Math 4680, Topics in Logic and Computation, Winter 2012

Lecture Notes 4: Soundness, Completeness, and Consequences for First-Order Logic

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1 Substitution

We write $t_1[t/x]$ for the result of substituting the term t for the variable x in the term t_1, and $\varphi[t/x]$ for the result of substituting t for x in the formula φ. Here, only *free* occurrences of x are substituted. More precisely, substitution is defined recursively as follows. On terms:

$$\begin{array}{lcll} x[t/x] & = & t & \\ y[t/x] & = & y & \text{if } x, y \text{ are different variables} \\ f(t_1,\ldots,t_n)[t/x] & = & f(t_1[t/x],\ldots,t_n[t/x]) & \end{array}$$

On formulas:

$$\begin{array}{llcll} 1. & P(t_1,\ldots,t_n)[t/x] & = & P(t_1[t/x],\ldots,t_n[t/x]) & \\ & (t_1 \approx t_2)[t/x] & = & t_1[t/x] \approx t_2[t/x] & \\ 2. & (\neg\varphi)[t/x] & = & \neg(\varphi[t/x]) & \\ & (\varphi \Box \psi)[t/x] & = & (\varphi[t/x]) \Box (\psi[t/x]) & \\ 3. & (\forall x.\varphi)[t/x] & = & \forall x.\varphi & \\ & (\forall y.\varphi)[t/x] & = & \forall y.(\varphi[t/x]) & \text{if } x, y \text{ are different variables} \\ & (\exists x.\varphi)[t/x] & = & \exists x.\varphi & \\ & (\exists y.\varphi)[t/x] & = & \exists y.(\varphi[t/x]) & \text{if } x, y \text{ are different variables} \end{array}$$

Substitution is a more subtle notion than meets the eye. In particular, one has to be careful that t does not contain any free variables which get captured when t is substituted into some formula. Consider the formula $\exists y.x \not\approx y$. In a structure with two or more elements, this statement is true for any x. On the other hand, if we substitute y for x, we obtain $\exists y.y \not\approx y$, which is false! We want to rule out situations like this.

We say that ***t is substitutable for x in φ*** if we can substitute t for x in φ without worrying about free variables of t intruding the scope of quantifiers in φ. More precisely, this concept is defined by recursion on φ:

1. If φ is atomic, then t is always substitutable for x in φ.

2. t is substitutable for x in $\neg\varphi$ iff t is substitutable for x in φ.

   t is substitutable for x in $\varphi \Box \psi$ iff t is substitutable for x in φ and t is substitutable for x in ψ.

3. t is substitutable for x in $\forall y.\varphi$ iff either

   (a) x is not free in $\forall y.\varphi$ or

   (b) y is not free in t and t is substitutable for x in φ.

   t is substitutable for x in $\exists y.\varphi$ iff either

   (a) x is not free in $\exists y.\varphi$ or

   (b) y is not free in t and t is substitutable for x in φ.

Convention. From now on, whenever we write $\varphi[t/x]$, it is always implicitly assumed that t is substitutable for x in φ. If t is not substitutable for x in φ, then we implicitly rename the bound variables in φ so that t becomes substitutable for x in φ.

In the proofs of soundness and completeness, we often need to relate substitutions to the interpretation of the involved terms in a structure. The following lemma provides the necessary facts.

Lemma 1 (Substitution Lemma)**.** *Suppose $\mathfrak{A}$ is a structure and s is a valuation. Suppose t a term, x a variable, and $\bar{s}(t) = a$. Let $s' = s(a|x)$. Then*

1. $\bar{s}(t_1[t/x]) = \bar{s'}(t_1)$ *for all terms* t_1.
2. $\models_{\mathfrak{A}} \varphi[t/x]\,[s]$ *iff* $\models_{\mathfrak{A}} \varphi\,[s']$, *for all formulas φ such that t is substitutable for x in φ.*

Proof. By induction on terms and formulas. □

2 Natural Deduction

The natural deduction rules for first-order logic are those for sentential logic, plus the rules given below. Note that since we are already using lower-case roman letters for variables, we are now using numbers to identify canceled hypotheses. Also, in the rules for quantifiers, whenever we write $\varphi[t/x]$, it is always implicitly assumed that t is substitutable for x in φ. Note the side conditions in the $(\forall\mathcal{I})$ and $(\exists\mathcal{E})$ rules. These conditions ensure that we have not made any unwarranted assumptions about the variable a.

