Can. J. Math., Vol. XLII, No. 4, 1990, pp. 731–746

SIMPLY CONNECTED LIMITS

ROBERT PARÉ

1. Introduction. The importance of finite limits in completeness conditions has been long recognized. One has only to consider elementary toposes, pretoposes, exact categories, etc., to realize their ubiquity. However, often pullbacks suffice and in a sense are more natural. For example it is pullbacks that are the essential ingredient in composition of spans, partial morphisms and relations. In fact the original definition of elementary topos was based on the notion of partial morphism classifier which involved only pullbacks (see **[6]**). Many constructions in topos theory, involving left exact functors, such as coalgebras on a cotriple and the gluing construction, also work for pullback preserving functors. And pullback preserving functors occur naturally in the subject, e.g. constant functors and the Σ_α. These observations led Rosebrugh and Wood to introduce partial geometric morphisms; functors with a pullback preserving left adjoint **[9]**. Other reasons led Kennison independently to introduce the same concept under the name semi-geometric functors **[5]**.

In many respects pullbacks have nice properties which make them easily manipulated. We are thinking of pasting of pullbacks and pasting cancellation properties, pullbacks of monos are always monos whereas pullbacks of epis being epis is a good exactness condition, and such properties. In other respects pullbacks are not so good. For example, the iteration of pullbacks is not so nice. What does the pullback completion of a category look like? The problem is similar to describing the completion of a category under binary products. We start with a category and add binary products, but this new category does not have binary products so we repeat the process infinitely many times. This gives the completion. But there is a neater way: simply add finite (non empty) products and we are done in one step. For pullbacks the situation is more complicated because, after we have added pullbacks, there are many new morphisms which are not easily described but give rise to new diagrams whose pullbacks must be added at the next stage, and so on.

These considerations prompted us to task the question: what limits can be constructed using only pullbacks? The somewhat surprising answer is: limits of diagrams which are connected, simply connected and satisfy a certain finiteness condition (this is our Theorem 2). The importance of π_0, the set of connected components of a category, in the study of limits has long been understood (see **[8]**), but now the fundamental groupoid of a category, π_1, is also seen to be

Received by the editors October 10, 1989.

This work was partially supported by an NSERC operating grant.

Subject classification: Primary 18A30, 18B40; Secondary 55Q05.

important. The connected, simply connected diagrams give a natural class limits worthy of study in its own right. These limits are precisely those constructible from fibred products (=infinite pullbacks). This is the content of Theorem 1. The paper is rounded out by Theorem 3 which gives an explicit description of the pullback completion of a category.

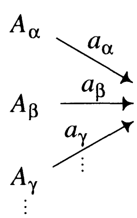
We would like to thank André Joyal, Max Kelly, and Richard Wood for helpful discussions about matters arising in this paper.

2. Categories with fibred products. Let **I** be a small category. In this section we prove the following theorem.

THEOREM 1. **I** *limits exist in any category with fibred products iff* **I** *is connected and simply connected*

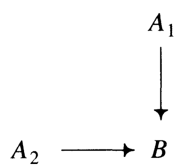
We first explain the various concepts involved.

By *fibred products* we mean limits of (possibly infinite) diagrams of the form



.

These are sometimes called (infinite) pullbacks but we reserve the term *pullback* for the case of two morphisms



.

Fibred products correspond to products in slice categories. The intersection of a family of subobjects is a naturally occurring example of fibred product.

The discrete category functor D : **Set** $\longrightarrow$ **Cat** has a left adjoint π_0 which associates to a category its set of *connected components*. **I** is said to be *connected* if $\pi_0(\mathbf{I}) = 1$. In concrete terms, **I** is connected if it is non-empty and any two objects I, I' can be connected by a finite path of morphisms, back and forth,

$$I \longleftarrow I_1 \longrightarrow I_2 \longleftarrow \ldots \longrightarrow I_{n-1} \longleftarrow I_n \longrightarrow I'. \qquad (*)$$

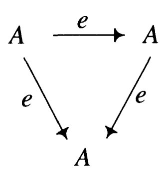
A basic property of π_0 is that it preserves finite products.

A *groupoid* is a category in which all morphisms are invertible. The full inclusion **Gpd** $\longrightarrow$ **Cat** of groupoid into the category of categories has both left and right adjoints. The right adjoint is, Iso, the functor which takes the groupoid of isomorphisms of a category. The left adjoint takes a category **I** and formally adds inverses for every morphism. The resulting groupoid is denoted $\pi_1(\mathbf{I})$ and is called the *fundamental groupoid* of **I**. The objects of $\pi_1(\mathbf{I})$ are those of **I** and the morphisms are equivalence classes of words in morphisms and their inverses. Such a word may be represented by a path such as (*) above. Two paths are equivalent if one can be deformed into the other by a "homotopy", a notion we shall not spell out in detail, preferring to work with the universal property of π_1. We say that **I** is *simply connected* if $\pi_1(\mathbf{I})$ has at most one morphism between any two objects, i.e. is an equivalence relation. Thus the condition of our theorem, **I** connected and simply connected, means that $\pi_1(\mathbf{I})$ is equivalent to $\mathbb{1}$.

As an example, let $\mathbb{E}$ be the monoid with one idempotent element apart from the identity, considered as a one-object category. Since in a groupoid **G** the only idempotents are identities, any functor $\mathbb{E} \longrightarrow \mathbf{G}$ is constant, so $\pi_1(\mathbb{E}) = \mathbb{1}$. Thus theorem 1 says that $\mathbb{E}$ limits exist, i.e. idempotents split, in any category with fibred products. We shall need this in our proof and in a slightly stronger form for theorem 2, so we give a direct proof.

