\quad = \sigma_{k+1}(A\otimes B)
\end{aligned}$$

where the second equality holds by functoriality of $\otimes$ and the third by proposition 2.4 and the induction hypothesis. ■

2.9. PROPOSITION. *For any $k, l\geqslant 0$ we have $\sigma_k(N^l)\sigma_l(N^k) = 1_{N^l\otimes N^k}$.*

PROOF. The proof is by induction of k.

When $k=0$ we have $\varrho^{-1}\lambda\sigma_l(I) = 1_{N^l\otimes I}$ by 2.7.

When $k=1$ we prove that $\sigma(N^l)\sigma_l(N) = 1_{N^l\otimes N}$ by induction on l. For $l=0$, we have $\sigma(I)\sigma_0(N) = \lambda^{-1}\varrho\varrho^{-1}\lambda = 1_{I\otimes N}$ by 2.3. The result for $l=1$ follows from 2.5. Now, assume we have the equality for $l \geqslant 1$ and consider

$$\begin{aligned}
&\sigma(N^{l+1})\sigma_{l+1}(N)\\
&\quad = (N^l\otimes\sigma(N))\bullet(\sigma(N^l)\otimes N)\bullet(\sigma_l(N)\otimes N)\bullet(N^l\otimes\sigma(N))\\
&\quad = (N^l\otimes\sigma(N))\bullet(N^l\otimes\sigma(N))\\
&\quad = 1.
\end{aligned}$$

The first equality follows from 2.4 and definition of σ_{l+1}, the second by induction hypothesis, and the third by 2.5.

For $k+1$, given the result for some $k \geqslant 1$,

$$\begin{aligned}
&\sigma_{k+1}(N^l)\sigma_l(N^{k+1})\\
&\quad = (\sigma_k(N^l)\otimes N)\bullet(N^k\otimes\sigma(N^l))\bullet(N^k\otimes\sigma_l(N))\bullet(\sigma_l(N^k)\otimes N)\\
&\quad = (\sigma_k(N^l)\otimes N)\bullet(\sigma_l(N^k)\otimes N)\\
&\quad = 1.
\end{aligned}$$

The first equality follows from the definition of σ_{k+1} and 2.4, the second by the case $k=1$, and the third by the induction hypothesis. ■

2.10. THEOREM. *If V is a monoidal category with a left natural numbers object N, then the full subcategory determined by the objects isomorphic to N^k for some $k \geqslant 0$ is a symmetric monoidal category with symmetry defined by $\sigma_{N^k,N^l} = \sigma_k(N^l)$.*

PROOF. In view of 2.7, 2.8, 2.9, the only thing to show is that σ_{N^k,N^l} is natural in both variables, which is the same as in each variable separately. That it is natural in the second variable follows easily from Proposition 2.2 and Definition 2.6 by induction. Naturality in the first variable follows from the equation $\sigma_{N^k,N^l} = (\sigma_{N^l,N^k})^{-1}$ which proposition 2.9 gives. ■

This theorem says that if we wish to study which morphisms can be defined $N^k \to N^l$ in a monoidal category with LNNO, we can restrict our attention to symmetric monoidal categories with NNO.

3. Natural numbers objects in a symmetric monoidal category.

Let V be a symmetric monoidal category with symmetry σ. We get the notion of a comonoid in V by reversing the arrows in the definition of monoid.

3.1. Definition. A *comonoid* in V is an object C together with two morphisms $\varepsilon\colon C \to I$ and $\delta\colon C \to C \otimes C$ such that

(coassociativity)
$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes C \arrow[d, "C \otimes \delta"] \\ C \otimes C \arrow[r, "\delta \otimes C"'] & C \otimes C \otimes C \end{tikzcd}$$

(counitary)
$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[dr, "\varrho^{-1}"'] & C \otimes C \arrow[d, "C \otimes \varepsilon"] \\ & C \otimes I \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\delta"] \arrow[dr, "\lambda^{-1}"'] & C \otimes C \arrow[d, "\varepsilon \otimes C"] \\ & I \otimes C \end{tikzcd}$$

commute. Furthermore, C is *cocommutative* if $\sigma\delta = \delta$. A *homomorphism* between two coalgebras C and D is a morphism $f\colon C \to D$ such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\delta"'] & D \arrow[d, "\delta"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\ & I & \end{tikzcd}$$

commute.