Rules for quantifiers:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \varphi[a/x]\end{matrix}}{\forall x.\varphi}\ (\forall\mathcal{I}) \qquad \text{if } a \text{ is a variable not free in } \Gamma \text{ or } \varphi$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \forall x.\varphi\end{matrix}}{\varphi[t/x]}\ (\forall\mathcal{E}) \qquad \text{if } t \text{ is a term}$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \varphi\end{matrix}}{\exists x.\varphi}\ (\exists\mathcal{I}) \qquad \text{if } t \text{ is a term}$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \exists x.\varphi\end{matrix} \qquad \begin{matrix}\Gamma, [\varphi[a/x]]_1\\ \vdots\\ \psi\end{matrix}}{\psi}\ {}_1\,(\exists\mathcal{E}) \qquad \text{if } a \text{ is a variable not free in } \Gamma, \varphi, \text{ or } \psi$$

Rules for equality:

$$\dfrac{}{t \approx t}\ (\mathit{refl}) \qquad \dfrac{s \approx t}{t \approx s}\ (\mathit{symm}) \qquad \dfrac{r \approx s \qquad s \approx t}{r \approx t}\ (\mathit{trans})$$

$$\dfrac{s \approx s'}{t[s/x] \approx t[s'/x]}\ (\mathit{cong}_1) \qquad \dfrac{s \approx s' \qquad \varphi[s/x]}{\varphi[s'/x]}\ (\mathit{cong}_2)$$

As before, we write $\Gamma \vdash \varphi$ if there is a natural deduction derivation, all of whose uncanceled hypotheses are in Γ, and whose conclusion is φ.

3 Soundness and Completeness

Theorem 2 (Soundness and Completeness)**.** *If Γ is a set of formulas, and φ is a formula, then*

$$\Gamma \vdash \varphi \qquad \textit{iff} \qquad \Gamma \models \varphi.$$

*The left-to-right implication is called **soundness**, and the right-to-left implication is called **completeness**.*

Proof. Soundness is proved by induction on the size of derivations, and by a case distinction on what the last rule in the derivation is. The substitution lemma is needed in the cases of the quantifier rules.

For the proof of completeness, see e.g. van Dalen's book [**?**]. □

4 Compactness and consequences

The following theorem is a trivial consequence of the soundness and completeness theorem, but it has many interesting and surprising applications. Recall that a set of formulas is called ***satisfiable*** if there exists a structure and a valuation that makes all formulas in the set true.

Theorem 3 (Compactness)**.** *Let Γ be a set of formulas. If every finite subset of Γ is satisfiable, then Γ is satisfiable.*

Proof. We prove the contrapositive. Suppose Γ is not satisfiable. Then $\Gamma \models \bot$. By completeness, $\Gamma \vdash \bot$. But natural deductions are finite, hence any deduction can only use finitely many hypotheses. It follows that $\Gamma' \vdash \bot$ for some finite $\Gamma' \subseteq \Gamma$. By soundness, $\Gamma' \models \bot$, and thus Γ' is not satisfiable, as desired. □

Several applications of the compactness theorem are demonstrated in the exercises of Problem Set 9. Here are some more examples of such applications:

Theorem 4. *Suppose Σ is a set of sentences. If Σ has arbitrarily large finite models, then it has an infinite model.*

Proof. Suppose Σ has arbitrarily large finite models. For every $n \in \mathbb{N}$, let λ_n be the sentence that states "there are at least n distinct object". Notice that λ_n is first-order definable, for instance

$$\lambda_3 = \exists x \exists y \exists z (x \not\approx y \wedge x \not\approx z \wedge y \not\approx z).$$

Consider the set of sentences $\Phi = \Sigma \cup \{\lambda_n \mid n \in \mathbb{N}\}$. Since Σ has arbitrarily large finite models, every finite subset of Φ has a model. By compactness, Φ has a model. But any model of Φ is infinite, and it is also a model of Σ. Thus, Σ has an infinite model. □

Recall that a class K of structures is called ***axiomatizable*** if $K = \mathrm{Mod}(\Sigma)$, for some set of sentences Σ. Also, K is called ***finitely axiomatizable*** if $K = \mathrm{Mod}(\sigma_1, \ldots, \sigma_n)$ for finitely many sentences $\sigma_1, \ldots, \sigma_n$.