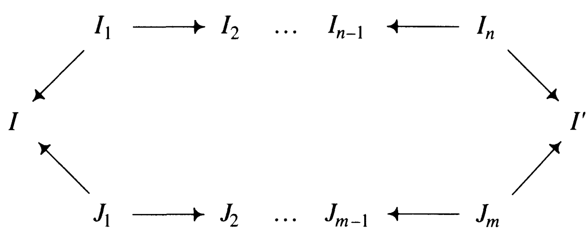
PROPOSITION 1. *Idempotents split in any caetgory with pullbacks.*

Proof. Let **A** be a category with pullbacks and $e : A \longrightarrow A$ an idempotent. Now $\mathbf{A}/A$ still has pullbacks but also has a terminal object, so has all finite limits. Thus the idempotent

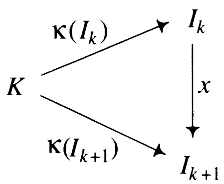


splits in $\mathbf{A}/A$, and applying the domain functor $\Sigma_A : \mathbf{A}/A \longrightarrow \mathbf{A}$, we get a splitting for e in **A**. □

As another illustration, let **I** be cofiltered, i.e. any finite diagram in **I** has a cone. Such an **I** is necessarily connected. A parallel pair $I \rightrightarrows I'$ in $\pi_1\mathbf{I}$ is represented by two paths



in **I**. The finite diagram, D, pictured above has a cone $\kappa : K \longrightarrow D$. If $x : I_k \longrightarrow I_{k+1}$, then



,

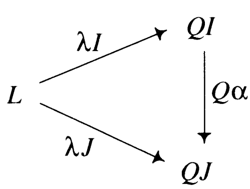
so $x = \kappa(I_{k+1})\kappa(I_k)^{-1}$ and $x^{-1} = \kappa(I_k)\kappa(I_{k+1})^{-1}$ in $\pi_1\mathbf{I}$. So, in evaluating each word everything cancels except the end terms, i.e. both morphisms in $\pi_1\mathbf{I}$ are $\kappa(\Gamma)\kappa(I)^{-1}$ or, put another way, there is exactly one morphism $I \longrightarrow I'$. Thus $\pi_1\mathbf{I} \simeq \mathbb{1}$ (we use $\simeq$ to denote equivalence of categories and $\cong$ for isomorphism). So theorem 1 asserts that cofiltered limits exist in any category with fibred products. In fact we have the following.

PROPOSITION 2. *A category has fibred products iff it has pullbacks and cofiltered limits.*

Proof. The necessity is obvious in view of the above discussion (assuming theorem 1, of course).

The sufficiency follows by an application of the fact that an arbitrary product can be computed as the cofiltered limit of its finite subproducts and that fibred products are the same as products in slice categories. □

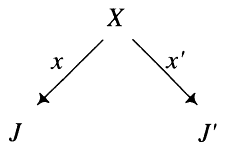
Proof of necessity for theorem 1. Assume that **I** limits exist in any category with fibred products. Given a family of maps $\langle f_x : A_x \longrightarrow B\rangle_{x\in X}$ in a groupoid it is easily seen that the cone $\langle f_x^{-1} : B \longrightarrow A_x\rangle_{x\in X}$ is a fibred product of the A_x over B. Thus the canonical functor $Q : \mathbf{I} \longrightarrow \pi_1\mathbf{I}$ has a limit, $\lambda : L \longrightarrow Q$. If $\alpha : I \longrightarrow J$ is a morphism of **I**, then



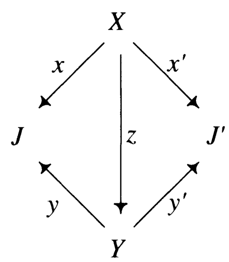
commutes, so $Q\alpha = (\lambda J)(\lambda I)^{-1}$. Since any morphism of $\pi_1\mathbf{I}$ is a composite of $Q\alpha$'s and $(Q\alpha)^{-1}$'s, it follows that any morphism $I \longrightarrow J$ in $\pi_1\mathbf{I}$ is equal to $(\lambda J)(\lambda I)^{-1}$, i.e. there is exactly one morphism between any two objects of $\pi_1\mathbf{I}$. □

Because $\pi_1 \mathbf{I}$ is somewhat difficult to describe explicitly, we approach the sufficiency of theorem 1 in a roundabout way. We first note that π_1 is easily described for categories with pullbacks. This is because the pullbacks can be used to shorten paths such as (*) to ones of length two, and at the same time to simplify "homotopies" between them.

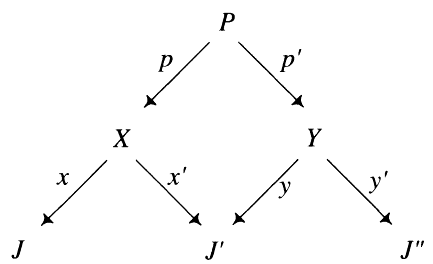
Let $\mathbf{J}$ be a category with pullbacks. Recall [1] that a *span* from J to J' is a pair of morphisms



in $\mathbf{J}$. We denote this span by $x'_* x^* : J \longrightarrow J'$. A morphism of spans $z : x'_* x^* \longrightarrow y'_* y^*$ is a commutative diagram



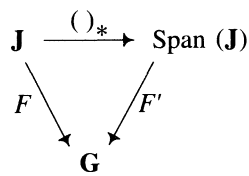
This makes spans from J to J' into a category. On the other hand, we can compose spans using pullback



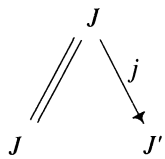
$(y'_* y^*)(x'_* x^*) = (y'p')_*(xp)^*$. This makes Span($\mathbf{J}$) into a bicategory with the same objects as $\mathbf{J}$.