It is well-known that the category $CC(V)$ of cocommutative comonoids in V is cartesian. The cartesian product of C and D is given by $C \otimes D$ with

$$p_1 = (C \otimes D \xrightarrow{C \otimes \varepsilon} C \otimes I \xrightarrow{\varrho} C)$$
$$p_2 = (C \otimes D \xrightarrow{\varepsilon \otimes D} I \otimes D \xrightarrow{\lambda} C)$$
$$\langle f, g \rangle = (E \xrightarrow{\delta} E \otimes E \xrightarrow{f \otimes g} C \otimes D)$$

for $f\colon E \to C$, $g\colon E \to D$. The terminal object is I with $\varepsilon = 1_I$ and $\delta = \lambda_I^{-1} (= \varrho_I^{-1})$. Cocommutativity is used to get the comonoid structure on $C \otimes D$.

The category $CC(V)$ is in some sense the best approximation to V by a cartesian category. More precisely, if ***Mon*** denotes the category of (small) monoidal categories with functors preserving the monoidal structure exactly, and ***Cart*** the category of (small) cartesian categories with chosen binary products and functors preserving the cartesian structure exactly, then CC: $\boldsymbol{Mon} \to \boldsymbol{Cart}$ is right adjoint to the forgetful functor $\boldsymbol{Cart} \to \boldsymbol{Mon}$. This fact was first observed by Fox [2]. See also [5] where the point of view that $CC(V)$ is the category in which the logic of V takes place is espoused.

Let N be an NNO in V. We wish to endow it with a comonoid structure.

3.2. DEFINITION. The morphisms ε: $N \to I$ and δ: $N \to N \otimes N$ are defined to be the unique morphisms filling in the diagrams

$$\begin{tikzcd} I \arrow[r, "0"] \arrow[d, equal] & N \arrow[r, "s"] \arrow[d, "\varepsilon"] & N \arrow[d, "\varepsilon"] \\ I \arrow[r, equal] & I \arrow[r, equal] & I, \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "0"] \arrow[d, "\varrho^{-1}"'] & N \arrow[r, "s"] \arrow[d, "\delta"] & N \arrow[d, "\delta"] \\ I \otimes I \arrow[r, "0 \otimes 0"'] & N \otimes N \arrow[r, "s \otimes s"'] & N \otimes N. \end{tikzcd}$$

3.3. PROPOSITION. *N equipped with ε and δ is a cocommutative comonoid.*

PROOF.

$$\begin{tikzcd} I \arrow[r, "0"] \arrow[d, "\varrho^{-1}"'] & N \arrow[r, "s"] \arrow[d, "\delta"] & N \arrow[d, "\delta"] \\ I \otimes I \arrow[r, "0 \otimes 0"] \arrow[d, "I \otimes \varrho^{-1}"'] & N \otimes N \arrow[r, "s \otimes s"] \arrow[d, "N \otimes \delta"] & N \otimes N \arrow[d, "N \otimes \delta"] \\ I \otimes I \otimes I \arrow[r, "0 \otimes 0 \otimes 0"'] & N \otimes N \otimes N \arrow[r, "s \otimes s \otimes s"'] & N \otimes N \otimes N \end{tikzcd}$$

commutes by definition of δ and functoriality of $\otimes$. So does the one with $N \otimes \delta$ and $I \otimes \varrho^{-1}$ replaced by $\delta \otimes N$ and $\varrho^{-1} \otimes I$ respectively, for the same reasons. The coherence theorem says that $I \otimes \varrho^{-1} = \varrho^{-1} \otimes I$ so uniqueness gives coassociativity of N.

Cocommutativity follows from uniqueness by considering the following commutative diagram

$$\begin{tikzcd} I \arrow[r, "0"] \arrow[d, "\varrho^{-1}"'] & N \arrow[r, "s"] \arrow[d, "\delta"] & N \arrow[d, "\delta"] \\ I \otimes I \arrow[r, "0 \otimes 0"'] \arrow[d, "\sigma"'] & N \otimes N \arrow[r, "s \otimes s"'] \arrow[d, "\sigma"] & N \otimes N \arrow[d, "\sigma"] \\ I \otimes I \arrow[r, "0 \otimes 0"'] & N \otimes N \arrow[r, "s \otimes s"'] & N \otimes N \end{tikzcd}$$

and noting that $\sigma \varrho_I^{-1} = \varrho_I^{-1}$ follows from the fact that $\varrho \sigma = \lambda$ and $\lambda_I = \varrho_I$.

