Math 4680, Topics in Logic and Computation, Winter 2012

Lecture Notes 5: Fitch style natural deduction

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1 Prawitz style vs. Fitch style

In previous lecture notes, we have used the "Prawitz style" presentation of natural deduction: a deduction is a certain kind of tree whose leaves represent hypotheses, and whose root represents a conclusion. These trees are complicated by two facts: first, sometimes hypotheses are discharged ("crossed out") during a proof, and second, the freshness conditions in the quantifier rules require that certain variables do not occur freely in the hypothesis and conclusions of certain subtrees, *excluding* hypotheses that have "already" been discharged at the time the quantifier rule is applied. There is also a certain practical disadvantage to writing proofs as trees: large proofs tend to be much wider than they are high, so one quickly runs out of space.

Here, we briefly describe an alternative notation for natural deduction derivations, called the "Fitch style" notation. It is more linear, in the sense that a proof is essentially a list of formulas, one on each line, and formulas on later lines are meant to be consequences of formulas on earlier lines. Instead of crossing out hypotheses, the Fitch style notation uses indentation to indicate a subderivation using a temporary hypothesis.

2 First examples

In its simplest form, a Fitch style natural deduction is just a list of numbered lines, each containing a formula, such that each formula is either a hypothesis (separated from the rest of the proof by a horizontal line), or else follows from previous formulas (indicated by a rule name and line numbers of relevant formulas). The very last line in the derivation contains the conclusion.

Here is an example of a derivation of $A \to B, B \to C, A \vdash C$:

$$\begin{array}{r|ll}
1 & A \to B & \\
2 & B \to C & \\
3 & A & \\ \cline{2-2}
4 & B & \to\text{E, 1, 2} \\
5 & C & \to\text{E, 2, 4}
\end{array}$$

Note that lines 1–3 contain hypotheses; each subsequent line is justified by a rule.

Sometimes a derivation contains a subderivation that depends on a hypothetical, or temporary, assumption. Such subderivations are indented and marked with another vertical line. For example, here is a derivation of $A \to B, B \to C \vdash A \vdash C$:

$$\begin{array}{r|l|ll}
1 & \multicolumn{2}{l}{A \to B} & \\
2 & \multicolumn{2}{l}{B \to C} & \\ \cline{2-3}
3 & & A & \\ \cline{3-3}
4 & & B & \to\text{E, 1, 2} \\
5 & & C & \to\text{E, 2, 4} \\
6 & \multicolumn{2}{l}{A \to C} & \to\text{I, 3–5}
\end{array}$$

On line 3, we "temporarily" assume A. Lines 4 and 5 are consequences. The subderivation ends on line 5 with conclusion C; therefore, one has proved $A \to C$ at the "next level up" on line 6.

3 The rules of Fitch style natural deduction

Conjunction introduction (∧I)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & B & \\ \vdots & \vdots & \\ p & A \wedge B & \wedge\mathrm{I}, m, n \end{array} \qquad \begin{array}{l|ll} m & B & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & A \wedge B & \wedge\mathrm{I}, m, n \end{array}$$

Conjunction elimination (∧E)

$$\begin{array}{l|ll} m & A \wedge B & \\ \vdots & \vdots & \\ n & A & \wedge\mathrm{E}, m \end{array} \qquad \begin{array}{l|ll} m & A \wedge B & \\ \vdots & \vdots & \\ n & B & \wedge\mathrm{E}, m \end{array}$$

Disjunction Introduction (∨I)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & A \vee B & \vee\mathrm{I}, m \end{array} \qquad \begin{array}{l|ll} m & B & \\ \vdots & \vdots & \\ n & A \vee B & \vee\mathrm{I}, m \end{array}$$

Disjunction Elimination (∨E)

$$\begin{array}{l|ll} m & A \vee B & \\ \vdots & \vdots & \\ n & \quad \mid \underline{A} & \\ \vdots & \quad \mid \vdots & \\ p & \quad \mid \varphi & \\ q & \quad \mid \underline{B} & \\ \vdots & \quad \mid \vdots & \\ r & \quad \mid \varphi & \\ s & \varphi & \vee\mathrm{E}, m, n\text{–}p, q\text{–}r \end{array}$$

Implication Introduction (→I)

$$\begin{array}{l|ll} m & \mid \underline{A} & \\ \vdots & \mid \vdots & \\ n & \mid B & \\ n+1 & A \to B & \to\mathrm{I}, m\text{–}n \end{array}$$

Implication Elimination (→E)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & A \to B & \\ \vdots & \vdots & \\ p & B & \to\mathrm{E}, m, n \end{array} \qquad \begin{array}{l|ll} m & A \to B & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & B & \to\mathrm{E}, m, n \end{array}$$

Negation Introduction (¬I)

$$\begin{array}{l|ll} m & \mid \underline{A} & \\ \vdots & \mid \vdots & \\ n & \mid \bot & \\ n+1 & \neg A & \neg\mathrm{I}, m\text{–}n \end{array}$$