PROPOSITION 3. *If* $\mathbf{J}$ *has pullbacks, then* $\pi_1(\mathbf{J})$ *is the category whose objects are those of* $\mathbf{J}$ *and whose morphisms are connected components of spans, i.e.* $\pi_1(\mathbf{J}) = \pi_{0*}$ Span($\mathbf{J}$) *the category obtained by applying* π_0 *to the hom categories of* Span($\mathbf{J}$).

Proof. Let **G** be a groupoid and $F : \mathbf{J} \longrightarrow \mathbf{G}$ any functor. Consider **G** as a locally discrete bicategory (i.e. only identity 2-cells). Then there is a unique extension of F to a homomorphism of bicategories $F' : \mathrm{Span}(\mathbf{J}) \longrightarrow \mathbf{G}$ such that



commutes. Here $(\ )_*$ is the homomorphism taking $j : J \longrightarrow J'$ to the span $j_*(1_J)^*$,



which can also be denoted j_* with impunity. Uniqueness follows from the fact that for any span $x'_*x^* : J \longrightarrow J'$, there is a morphism of spans $\delta : x'_* \longrightarrow (x'_*x^*)x_*$ induced by the diagonal into the pullback of x with itself. Thus, if F' exists, we must have

$$\begin{aligned} F(x') = F'(x'_*) &= F'((x'_*x^*)x_*) \\ &= F'(x'_*x^*)F'(x_*) \\ &= F'(x'_*x^*)F(x). \end{aligned}$$

So $F'(x'_*x^*)$ must be $F(x')F(x)^{-1}$. It is an easy matter to check that this does work.

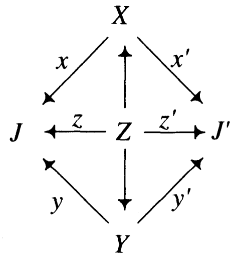
Let π_{0*} Span(**J**) be the category obtained by applying the functor π_0 to each of the hom categories of Span(**J**). This is indeed a category as $\pi_0 : \mathbf{Cat} \longrightarrow \mathbf{Set}$ preserves finite products. The universal property of π_0, namely that it is left adjoint to the discrete category functor $D : \mathbf{Set} \longrightarrow \mathbf{Cat}$, now tells us that any homomorphism $F' : \mathrm{Span}(\mathbf{J}) \longrightarrow \mathbf{G}$ lifts to a unique functor $F'' : \pi_{0*}\ \mathrm{Span}(\mathbf{J}) \longrightarrow \mathbf{G}$ (since **G** is locally discrete).

Thus π_{0*} Span(**J**) has the universal property of $\pi_1(\mathbf{J})$ once we notice that it is a groupoid. This follows from the fact that there are morphisms

$$(x_*x'^*)(x'_*x^*) \longleftarrow x_*x^* \longrightarrow 1_J$$

induced by the pullback diagonal and x itself. Thus in π_{0*} Span(**J**), $x_*x'^*$ is $(x'_*x^*)^{-1}$. □

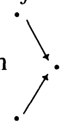
Remarks. (1) Since the hom categories of Span(**J**) have pullbacks (in fact Span(**J**) has local pullbacks), two spans x'_*x^* and y'_*y^* are equivalent in $\pi_1(\mathbf{J})$ if and only if there is a fill-in span of spans



In particular, two morphisms $x,\ y : J \rightrightarrows J'$ of **J** become equal in $\pi_1(\mathbf{J})$ if and only if there exists z such that $xz = yz$.

(2) The above proposition can be understood conceptually as follows. We are trying to make **J** into a groupoid in a universal way. Instead of adding inverses directly to **J**, we first add a right adjoint for each morphism (subject to the Beck condition). This is the well-known construction Span(**J**). Every morphism of a groupoid has a right adjoint, its inverse, and the Beck condition is automatic, so we haven't done anything to **J** we don't ultimately want, except that now we have a bicategory. We now apply the 2-functor π_{0*} : **Bicat** $\longrightarrow$ **Cat** which is the universal way of making a bicategory into a category. But now, in π_{0*} Span(**J**), every morphism still has a right adjoint, i.e. a two-sided inverse, thus we've got our groupoid π_1 **J**.

COROLLARY 1. *If* **J** *has pullbacks, then* $\pi_1\, \mathbf{J} \simeq \mathbb{1}$ *if and only if* **J** *is cofiltered.*

Proof. A category is cofiltered if it is connected, every diagram  can be completed to a commutative square, and every parallel pair $\cdot \rightrightarrows \cdot$ can be equalized. If $\pi_1\, \mathbf{J} \simeq$, then **J** is connected, the pullbacks insure the second condition, and remark (1) shows that the third follows, so **J** is cofiltered.

On the other hand, the converse follows by the discussion preceding Proposition 2. □

Let us now return to a general **I**. Let FibP(**I**) be a completion of **I** with respect to fibred products. By this we understand a category with fibred products, and a functor $H : \mathbf{I} \longrightarrow$ FibP(**I**) which induces an *equivalence* between the category of fibred product preserving functors FibP(**I**) $\longrightarrow$ **A** and the functor category $\mathbf{A}^{\mathbf{I}}$, for every category **A** with fibred products. Of course FibP(**I**) is not usually a small category. Its existence is proved in [**10**]. We shall give a concrete description of it in the last section.