Finally, the right counit law can be read off the diagram

$$\begin{tikzcd}
I \arrow[r, "0"] \arrow[d, "\varrho^{-1}"'] & N \arrow[r, "S"] \arrow[d, "\delta"] & N \arrow[d, "\delta"] \\
I \otimes I \arrow[r, "0 \otimes 0"'] \arrow[d, equal] & N \otimes N \arrow[r, "S \otimes S"'] \arrow[d, "N \otimes \varepsilon"] & N \otimes N \arrow[d, "N \otimes \varepsilon"] \\
I \otimes I \arrow[r, "0 \otimes I"'] \arrow[d, "\varrho"'] & N \otimes I \arrow[r, "S \otimes I"'] \arrow[d, "\varrho"] & N \otimes I \arrow[d, "\varrho"] \\
I \arrow[r, "0"'] & N \arrow[r, "S"'] & N
\end{tikzcd}$$

and the left counit law follows by cocommutativity. ■

3.4. Corollary. *N^k is a cocommutative comonoid in a canonical way, for each $k > 0$.*

Proof. N^k is the cartesian product of k copies of N in $CC(V)$. ■

3.5. Theorem. *N with 0: $I \to N$ and S: $N \to N$ is a natural numbers object in the category $CC(V)$ of cocommutative comonoids in V.*

Proof. We have already seen that N is a cocommutative comonoid, and the definitions of ε and δ (3.2) are such that 0: $I \to N$ and S: $N \to N$ are comonoid homomorphisms.

Now let f: $A \to B$ and g: $B \to B$ be comonoid homomorphisms and let h be the unique morphism making

$$\begin{tikzcd}
I \otimes A \arrow[r, "0 \otimes A"] \arrow[d, "\lambda"'] & N \otimes A \arrow[r, "S \otimes A"] \arrow[d, "h"] & N \otimes A \arrow[d, "h"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & B
\end{tikzcd}$$

commute. We wish to show that h is a comonoid homomorphism.

First, note that

$$\begin{tikzcd}
I \otimes A \arrow[r, "0 \otimes A"] \arrow[d, "\lambda"'] & N \otimes A \arrow[r, "S \otimes A"] \arrow[d, "h"] & N \otimes A \arrow[d, "h"] \\
A \arrow[r, "f"'] \arrow[d, "\varepsilon"'] & B \arrow[r, "g"'] \arrow[d, "\varepsilon"] & B \arrow[d, "\varepsilon"] \\
I \arrow[r, equal] & I \arrow[r, equal] & I
\end{tikzcd}$$

commutes, as does

$$\begin{tikzcd}
I\otimes A \arrow[r, "0\otimes A"] \arrow[d, "I\otimes\varepsilon"'] & N\otimes A \arrow[r, "S\otimes A"] \arrow[d, "\varepsilon\otimes\varepsilon"] & N\otimes A \arrow[d, "\varepsilon\otimes\varepsilon"] \\
I\otimes I \arrow[r, equal] \arrow[d, "\lambda"'] & I\otimes I \arrow[r, equal] \arrow[d, "\lambda"] & I\otimes I \arrow[d, "\lambda"] \\
I \arrow[r, equal] & I \arrow[r, equal] & I
\end{tikzcd}$$

and, since $\lambda(I\otimes\varepsilon)=\varepsilon\lambda$, we see that $\lambda(\varepsilon\otimes\varepsilon)=\varepsilon h$, i.e. h preserves the counit.

