MATH/CSCI 2113, DISCRETE STRUCTURES II, Winter 2010

Handout 4: Problems on Recurrences
Wednesday, February 10, 2010

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Problem 1. For each of the following linear recurrences, find the first 6 elements. Then find a closed formula.

(a) $A_n = 3A_{n-1} + 4A_{n-2}$, with $A_0 = 1$ and $A_1 = 2$.

(b) $B_n = 4B_{n-1} - 4B_{n-2}$, with $B_0 = 1$ and $B_1 = 1$.

Problem 2. Sophie has a supply of colored blocks. Red blocks are 1in high, and blue and green blocks are 2in high. (a) In how many different ways can you build a tower of height 3in? 4in? 5in? Example: RBGR is a tower of height 6in. (b) Let C_n be the number of ways of building a tower of height n inches. Find a recurrence formula for C_n. Hint: do a case distinction on the bottom-most block. Don't forget specifying the initial values. (c) Find a closed formula for C_n.

Problem 3. Using the method of generating functions, solve the following recurrences:

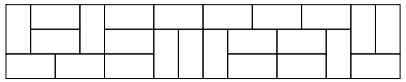
(a) $a_{n+1} = 3a_n + 1$, for $n \geqslant 0$, with $a_0 = 0$.

(b) $a_{n+1} = 3a_n + 1$, for $n \geqslant 0$, with $a_0 = 1$.

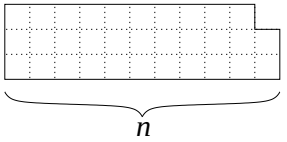
(c) $a_{n+2} = 2a_{n+1} - a_n$, for $n \geqslant 0$, with $a_0 = 0$ and $a_1 = 1$.

Problem 4. Consider strings in the alphabet $\{A, B, C\}$. Let us call such a string "legal" if it does not contain two consecutive A's. Let b_n be the number of legal strings of length n. (a) Find a recursive definition of b_n. (b) Find a closed formula. (c) How many strings of length 8 are legal?

Problem 5. Consider the problem of tiling a rectangle of size $3 \times n$ with dominoes of size 2×1, for example as in the following picture:



Let A_n be the number of such tilings for a rectangle of size $3 \times n$. Also, let B_n be the number of tilings of the following figure:



(a) Find a recurrence expressing A_n in terms of B_{n-1} and A_{n-2}, for $n \geqslant 2$. (b) Find a recurrence expressing B_n in terms of A_{n-1} and B_{n-2}. (c) Find the initial values for this recurrence. (d) How many ways are there to tile a 3×10 rectangle? (e) Solve the recurrence for A_n. (f) Extra challenge: Can you do a similar recurrence for the problem of tiling rectangles of size $4 \times n$ with dominoes? Note: there will be more than two shapes to consider.

Problem 6. For each of the following recurrence, find the generating function and a closed formula for the sequence.

(a) $a_{n+1} = 2a_n + n \quad (n \geqslant 0; a_0 = 1)$.

(b) $a_{n+2} = a_{n+1} + 3a_n \quad (n \geqslant 0; a_0 = 1, a_1 = 2)$.

(c) $a_{n+1} = a_n + n^2 \quad (n \geqslant 0; a_0 = 0)$.