LEMMA 1. *Let* $G : \mathbf{I} \longrightarrow \mathbf{A}$ *be any functor where* **A** *has fibred products, and* F : FibP(**I**) $\longrightarrow$ **A** *a fibred product preserving extension of* G*. Then there is a vertex preserving bijection between cones on* G *and cones on* F*.*

Proof. For any A in **A**, the constant factor Δ_A : FibP(**I**) $\longrightarrow$ **A** preserves fibred products, so the full and faithfulness of the equivalence induced by H gives a bijection between natural transformations $\Delta_A \longrightarrow F$ and $\Delta_A \longrightarrow G$. □

COROLLARY 2. *With the same notation and hypotheses as above,* $\varprojlim F$ *exists if and only if* $\varprojlim G$ *does, and in that case* $\varprojlim F \cong \varprojlim G$. □

PROPOSITION 4. **I** *limits exist in any category with fibred products if and only if* FibP(**I**) *has an initial object.*

Proof. (Necessity) Since FibP(**I**) has fibred products, the limit of $H : \mathbf{I} \longrightarrow$ FibP(**I**) exists, but then by the above corollary, $\varprojlim 1_{\mathrm{FibP(I)}}$ also exists so FibP(**I**) has an initial object.

(Sufficiency) For any **A** with fibred products and any $G : \mathbf{I} \longrightarrow \mathbf{A}$, pick a fibred product preserving extension F : FibP(**I**) $\longrightarrow$ **A**. Since FibP(**I**) has an initial object, $\varprojlim F$ always exists and so, by the corollary, does $\varprojlim G$. □

COROLLARY 3. *If* **I** *limits exist in any category with fibred products then any fibred product preserving functor also preserves* **I** *limits.*

Proof. This is an immediate consequence of [**4** (1.3)], however it can be seen immediately from the above proof. If $U : \mathbf{A} \longrightarrow \mathbf{B}$ is fibred product preserving, then UF is an extension of UG (same notation as above). Then $\varprojlim UG \cong \varprojlim UF \cong UF(1) \cong U \varprojlim F \cong U \varprojlim G$. □

PROPOSITION 5. $H : \mathbf{I} \longrightarrow$ FibP(**I**) *induces an equivalence* $\pi_1(\mathbf{I}) \longrightarrow \pi_1(\mathrm{FibP}(\mathbf{I}))$.

Proof. Strictly speaking the two π_1 of the statement are different: the first is as we have been using all along, for small categories, whereas the second is applied to a large category. It is easily seen, either by direct calculation or by referring to universes, that π_1 also exists for large categories and large groupoids, and when it is applied to a small category gives the one we have been using so far.

With this out of the way, note that not only does a groupoid have all fibred products but their universal property is automatic, i.e. provided the diagram commutes it is a fibred product. Thus any functor into a groupoid is fibred product preserving. So if **G** is any groupoid, H induces an equivalence

$$\mathbf{CAT}(\mathrm{FibP}(\mathbf{I}),\ \mathbf{G}) \longrightarrow \mathbf{CAT}(\mathbf{I},\ \mathbf{G})$$

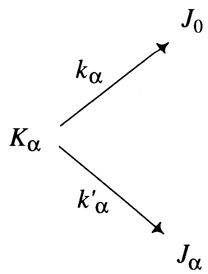
which is the same as an equivalence

$$\mathbf{GPD}(\pi_1\, \mathrm{FibP}(\mathbf{I})),\ \mathbf{G}) \longrightarrow \mathbf{GPD}(\pi_1(\mathbf{I}),\ \mathbf{G}).$$

This makes $\pi_1(H) : \pi_1(\mathbf{I}) \longrightarrow \pi_1(\mathrm{FibP}(\mathbf{I}))$ into an equivalence. □

Proof of sufficiency for theorem 1. Assume that $\pi_1(\mathbf{I}) \simeq \mathbb{1}$. Then by the preceding proposition, $\pi_1(\mathrm{FibP}(\mathbf{I})) \cong \mathbb{1}$. We shall first show that FibP(**I**) is κ-cofiltered for any regular cardinal κ (see [**7**]).

We already know by the corollary to proposition 3 that it is cofiltered. Let $\langle J_\alpha \rangle$ be a family of $< \kappa$ objects. Choose some object J_0. By filteredness we have a diagram



for each α. Now take the fibred product $P \xrightarrow{p_\alpha} K_\alpha$ of the k_α. Then P has a morphism, $k'_\alpha p_\alpha$, to each J_α.

Next, let j_α, $j'_\alpha : J_0 \rightrightarrows J_\alpha$ be a $< \kappa$ family of parallel pairs which we wish to equalize simultaneously. By filteredness, each pair can be equalized by some $K_\alpha \xrightarrow{k_\alpha} J_0$. Now take the fibred product $P \xrightarrow{p_\alpha} K_\alpha$ of the k_α. Then the morphism $k_\alpha p_\alpha : P \longrightarrow J_0$ is independent of α and equalizes all pairs.

From the properties proved in the two preceding paragraphs it is easily seen that any small diagram in FibP(**I**) has a cone. Thus H : **I** $\longrightarrow$ FibP(**I**) has one, and by lemma 1, $1_{\mathrm{FibP}(\mathbf{I})}$ has one too. Thus FibP(**I**) has an initial object as, by proposition 1, FibP(**I**) has split idempotents. The result follows by proposition 4. □

For those who would prefer to avoid the large category FibP(**I**) , as we do, we give the following results which may be of independent use.