Negation Elimination (¬E)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ n & \neg A & \\ \vdots & \vdots & \\ p & \bot & \neg\mathrm{E}, m, n \end{array} \qquad \begin{array}{l|ll} m & \neg A & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & \bot & \neg\mathrm{E}, m, n \end{array}$$

Contradiction Elimination (⊥E)

$$\begin{array}{l|ll} m & \bot & \\ \vdots & \vdots & \\ n & C & \bot\mathrm{E}, m \end{array}$$

Proof by Contradiction (C)

$$\begin{array}{l|ll} m & \mid \underline{\neg A} & \\ \vdots & \mid \vdots & \\ n & \mid \bot & \\ n+1 & A & \mathrm{C}, m\text{–}n \end{array}$$

Repetition (R)

$$\begin{array}{l|ll} m & A & \\ \vdots & \vdots & \\ \vdots & \cdots \mid \vdots & \\ n & \quad \mid A & \mathrm{R}, m \end{array}$$

Forall-introduction (∀I)

$$\begin{array}{l|ll} m & u \mid \vdots & \\ \vdots & \quad \mid \vdots & \\ n & \quad \mid A[u/x] & \\ n+1 & \forall x\, A & \forall\mathrm{I}, m\text{–}n \end{array}$$

Forall-elimination (∀E)

$$\begin{array}{l|ll} m & \forall x\, A & \\ \vdots & \vdots & \\ n & A[t/x] & \forall\mathrm{E}, m \end{array}$$

Exists-Introduction (∃I)

$$\begin{array}{l|ll} m & A[t/x] & \\ \vdots & \vdots & \\ n & \exists x\, A & \exists\mathrm{I}, m \end{array}$$

Exists-Elimination (∃E)

$$\begin{array}{l|ll} p & \exists x\, A & \\ \vdots & \vdots & \\ m & u \mid \underline{A[u/x]} & \\ \vdots & \quad \mid \vdots & \\ n & \quad \mid \varphi & \\ n+1 & \varphi & \exists\mathrm{E}, p, m\text{–}n \end{array}$$

4 Remarks

The biconditional ($\leftrightarrow$)

To simplify our formal proof system, we do not introduce any special rules for the connective $\leftrightarrow$. Instead, we simply regard the formula $A \leftrightarrow B$ as an *abbreviation* for $(A \to B) \wedge (B \to A)$.

Falsity ($\bot$)

The symbol $\bot$ stands for "contradiction" or "falsity". The formula $\bot$ is always false, and it is used in the rules for negation and contradiction above.

Negation ($\neg$)

As we have done before, it is possible to regard negation $\neg A$ as an abbreviation for $A \to \bot$. In this case, the negation introduction and elimination rules are simply instances of the implication introduction and elimination rules.

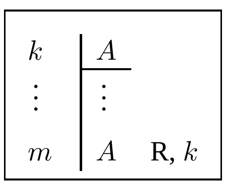
Repetition (R)

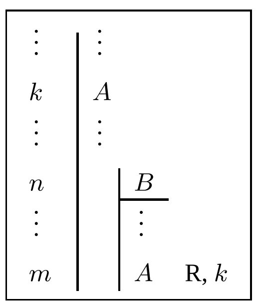
Let A be a formula written at line k (either as a hypothesis, or as a formula already proven). Then one can repeat A at line m if:

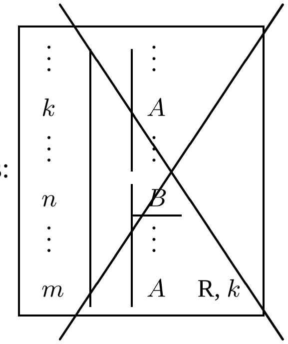
(1) $k < m$, and

(2) every vertical from line k continues without interruption to line m.

Examples of repetition:

This is ok: 

This too: 

But not this: 

Quantifiers

In the rules for quantifiers:

- in $\forall$E and $\exists$I, t is any term.
- in $\forall$I and $\exists$E, u is a *fresh variable*. Here "fresh" means that this variable does not occur anywhere else in the derivation. It may only occur in the subderivation from lines m–n. The "u" that is written between the vertical lines on line m is called a *guard* — it serves as a reminder that u must be fresh in this subderivation. In particular, this means that no formula containing u can be imported (repeated) into lines m–n from outside lines m–n. Also, this means that u cannot occur in the formula φ in lines n and $n+1$ of $\exists$E.