Finally, the preservation of comultiplication can be seen by comparing the following commutative diagrams

$$\begin{tikzcd}
I\otimes A \arrow[r, "0\otimes A"] \arrow[d, "\lambda"'] & N\otimes A \arrow[r, "S\otimes A"] \arrow[d, "h"] & N\otimes A \arrow[d, "h"] \\
A \arrow[r, "f"] \arrow[d, "\delta"'] & B \arrow[r, "g"] \arrow[d, "\delta"] & B \arrow[d, "\delta"] \\
A\otimes A \arrow[r, "f\otimes f"'] & B\otimes B \arrow[r, "g\otimes g"'] & B\otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
I\otimes A \arrow[r, "0\otimes A"] \arrow[d, "\lambda^{-1}\otimes\delta"'] & N\otimes A \arrow[r, "S\otimes A"] \arrow[d, "\delta\otimes\delta"] & N\otimes A \arrow[d, "\delta\otimes\delta"] \\
I^2\otimes A^2 \arrow[r, "0^2\otimes A^2"] \arrow[d, "\sigma"'] & N^2\otimes A^2 \arrow[r, "S^2\otimes A^2"] \arrow[d, "\sigma"] & N^2\otimes A^2 \arrow[d, "\sigma"] \\
(I\otimes A)^2 \arrow[r, "(0\otimes A)^2"] \arrow[d, "\lambda^2"'] & (N\otimes A)^2 \arrow[r, "(S\otimes A)^2"] \arrow[d, "h^2"] & (N\otimes A)^2 \arrow[d, "h^2"] \\
A^2 \arrow[r, "f^2"'] & B^2 \arrow[r, "g^2"'] & B^2
\end{tikzcd}$$

and noting the equality of the two left-hand arrows. ■

It follows from [12] that every primitive recursive function has a counterpart in $CC(V)$, or to put it differently, every primitive recursive function $f\colon N^k\to N^r$ can be defined in any monoidal category with an LNNO and this f is actually a comonoid homomorphism. Furthermore, any equalities provable in the theory of primitive recursive functions hold in any such V.

The reader may wonder what operation on polynomials corresponds to multiplication on N in ***R-mod*** since, as we saw, multiplication of polynomials corresponds to addition $+: N\otimes N \to N$. It is the operation, denoted by $*: R[x]\otimes R[x]\to R[x]$, which is the bilinear extension of $x^i * x^j = x^{ij}$. Thus, for example, $(2x^3+x-1)*(7x^2-2) = 14x^6+7x^2-11$. This strange operation is unitary, commutative and associative, as multiplication of natural numbers must be. It also distributes over multiplication of polynomials but in an internal sense, i.e. the following diagram commutes

$$\begin{tikzcd}
R[x]\otimes R[x]\otimes R[x] \arrow[rr, "R[x]\otimes *"] \arrow[d, "\delta\otimes R[x]\otimes R[x]"'] & & R[x]\otimes R[x] \arrow[d, "*"] \\
R[x]\otimes R[x]\otimes R[x]\otimes R[x] \arrow[d, "\sigma_{23}"'] & & R[x] \\
R[x]\otimes R[x]\otimes R[x]\otimes R[x] \arrow[rr, "*\otimes *"'] & & R[x]\otimes R[x], \arrow[u, "*"']
\end{tikzcd}$$

in which σ_{23} denotes the isomorphism which switches the second and third factors, and δ is the comultiplication on $R[x]$, i.e. $\delta(x^n) = x^n\otimes x^n$. Of course, since $*$ is defined on the tensor product, it is bilinear so it distributes over addition in the usual sense. Thus replacing multiplication of polynomials by $*$ we get a new ring structure on $R[x]$.

Many symmetric monoidal categories which arise in practice are closed and admit a cofree comonoid construction. This is true, for example, for ***Ab***, ***Ban***, and ***GrAb*** (graded Abelian groups) (see [3]). For such categories, the previous theorem has the following converse.

3.6. Proposition. *Let V be a monoidal closed category for which the forgetful functor $U: CC(V)\to V$ has a right adjoint* !( ). *If $(N, 0, S)$ is a natural numbers object for $CC(V)$ then $(UN, U0, US)$ is a natural numbers object for V.*