Note that the proof of the sufficiency of theorem 1 only uses that FibP(**I**) is κ-cofiltered for some infinite regular κ greater than the cardinality of (the set of morphisms of) **I**, and to get this all that is needed about FibP(**I**) is that it have $< \kappa$-fibred products, i.e. fibred products of diagrams of cardinality $< \kappa$. So introduce FibP_κ(**I**) a completion of **I** with respect to $< \kappa$-fibred products (see [**10**]). It is small. Now, lemma 1, corollary 2, the sufficiency of proposition 4, and proposition 5 all work for FibP$_\kappa$(**I**) for any κ with the same proofs. Then in the sufficiency proof of theorem 1, we get that FibP$_\kappa$(**I**) is κ-filtered and the rest goes through if we take $\kappa >$ #**I**, the cardinality of **I**. Thus we avoid large categories and we get the following (not surprising) result.

Proposition 6. *Let κ be an infinite regular cardinal* $>$ #**I**. **I** *limits exist in any category with fibred products iff* **I** *limits exist in any category with $< \kappa$-fibred products.* □

3. Categories with Pullbacks. In this section we characterize those limits whose existence follows from that of pullbacks.

THEOREM 2. **I** *limits exist in any category with pullbacks if and only if* **I** *is connected, simply connected and L-finite.*

The concept of L-finiteness is intended to express the fact that, as far as limits are concerned, **I** can be thought of as finite. It was Max Kelly who suggested that the following definition might be appropriate for our purposes.

DEFINITION. *Say that* **I** *is L-finite if* 1 *is finitely presentable in* $\mathbf{Set}^{\mathbf{I}}$.

Recall from [**2**] or [**7**] that this means that the representable functor $\mathbf{Set}^{\mathbf{I}}(1,\text{-})$: $\mathbf{Set}^{\mathbf{I}} \longrightarrow \mathbf{Set}$ preserves filtered colimits. But this functor is $\varprojlim : \mathbf{Set}^{\mathbf{I}} \longrightarrow \mathbf{Set}$, so **I** being L-finite means that **I** limits commute with filtered colimits in **Set**. Turning things around we see that L-finiteness is equivalent to saying that $\varinjlim : \mathbf{Set}^{\mathbf{J}} \longrightarrow \mathbf{Set}$ preserves **I** limits for all filtered categories **J**. It is well-known that any finite (in fact finitely generated) category has this property.

Recall that a functor $\Upsilon : \mathbf{I}_0 \longrightarrow \mathbf{I}$ is *initial* if for any diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$, $\varprojlim \Gamma$ exists if and only if $\varprojlim \Gamma\Upsilon$ exists and then $\varprojlim \Gamma \cong \varprojlim \Gamma\Upsilon$. If we replace Γ by $\mathbf{A}(A, \Gamma(\ )) : \mathbf{I} \longrightarrow \mathbf{Set}$ in this, we see that for initial functors Υ there is a vertex preserving bijection between cones on Γ and cones on $\Gamma\Upsilon$. This is equivalent to initiality.

It is proved in [**8**] that a functor $\Upsilon : \mathbf{I}_0 \longrightarrow \mathbf{I}$ is initial if and only if for each $I \in \mathbf{I}$, the comma category (Υ, I) is connected, i.e. $\pi_0(\Upsilon, I) \cong 1$.

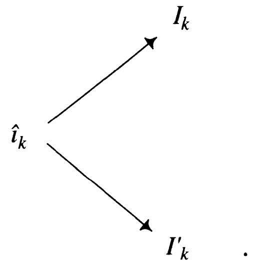
PROPOSITION 7. *The following are equivalent:*
(1) **I** *is L-finite;*
(2) **I** *has a finitely generated initial subcategory;*
(3) **I** *admits a final functor from a finite category.*

Proof. (1) $\Rightarrow$ (2): **I** is the directed colimit of its finitely generated subcategories, $\mathbf{I} = \varinjlim_{\alpha} \mathbf{I}_\alpha$. If $Y : \mathbf{I}^{op} \longrightarrow \mathbf{Set}^{\mathbf{I}}$ is the Yoneda functor, then

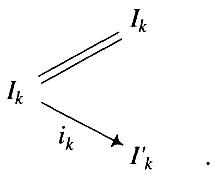
$$\varinjlim_{\alpha} (\varinjlim Y|_{\mathbf{I}_\alpha}) \cong \varinjlim Y \cong 1,$$

the last isomorphism being the canonical presentation of 1 as a colimit of representables. Since 1 is finitely presentable, there is an α such that $\varinjlim Y|_{\mathbf{I}_\alpha}$ is a retract of 1, i.e. $\varinjlim Y|_{\mathbf{I}_\alpha} \cong 1$. If $\Upsilon_\alpha : \mathbf{I}_\alpha \longrightarrow \mathbf{I}$ denotes the inclusion, then this isomorphism says exactly $\pi_0(\Upsilon_\alpha,\text{-}) \cong 1$, i.e. Υ_α is initial.

(2) $\Rightarrow$ (3): Let $\Upsilon : \mathbf{I}' \longrightarrow \mathbf{I}$ be a finitely generated initial subcategory of **I**. Build a new category $\mathbf{I}''$ (closely related to the Kan subdivision category [**3**]) whose objects are those of $\mathbf{I}'$ together with a new object $\hat{i}_k$ for each i_k in some generating set for $\mathbf{I}'$. The only morphisms of $\mathbf{I}''$ are identities, and for each $i_k : I_k \longrightarrow I'_k$ two others



There are no non-trivial compositions in $\mathbf{I}''$ which is finite. There is a functor $\Upsilon' : \mathbf{I}'' \longrightarrow \mathbf{I}'$ which takes the above diagram to



One easily sees that cones on $\Gamma\Upsilon'$ are the same as cones on Γ, for any Γ, so Υ' is initial.