Theorem 5. *The class of all infinite structures is axiomatizable, but not finitely axiomatizable.*

Proof. Let K be the class of infinite structures. The set $\{\lambda_n \mid n \in \mathbb{N}\}$ axiomatizes K. Suppose, on the other hand, that K was finitely axiomatizable. Then there exist sentences $\sigma_1, \ldots, \sigma_n$ such that $K = \mathrm{Mod}(\sigma_1, \ldots, \sigma_n)$. Let $\sigma = \sigma_1 \wedge \ldots \wedge \sigma_n$, then $K = \mathrm{Mod}(\sigma)$. Thus, a structure $\mathfrak{A}$ is infinite iff $\models_{\mathfrak{A}} \sigma$. Equivalently, a structure $\mathfrak{A}$ is finite iff $\models_{\mathfrak{A}} \neg\sigma$. But then the class of finite structures would be axiomatizable, contradicting Theorem 4. □

The following theorem is often useful in proving that a certain class of structures is *not* finitely axiomatizable:

Theorem 6. *If K is a finitely axiomatizable class of structures, and if $K = \mathrm{Mod}(\Sigma)$, then there exists a finite subset $\Sigma' \subseteq \Sigma$ such that $K = \mathrm{Mod}(\Sigma')$.*

Proof. By assumption, K is finitely axiomatizable. Let $\tau_1, \ldots, \tau_n$ be sentences such that $K = \mathrm{Mod}(\tau_1, \ldots, \tau_n)$. Then $K = \mathrm{Mod}(\tau)$, where $\tau = \tau_1 \wedge \ldots \wedge \tau_n$. Now every model of Σ is in the class K, and hence satisfies τ. It follows that the set $\Sigma \cup \{\neg\tau\}$ is unsatisfiable. By compactness, there exists a finite subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' \cup \{\neg\tau\}$ is unsatisfiable. This means that every model of Σ' is not a model of $\neg\tau$, or in other words, every model of Σ' is a model of τ. Also, every model of Σ is certainly a model of Σ'. We thus have $K = \mathrm{Mod}(\Sigma) \subseteq \mathrm{Mod}(\Sigma') \subseteq \mathrm{Mod}(\tau) = K$. It follows that $K = \mathrm{Mod}(\Sigma')$ as desired. □

If K is a class of structures, let us write K^c for the complement of the class. That is, a structure $\mathfrak{A}$ is in K^c iff it is not in K.

Theorem 7. *A class K of structures is finitely axiomatizable if and only if both K and K^c are axiomatizable.*

Proof. "⇒": Suppose K is finitely axiomatizable. Then surely K is axiomatizable. To show that K^c is axiomatizable, let $K = \mathrm{Mod}(\sigma_1, \ldots, \sigma_n)$. Let $\sigma = \sigma_1 \wedge \ldots \wedge \sigma_n$. Then $\mathfrak{A} \in K$ iff $\models_{\mathfrak{A}} \sigma$. Consequently $\mathfrak{A} \in K^c$ iff $\not\models_{\mathfrak{A}} \sigma$, iff $\models_{\mathfrak{A}} \neg\sigma$. Thus, $K^c = \mathrm{Mod}(\neg\sigma)$.

"⇐": Suppose both K and K^c are axiomatizable. Let $K = \mathrm{Mod}(\Sigma)$ and $K^c = \mathrm{Mod}(\Gamma)$. Since no structure is in K and K^c, the set $\Sigma \cup \Gamma$ is unsatisfiable. By compactness, there exists a finite subset $\Sigma' \cup \Gamma'$ which is unsatisfiable. Clearly every model of Σ is a model of Σ'. Conversely, let $\mathfrak{A}$ be a model of Σ'. Then $\mathfrak{A}$ does not satisfy Γ', and hence not Γ. Thus $\mathfrak{A} \notin K^c$, thus $\mathfrak{A} \in K$. We have:

$$K = \mathrm{Mod}(\Sigma) \subseteq \mathrm{Mod}(\Sigma') \subseteq K,$$

and hence $K = \mathrm{Mod}(\Sigma')$. Thus K is finitely axiomatizable, as desired. □

5 Size of models

The ***cardinality*** of a set is the number of elements in the set. Different infinite sets can have different cardinalities; for instance, the set of natural numbers has a smaller cardinality than the set of real numbers. We say the cardinality of a structure $\mathfrak{A}$ is the cardinality of its carrier $|\mathfrak{A}|$. The cardinality of a language L is the cardinality of L, considered as a set of sentences.