Proof. Since V is closed it is sufficient for $(UN, U0, US)$ to be a natural numbers object that, for every $b: I\to B$, $g: B\to B$ in V, there exists a unique $h: UN\to B$ such that

$$\begin{tikzcd}
U1 \arrow[r, dashed, "U0"] \arrow[d, equal] & UN \arrow[r, "US"] \arrow[d, "h"] & UN \arrow[d, "h"] \\
I \arrow[r, "b"'] & B \arrow[r, "g"'] & B
\end{tikzcd}$$

commute. But this is equivalent to the existence of a unique comonoid homomorphism $\bar{h}$ such that

$$
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[d, "\eta"'] & N \arrow[r, "S"] \arrow[d, "\bar{h}"] & N \arrow[d, "\bar{h}"] \\
!I \arrow[r, "!b"'] & !B \arrow[r, "!g"'] & !B
\end{tikzcd}
$$

commutes (η: $1 \to !U1$ is the unit of the adjunction $!(\ \ ) \dashv U$). But this exists because N is an NNO in $CC(V)$. ∎

4. Free monoidal categories with LNNO

We end with a short discussion of the free monoidal category with LNNO generated by the empty category (or graph). The free cartesian category with NNO generated by the empty category is of considerable interest if we wish to study primitive recursive functions. Its morphisms are equivalence classes of formulas for primitive recursive functions. Since it is free on the empty category, it is initial in the category of cartesian categories with NNO. Thus any relation between primitive recursive functions which holds in it must hold in any cartesian category with NNO. See [9] and [12] for more details.

One might ask whether an initial monoidal category with LNNO exists, and, if it does, what is its structure. Theorem 4.2 below says that it is the same as the initial cartesian category with NNO. This says that for the theory of primitive recursive functions we can dispense with projections and symmetry.

Let ***Monnat*** denot the category whose objects are monoidal categories with LNNO and whose morphisms are functors precisely preserving all of the structure, i.e. $\otimes$, I, α, λ, ϱ, N, 0, S. There is an obvious forgetful functor, Υ: ***Monnat*** $\to$ ***Cat***, to the category of small categories. A straightforward application of Freyd's general adjoint functor theorem gives the following.

4.1. PROPOSITION. *Every category generates a free monoidal category with* LNNO, *i.e.* Υ *has a left adjoint* Φ: ***Cat*** $\to$ ***Monnat***. ∎

Of course the free monoidal category with LNNO will not be cartesian or even symmetric in general but the initial one, i.e. $\Phi(\emptyset)$, is quite special.

4.2. THEOREM. *The initial monoidal category with* LNNO *is isomorphic to the initial cartesian category with* NNO.

PROOF. Let V_0 denote the initial monoidal category with LNNO and C_0^1 the initial cartesian category with NNO.

The full subcategory, $\bar{V}_0$, of V_0 determined by those objects isomorphic to some N^k is also monoidal and has an LNNO, so there is a unique functor F: $V_0 \to \bar{V}_0$ preserving all the structure. When F is composed with the inclusion we get 1_{V_0} by initiality, so $\bar{V}_0 = V_0$. Thus every object of V_0 is isomorphic to one of the form N^k and, by Theorem 2.10, V_0 is symmetric. Now, by Theorem 3.5,

$CC(V_0)$ is a cartesian category with NNO so there is a unique functor $G\colon C_0 \to CC(V_0)$ preserving the cartesian structure and N. Since C_0 may be viewed as a monoidal category with LNNO, there is a unique morphism in ***Monnat***, $H\colon V_0 \to C_0$. Finally, the forgetful functor $U\colon CC(V_0) \to V_0$ is also in ***Monnat***. Now, by initiality, the composite

$$V_0 \xrightarrow{H} C_0 \xrightarrow{G} CC(V_0) \xrightarrow{U} V_0$$

is the identity. The composite

$$C_0 \xrightarrow{G} CC(V_0) \xrightarrow{U} V_0 \xrightarrow{H} C_0,$$

being a morphism in ***Monnat***, preserves N, 0, S, 1, $\times$, but it also preserves the unique morphism $\top_A\colon A \to 1$ (since it is unique) and projections, $\pi_{A,B}\colon A \times B \to A$ and $\pi'_{A,B}\colon A \times B \to B$, since

$$\pi_{A,B} = (A \times B \xrightarrow{A \times \top_B} A \times 1 \xrightarrow{e} A)$$

and similarly for $\pi'_{A,B}$.

Thus the composite HUG is a cartesian morphism and so it is the identity by initiality of C_0. Therefore $V_0 \cong C_0$. ■

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