(3) $\Rightarrow$ (1): Let $\Upsilon : \mathbf{I}' \longrightarrow \mathbf{I}$ be initial with $\mathbf{I}'$ finite. Then 1 is a finite colimit of representables, $1 \cong \varinjlim Y \cong \varinjlim Y\Upsilon$, and is therefore finitely presentable. □

Remark. Richard Wood points out that the definition of L-finite is formally the same as that of K-finite in a topos. The singleton function $X \longrightarrow \Omega^X$ is replaced by the Yoneda embedding $\mathbf{I}^{op} \longrightarrow \mathbf{Set}^{\mathbf{I}}$ and $K(X)$ by $L(\mathbf{I})$ the closure of the representables under finite colimits. Then $L(\mathbf{I})$ consists of the finitely presentable objects in $\mathbf{Set}^{\mathbf{I}}$. Thus $\mathbf{I}$ is L-finite if and only if $1 \in L(\mathbf{I})$. This tantalizing idea, together with the above proposition, suggests that we have indeed the right notion.

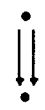
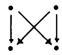
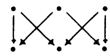
Proof of theorem 2. (Necessity) If $\mathbf{I}$ limits exist in any category with pullbacks then $\mathbf{I}$ is connected and simply connected by theorem 1. For any filtered category $\mathbf{J}$, $\varinjlim : \mathbf{Set}^{\mathbf{J}} \longrightarrow \mathbf{Set}$ preserves pullbacks so by [**4** (1.3)] it also preserves $\mathbf{I}$ limits. Thus, $\mathbf{I}$ is L-finite.

(Sufficiency) Let $Pb(\mathbf{I})$ be a completion of $\mathbf{I}$ with respect to pullbacks, i.e. $\mathrm{FibP}_{\aleph_0}(\mathbf{I})$ in the notation of §2. Then, as discussed before proposition 6, $\mathbf{I}$ limits exist in any category with pullbacks if and only if $Pb(\mathbf{I})$ has an initial object. Furthermore $\pi_1(Pb(\mathbf{I})) \simeq \pi_1(\mathbf{I})$ so, as $\pi_1(\mathbf{I}) \simeq \mathbb{1}$, so is $\pi_1(Pb(\mathbf{I}))$ and therefore by corollary 1, $Pb(\mathbf{I})$ is cofiltered.

Since we are assuming that $\mathbf{I}$ is L-finite, we can find a finite category $\mathbf{I}_0$ and an initial functor $\Upsilon : \mathbf{I}_0 \longrightarrow \mathbf{I}$. Then, by cofilteredness

$$\mathbf{I}_0 \xrightarrow{\ \Upsilon\ } \mathbf{I} \xrightarrow{\ H\ } Pb(\mathbf{I})$$

has a cone. By initiality, this gives a cone for H and so by the corresponding version of lemma 1, $1_{Pb(\mathbf{I})}$ also has a cone. Since idempotents split in $Pb(\mathbf{I})$ (proposition 1), it has an initial object. □

Examples. The categories ,  and  all have a fundamental groupoid equivalent to $\mathbb{Z}$, and so by theorem 2, their limits cannot be calculated using pullbacks alone.

On the other hand, the fundamental groupoid of the category $\cdot \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \cdot \overset{\gamma}{\longrightarrow} \cdot$, with $\gamma\alpha = \gamma\beta$, is equivalent to $\mathbb{1}$. So limits of diagrams $A_0 \underset{g}{\overset{f}{\rightrightarrows}} A_1 \overset{h}{\longrightarrow} A_2$ with $hf = hg$ can be calculated using pullbacks. But the limit of this diagram is exactly the equalizer of f and g. This gives rise to the following curiosity. A category with pullbacks and coequalizers has equalizers. Another consequence is that if a category has a terminal object and pullbacks, then it of course has equalizers, but these equalizers will be preserved by any functor preserving pullbacks (not necessarily 1).

Finally, we mention that everything in this section can be done for $< \kappa$-fibred products for any infinite regular cardinal κ. The translation is a straightforward matter of replacing the word finite by $< \kappa$ and inserting a κ here and there.

4. Completions. In this section we prove the following theorem, which we take as further evidence that the connected, simply connected limits are a natural class to consider. In order to save unnecessary dualities in the middle of proofs, we state our results for the fibred coproduct (resp. pushout) completion FibCp($\mathbf{I}$) (resp. $Po(\mathbf{I})$) of a small category $\mathbf{I}$. Of course FibP($\mathbf{I}$) = FibCp($\mathbf{I}^{op})^{op}$ and $Pb(\mathbf{I}) = Po(\mathbf{I}^{op})^{op}$.

THEOREM 3. FibCp($\mathbf{I}$) *can be taken to be the full subcategory of* $\mathbf{Set}^{\mathbf{I}^{op}}$ *determined by the connected and simply connected presheaves;* $Po(\mathbf{I})$ *the subcategory of that determined by the finitely presentable presheaves.*

A presheaf $\Phi : \mathbf{I}^{op} \longrightarrow \mathbf{Set}$ has a *category of elements* El(Φ) whose objects are pairs $(I, x \in \Phi I)$ and whose morphisms $(I, x \in \Phi I) \longrightarrow (I', x' \in \Phi I')$ are morphisms $i : I \longrightarrow I'$ in $\mathbf{I}$ such that $\Phi(i)(x') = x$. El(Φ) with its forgetful functor to $\mathbf{I}$ is sometimes called the *diagram* of Φ as it gives a geometric picture of Φ. We say that Φ is *connected and simply connected* if El(Φ) is.