Remark. If $\mathcal{P}$ and $\mathcal{F}$ are the sets of predicate symbols, respectively function symbols, of the language L, then the cardinality of L is $\kappa = \max(\mathrm{card}\,\mathcal{P} \cup \mathcal{F}, \aleph_0)$. Here, $\aleph_0$ is the cardinality of a countable set.

To see why this is true, first, notice that the alphabet $\mathcal{A}$ of L consists of the symbols from $\mathcal{P}$ and $\mathcal{F}$, finitely many special symbols such as parentheses and logical connectives, and countably many variables. Thus, the cardinality of $\mathcal{A}$ is κ. Let $\mathcal{A}^*$ be the set of finite strings in the alphabet $\mathcal{A}$. One can regard these strings as finite tuples, thus $\mathcal{A}^* = \{\epsilon\} \cup \mathcal{A} \cup \mathcal{A} \times \mathcal{A} \cup \mathcal{A}^3 \cup \mathcal{A}^4 \cup \ldots$. Here ϵ is the empty string. But notice that the cardinality of each $\mathcal{A}^n$ is the same as the cardinality of $\mathcal{A}$, when $n \geqslant 1$. Thus the cardinality of $\mathcal{A}^*$ is at most $\mathcal{A} \times \aleph_0$, which is in turns the cardinality of $\mathcal{A}$. Since $L \subseteq \mathcal{A}^*$, it follows that $\mathrm{card}\, L \leqslant \mathrm{card}\, \mathcal{A}^* \leqslant \mathrm{card}\, \mathcal{A}$. On the other hand, clearly $\mathrm{card}\, \mathcal{A} \leqslant \mathrm{card}\, L$. Thus it follows that L has the same cardinality as its alphabet $\mathcal{A}$.

Theorem 8 (Löwenheim-Skolem-Tarski)**.** *Let Γ is a satisfiable set of formulas in a language of cardinality κ. Then*

1. *Γ is satisfiable in some structure of cardinality $\leqslant \kappa$.*

2. *If Γ is satisfiable in some infinite structure, then for every cardinality $\lambda \geqslant \kappa$, there exists a structure of cardinality λ in which Γ is satisfiable.*

Proof. 1. This follows from the proof of the completeness theorem. In the proof of the completeness theorem, we proceeded as follows: First, we replace all free variables in Γ by new constants, to obtain a set of sentences, which we close under derivability to obtain a theory T. The language of T contains at most countably many new constants, so it has the same cardinality as the language of Γ. Let L be the language of T. Next, we constructed a Henkin theory T_ω by adding a constant symbol for each existential sentence of L, countably many times. The resulting language L_ω still has the same cardinality as L. We defined A to be the set of closed terms of L_ω. Clearly, the cardinality of A is at most that of L_ω. Finally, we constructed a structure $\mathfrak{A}$ in which T, thus Γ, is satisfiable. We let the carrier $|\mathfrak{A}|$ be a certain quotient of A. Thus, $\operatorname{card}|\mathfrak{A}| \leqslant \operatorname{card} A \leqslant \operatorname{card} L_\omega = \operatorname{card} L = \kappa$.

2. Suppose now that Γ is satisfiable in some infinite structure. Let L be the language of Γ. Let $\lambda \geqslant \kappa$ be a cardinal. Consider the language L' obtained from L by adding λ many new constant symbols $\{\mathbf{c}_x \mid x \in \lambda\}$. Consider the set of formulas

$$\Phi = \Gamma \cup \{\mathbf{c}_x \not\approx \mathbf{c}_y \mid x \neq y \in \lambda\}.$$

Notice that since Γ is satisfiable in some infinite structure $\mathfrak{A}$, every finite subset Φ' of Φ is also satisfiable, namely by mapping the finitely many $\mathbf{c}_x$ that are mentioned in Φ' to different elements of $\mathfrak{A}$. By compactness, it follows that Φ is satisfiable. By part 1., Φ is satisfiable in some structure $\mathfrak{B}$ of cardinality $\leqslant \lambda$ (notice that λ is the cardinality of the language L'). On the other hand, since $\mathfrak{B}$ is a model of $\mathbf{c}_x \not\approx \mathbf{c}_y$, for any distinct $x, y \in \lambda$, $\mathfrak{B}$ has cardinality at least λ. It follows that the cardinality of $\mathfrak{B}$ is exactly λ. Further, Γ is satisfiable in $\mathfrak{B}$. $\square$

Recall that two structures $\mathfrak{A}$ and $\mathfrak{B}$ are called ***elementarily equivalent*** if $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$. Concretely, this means that $\mathfrak{A}$ and $\mathfrak{B}$ make precisely the same sentences true. If $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, we write $\mathfrak{A} \equiv \mathfrak{B}$.