Joyal pointed out that the following result, which we learned from Richard Wood ten years ago, could be used to considerably simplify our proof of theorem 3.

PROPOSITION 8. *Taking the category of elements of a presheaf is a functor* El : $\mathbf{Set}^{\mathbf{I}^{op}} \longrightarrow \mathbf{Cat}$ *which has a right adjoint.*

Proof. Using the Yoneda lemma we can determine what the right adjoint R has to be; it is then a matter of calculation to check that it works. For a category

C, $R(\mathbf{C}) : \mathbf{I}^{op} \longrightarrow \mathbf{Set}$ is defined by $R(\mathbf{C})(I) = \mathbf{Cat}(\mathbf{I}/I,\ \mathbf{C})$ and made functorial by composing with the composition functor $\Sigma_i : \mathbf{I}/I \longrightarrow \mathbf{I}/I'$. □

Before giving the proof of theorem 3, we prove the following result which looks obvious and for which "proofs' that are 95% convincing are easily given but which is maddeningly difficult to *prove*, even for pushouts.

Proposition 9. *A fibred coproduct of groupoids equivalent to* $\mathbb{1}$ *is itself equivalent to* $\mathbb{1}$.

Proof. Let $\langle \mathbf{G} \stackrel{\Phi_\alpha}{\to} \mathbf{G}_\alpha \rangle$ be a family of groupoid homomorphisms with $\mathbf{G} \simeq \mathbb{1}$ and $\mathbf{G}_\alpha \simeq \mathbb{1}$ for all α, and let $\langle \mathbf{G}_\alpha \stackrel{\Psi_\alpha}{\to} \mathbf{P} \rangle$ be the family of fibred coproduct injections. The property we shall use is that a groupoid is equivalent to   if and only if the identity functor on it has a cone.

Choose A in $\mathbf{G}$ and let $X = \Psi_\alpha \Phi_\alpha A$. Each Ψ_α has a cone $\gamma_\alpha : X \longrightarrow \Psi_\alpha$ defined by $\gamma_\alpha B = \Psi_\alpha(!) : \Psi_\alpha \Phi_\alpha A \longrightarrow \Psi_\alpha B$ where ! denotes the unique morphism $\Phi_\alpha A \longrightarrow B$. This cone gives a functor $\Gamma_\alpha : \mathbf{G}_\alpha \longrightarrow \mathbf{P^2}$, and $\Gamma_\alpha \Phi_\alpha = \Gamma_\beta \Phi_\beta$ for all α, β. Thus there exists a unique $\Gamma : \mathbf{P} \longrightarrow \mathbf{P^2}$ such that $\Gamma \Psi_\alpha = \Gamma_\alpha$. Composing with the domain and codomain functors $\mathbf{P^2} \longrightarrow \mathbf{P}$ we see that Γ corresponds to a cone $\gamma : X \longrightarrow 1_\mathbf{P}$, so $\mathbf{P} \simeq \mathbb{1}$. □

Corollary 4. *A fibred coproduct of connected simply connected categories is connected and simply connected.*

Proof. Apply $\pi_1 : \mathbf{Cat} \longrightarrow \mathbf{Gpd}$ which preserves fibred coproducts as it is a left adjoint. □

The following examples show that some care must be taken with proposition 9, as various similar statements are false. Let $\mathbb{I}$ be the chaotic category (i.e. $\simeq \mathbb{1}$) with two objects 0 and 1, and $\mathbb{Z}$ the additive group of integers considered as a one object category. Then,

$$\begin{array}{ccc} \mathbb{1} + \mathbb{1} & \xrightarrow{(0,1)} & \mathbb{I} \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & \mathbb{Z} \end{array}$$

is an example of a pushout of equivalence relations which is not an equivalence relation. The following shows that a pushout of an equivalence of categories need not be an equivalence:

$$\begin{array}{ccc} \mathbb{I} & \longrightarrow & \mathbb{1} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{1} \end{array}$$

where the vertical map on the left sends $0 \longrightarrow 1$ to $1 \in \mathbb{Z}$. Finally, the proposition is false for coequalizers as

$$\mathbb{1} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{I} \to \mathbb{Z}$$

shows.

Proof of theorem 3. Let $\mathbf{CSC} \longrightarrow \mathbf{Set}^{\mathbf{I}^{op}}$ be the full subcategory of connected simply connected presheaves. A representable functor is connected simply connected, its category of elements being a slice category which of course has a terminal object. So the Yoneda functor factors through $\mathbf{CSC}$, $H : \mathbf{I} \longrightarrow \mathbf{CSC}$.

Let $F : \mathbf{I} \longrightarrow \mathbf{A}$ be any functor into a category with fibred coproducts. The left Kan extension $\mathrm{Lan}_H F : \mathbf{CSC} \longrightarrow \mathbf{A}$ is given at Φ by a colimit taken over $\mathrm{El}(\Phi)$, which exists by theorem 1. So we get an extension

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{H} & \mathbf{CSC} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \mathrm{Lan}_H F} & \\ & \mathbf{A} & \end{array} .$$

Any other extension which preserved filtered coproducts would also preserve connected simply connected colimits, and so would be isomorphic to $\mathrm{Lan}_H F$, as every presheaf in $\mathbf{CSC}$ is such a colimit of representables. The universal property of Kan extension gives the full and faithfulness of the equivalence which expresses the universal property of FibCP($\mathbf{I}$).