Corollary 9. *(a) Let Σ be a set of sentences in a countable language. If Σ has an infinite model, then Σ has models of every infinite cardinality.*

(b) Let $\mathfrak{A}$ be an infinite structure for a language of cardinality κ. Then for any infinite cardinal $\lambda \geqslant \kappa$, there is a structure $\mathfrak{B}$ of cardinality λ such that $\mathfrak{B} \equiv \mathfrak{A}$.

Proof. (a) Take $\Gamma = \Sigma$ and $\kappa = \aleph_0$ in Theorem 8(2). (b) Take $\Gamma = \mathrm{Th}(\mathfrak{A})$ in Theorem 8(2) to obtain a model $\mathfrak{B}$ of $\mathrm{Th}(\mathfrak{A})$ of cardinality λ. Then $\mathrm{Th}(\mathfrak{A}) \subseteq \mathrm{Th}(\mathfrak{B})$. On the other hand, if σ is some sentence that is true in $\mathfrak{B}$, then $\neg\sigma$ is not true in $\mathfrak{B}$, thus $\neg\sigma$ is not true in $\mathfrak{A}$, hence σ is true in $\mathfrak{A}$. If follows that $\mathrm{Th}(\mathfrak{B}) \subseteq \mathrm{Th}(\mathfrak{A})$. Hence $\mathfrak{B} \equiv \mathfrak{A}$. $\square$

Note that the preceding theorem and corollary are surprising. They imply, for instance, that there is an uncountable structure which satisfies precisely the same first-order sentences as the natural numbers. On the other hand, there is some countable structure which is elementarily equivalent to the reals.

6 Complete and κ-categorical theories

Recall that a set of sentences is called a ***theory*** if for all sentences σ, $T \vdash \sigma$ implies $\sigma \in T$. Also recall that the theory $\mathrm{Th}(\mathfrak{A})$ of a structure $\mathfrak{A}$ is the set of sentences that are satisfied in $\mathfrak{A}$. (It follows from soundness that this is indeed a theory). Further, if K is a class of structures, then $\mathrm{Th}(K)$ is defined to be the set of sentences that are satisfied in *all* structures in K.

Definition. A theory T is ***complete*** if for every sentence σ, either $\sigma \in T$ or $\neg\sigma \in T$.

Lemma 10. *1. If $T \subseteq T'$ and T is complete and T' is consistent, then $T = T'$.*

2. A theory is complete iff it is maximally consistent.

3. For any structure $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A})$ is complete.

4. Suppose K is a non-empty class of structures. Then $\mathrm{Th}(K)$ is complete iff for all $\mathfrak{A}, \mathfrak{B} \in K$, $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. 1. Suppose $T \subseteq T'$ and T is complete and T' is consistent. Suppose there was some sentence $\sigma \in T'$ such that $\sigma \notin T$. Then $\neg\sigma \in T$ since T is complete. Since $T \subseteq T'$, it follows that $\neg\sigma \in T'$. But then $\sigma, \neg\sigma \in T'$, which implies that T' is inconsistent, a contradiction. Hence $T = T'$.

2. Left-to-right. Suppose T is complete. Then it is maximally consistent by 1. Right-to-left: Suppose T is maximally consistent. Suppose $\sigma \notin T$. Then $T \cup \{\sigma\}$ is inconsistent by maximality of T. Hence $T, \sigma \vdash \bot$, and thus $T \vdash \neg\sigma$ by the $(\neg\mathcal{I})$ rule. Since T is a theory, it follows that $\neg\sigma \in T$. Hence T is complete.

3. This is trivial. For any sentence σ, either $\models_{\mathfrak{A}} \sigma$ or $\models_{\mathfrak{A}} \neg\sigma$, by definition of $\models$. Thus $\sigma \in \mathrm{Th}(\mathfrak{A})$ or $\neg\sigma \in \mathrm{Th}(\mathfrak{A})$.