It remains to show that $\mathbf{CSC}$ is closed under fibred coproducts. Let $\langle \Phi \longrightarrow \Psi_\alpha \rangle$ be a diagram of connected simply connected presheaves, and let Θ be its fibred coproduct. By proposition 8, $\mathrm{El}(\Theta)$ is the fibred coproduct of $\mathrm{El}(\Psi_\alpha)$ over $\mathrm{El}(\Phi)$, and corollary 3 tells us that $\mathrm{El}(\Theta)$ is connected simply connected, i.e. Θ is connected and simply connected.

The proof for $Po(\mathbf{I})$ is the same except for one extra fact which we need: the finitely presentable presheaves are precisely the ones whose category of elements are co$L$-finite. We must show that the functor

$$\mathbf{Set}^{\mathrm{El}(\Phi)^{op}}(1, -) : \mathbf{Set}^{\mathrm{El}(\Phi)^{op}} \longrightarrow \mathbf{Set}$$

preserves filtered colimits. This functor is the same as

$$\mathbf{Set}^{\mathbf{I}^{op}} /_{\Phi} (1_\Phi, -) : \mathbf{Set}^{\mathbf{I}^{op}} /_{\Phi} \longrightarrow \mathbf{Set}$$

which can be calculated as a pullback

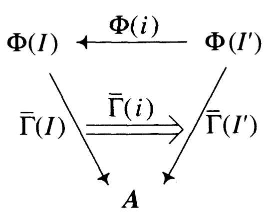
$$\begin{array}{ccc}
\mathbf{Set}^{\mathbf{I}^{op}}/_{\Phi}(1_{\Phi},-) & \longrightarrow & \mathbf{Set}^{\mathbf{I}^{op}}(\Phi,\Sigma_{\Phi}(-)) \\
\downarrow & & \downarrow \\
1 & \xrightarrow[\ulcorner 1_{\Phi} \urcorner]{} & \mathbf{Set}^{\mathbf{I}^{op}}(\Phi,\Phi)
\end{array}$$

where the two bottom functors are constant and $\Sigma_{\Phi} : \mathbf{Set}^{\mathbf{I}^{op}}/_{\Phi} \longrightarrow \mathbf{Set}^{\mathbf{I}^{op}}$ is the forgetful functor. Since Σ_{Φ} preserves all colimits and Φ is finitely presentable, the top right-hand functor preserves filtered colimits. Constant functors also preserve filtered colimits, and pullbacks of filtered colimits preserving functors are also filtered colimit preserving. The result follows. □

We end this section with a closely related fact.

PROPOSITION 10. *Let* $\Phi : \mathbf{I}^{op} \longrightarrow \mathbf{Cat}$ *be a pseudo-functor and* $\mathbf{J} \longrightarrow \mathbf{I}$ *the fibration associated to it by the Grothendieck construction. If* $\mathbf{I}$ *and each* $\Phi(I)$ *are connected and simply connected (and L-finite) then so is* $\mathbf{J}$.

Proof. $\mathbf{J}$ is the lax-colimit of Φ, which means that a diagram $\Gamma : \mathbf{J} \longrightarrow \mathbf{A}$ is the same as an $\mathbf{I}$ diagram of diagrams $\bar{\Gamma} : \mathbf{I} \longrightarrow \mathrm{Diag}(\mathbf{A})$, where $\bar{\Gamma}(I) : \Phi(I) \longrightarrow \mathbf{A}$ and $\bar{\Gamma}(i : I \longrightarrow I')$ is of the form



.

A triangle such as this induces a morphism $\varprojlim \bar{\Gamma}(I) \longrightarrow \varprojlim \bar{\Gamma}(I')$, and a calculation shows that

$$\varprojlim_{J \in \mathbf{J}} \Gamma(J) \cong \varprojlim_{I \in \mathbf{I}} \ \varprojlim_{J \in \Phi I} \bar{\Gamma}(I)(J).$$

Under the hypotheses of the proposition, theorem 1 (resp. theorem 2) says that the right-hand side exists in any $\mathbf{A}$ with fibred products (resp. pullbacks), so the left-hand side does too. The result now follows from another application of theorem 1 (resp. theorem 2). □

REFERENCES

1. J. Bénabou, *Introduction to bicategories*, Lecture Notes in Math., no. 47 (1967), 1–77, Springer-Verlag.
2. P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, (Lecture Notes in Math., no. 221, Springer-Verlag, 1971).

3. D. M. Kan, *Adjoint functors*, Trans. Amer. Math. Soc., *87* (1958), 294–329.
4. G. M. Kelly and R. Paré, *A note on the Albert-Kelly paper "The closure of a class of colimits"*, J. Pure Appl. Algebra *51* (1988), 19–25.
5. J. Kennison, *Semi-geometric Functors and Limits of Topoi*, preprint.
6. A. Kock and G. Wraith, *Elementary toposes*, (Aarhus Lecture Notes 30 (1971)).
7. M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, (Contemporary Mathematics, vol. 104, AMS, Providence, 1989).
8. R. Paré, *Connected Components and Colimits*, J. Pure Appl. Algebra *3* (1973), 21–42.
9. R. Rosebrugh and R. Wood, *Pullback preserving functors*, preprint.
10. R. H. Street, *The comprehensive construction of free colimits*, Sydney Category Seminar Reports (Macquarie University, 1979).

Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax, N.S.
B3H 3J5