4. Left-to-right: Suppose $\mathrm{Th}(K)$ is complete. Consider any $\mathfrak{A} \in K$. Then $\mathrm{Th}(K) \subseteq \mathrm{Th}(\mathfrak{A})$. But $\mathrm{Th}(K)$ is complete and $\mathrm{Th}(\mathfrak{A})$ is consistent, hence $\mathrm{Th}(K) = \mathrm{Th}(\mathfrak{A})$ by 2. Similarly $\mathrm{Th}(K) = \mathrm{Th}(\mathfrak{B})$ for any $\mathfrak{B} \in K$, hence $\mathfrak{A} \equiv \mathfrak{B}$. Right-to-left: Suppose $\mathfrak{A} \equiv \mathfrak{B}$ for all $\mathfrak{A}, \mathfrak{B} \in K$. Pick some $\mathfrak{A} \in K$. Then $\mathrm{Th}(K) = \mathrm{Th}(\mathfrak{A})$. But $\mathrm{Th}(\mathfrak{A})$ is complete by 3. □

One useful fact about complete theories is that they are often decidable.

Theorem 11. *Suppose T is a theory with an axiom set Σ that can be effectively listed by an algorithm. If T is complete, then T is decidable.*

Proof. Essentially, the decicion procedure for T is the following: Suppose you want to decide whether a given sentence σ is in T. systematically enumerate all the valid natural deductions whose hypotheses are among Σ. Since T is complete, eventually either σ or $\neg\sigma$ appears as the conclusion of one of these deductions. Depending on which is the case, the procedure will output "yes" or "no". This is always guaranteed to happen after a finite amount of time. □

The following test is sometimes useful for proving that certain theories are complete. If κ is a cardinality, then we say that a theory T is ***κ-categorical*** if all models of T of cardinality κ are isomorphic.

Theorem 12 (Łoś-Vaught Test)**.** *Suppose T only has infinite models, and T is κ-categorical for some κ not less than the cardinality of L. Then T is complete.*

Proof. Suppose T is not complete. Then there exists a sentence σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$. By completeness, there exist models $\mathfrak{A}$ and $\mathfrak{B}$ of T such that $\not\models_{\mathfrak{A}} \sigma$ and $\not\models_{\mathfrak{B}} \neg\sigma$. In other words, $\models_{\mathfrak{A}} \neg\sigma$ and $\models_{\mathfrak{B}} \sigma$. $\mathfrak{A}$ and $\mathfrak{B}$ are infinite by assumption. By Corollarycor-LST, there exist structures $\mathfrak{A}'$ and $\mathfrak{B}'$ of cardinality κ which are elementarily equivalent to $\mathfrak{A}$, respectively $\mathfrak{B}$. Thus $\models_{\mathfrak{A}'} \neg\sigma$ and $\models_{\mathfrak{B}'} \sigma$. Since both $\mathfrak{A}'$ and $\mathfrak{B}'$ are models of T, this contradicts the fact that T is κ-categorical.

Applications:

Example 13. We proved in class that any two countable dense linear orders without endpoints are isomorphic. In other words, the theory T of countable dense linear orders without endpoints is $\aleph_0$-categorical. Also, T has no finite models. It follows that T is complete.

Example 14. It is a theorem in algebra that two algebraically closed fields are isomorphic if they have the same characteristic and the same transcendence degree. It follows that any two algebraically closed fields of characteristic 0 are isomorphic if they have the same cardinality. In our terminology, the theory of algebraically closed fields of characteristic 0 is κ-categorical for any uncountable cardinal κ. Also, this theory has no finite models. Hence it is complete by the Łoś-Vaught Test. One consequence of this fact is that any two such fields are elementarily equivalent. Thus, any sentence that is true for the complex numbers is true in every algebraically closed field of characteristic 0. Another consequence of completeness is that the theory of the complex numbers is decidable. This means, for any first-order statement about the complex numbers, there is a decision procedure which decides whether the statement is true or false.

A decision procedure for the first-order theory of complex numbers is a very powerful tool to have. However, this does not mean that we can decide *any* statement about the complex numbers. Only *first-order* statements are affected. There are many interesting statements about the complex numbers that are not expressible in first-order, for instance, any statements that refer to arbitrary subsets of the complex numbers.