5 Longer examples

Without using the "logical equivalence" rule, we derive one direction of Morgan's law for disjunction,

$$\neg(A \vee B) \vdash \neg A \wedge \neg B.$$

1	$\neg(A \vee B)$	
2	$\quad A$	
3	$\quad A \vee B$	$\vee$I, 2
4	$\quad \neg(A \vee B)$	R, 1
5	$\quad \bot$	$\neg$E, 3, 4
6	$\neg A$	$\neg$I, 2–5
7	$\quad B$	
8	$\quad A \vee B$	$\vee$I, 7
9	$\quad \neg(A \vee B)$	R, 1
10	$\quad \bot$	$\neg$E, 8, 9
11	$\neg B$	$\neg$I, 7–10
12	$\neg A \wedge \neg B$	$\wedge$I, 6, 11

The next two examples use quantifiers.

1	$\forall x(A(x) \rightarrow B(x))$	
2	$\quad \exists y\, A(y)$	
3	$\quad u \quad A(u)$	
4	$\qquad \forall x(A(x) \rightarrow B(x))$	R, 1
5	$\qquad A(u) \rightarrow B(u)$	$\forall$E, 4
6	$\qquad B(u)$	$\rightarrow$E, 3, 5
7	$\qquad \exists y\, B(y)$	$\exists$I, 6
8	$\quad \exists y\, B(y)$	$\exists$E, 2, 3–7
9	$\exists y\, A(y) \rightarrow \exists y\, B(y)$	$\rightarrow$I, 2–8

1	$\quad \forall x\, P(x, x)$	
2	$\quad u \quad \forall x\, P(x, x)$	R, 1
3	$\qquad P(u, u)$	$\forall$E, 2
4	$\qquad \exists z\, P(u, z)$	$\exists$I, 3
5	$\quad \forall y\, \exists z\, P(y, z)$	$\forall$I, 2–4
6	$\forall x\, P(x, x) \rightarrow \forall y\, \exists z\, P(y, z)$	$\rightarrow$I, 1–5

6 Non-examples

Non-example 1

1	$\forall x(A(x) \rightarrow B(x))$	
2	$\quad \exists y\, A(y)$	
3	$\quad u \quad A(u)$	
4	$\qquad \forall x(A(x) \rightarrow B(x))$	R, 1
5	$\qquad A(u) \rightarrow B(u)$	$\forall$E, 4
6	$\qquad B(u)$	$\rightarrow$E, 3, 5
7	$\quad B(u)$	$\exists$E, 2, 3–6
8	$\exists y\, A(y) \rightarrow B(u)$	$\rightarrow$I, 2–7

WRONG, because u is not fresh in lines 3–6 (u must not occur in lines 6,7,8).

Non-example 2

1	$\quad \forall x\, P(x, x)$	
2	$\quad P(u, u)$	$\forall$E, 1
3	$\quad u \quad P(u, u)$	R, 2
4	$\qquad \exists z\, P(u, z)$	$\exists$I, 3
5	$\quad \forall y\, \exists z\, P(y, z)$	$\forall$I, 2–4
6	$\forall x\, P(x, x) \rightarrow \forall y\, \exists z\, P(y, z)$	$\rightarrow$I, 1–5

WRONG, because u is not fresh in lines 3–4 (u cannot be repeated past the guard from line 2 to line 3).

Non-example 3

1	$\forall x(A(x) \to \exists y\, B(x,y))$		**WRONG, because the substitution in line 3 improperly captured the variable y in the scope of a quantifier.**
2	$A(y)$		
3	$A(y) \to \exists y\, B(y,y)$	$\forall$E, 1	
4	$\exists y\, B(y,y)$	$\to$E, 2, 3	

Example

1	$\forall x(A(x) \to \exists y\, B(x,y))$		**CORRECT, because now the variable y does not get captured in the substitution in line 4.**
2	$A(y)$		
3	$\forall x(A(x) \to \exists z\, B(x,z))$	rename bound variables, 1	
4	$A(y) \to \exists z\, B(y,z)$	$\forall$E, 1	
5	$\exists z\, B(y,z)$	$\to$E, 2, 4	

7 Equivalence of Fitch style and Prawitz style

For the purpose of logic, what matters about a proof system is the derivability relation, i.e.,

$$\Gamma \vdash \varphi,$$

meaning there is a derivation of φ from a set of hypotheses Γ. Although Fitch style and Prawitz style natural deduction looks quite different, they both describe the same derivability relation. The rules of both systems can be translated into rules about the derivability relation in identical ways. In both cases, it is easy to prove by induction that the derivability relation is the smallest relation satisfying:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}(\wedge I) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}(\wedge E) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}(\wedge E)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}(\vee I) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}(\vee I) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash \varphi \quad \Gamma, B \vdash \varphi}{\Gamma \vdash \varphi}(\vee E)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}(\to I) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}(\to E)$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}(\neg I) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \bot}(\neg E)$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash C}(\bot E) \qquad \frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}(C)$$

$$\frac{\Gamma \vdash A[u/x]}{\Gamma \vdash \forall x.A}(\forall I) \qquad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]}(\forall E)$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x.A}(\exists I) \qquad \frac{\Gamma \vdash \exists x.A \quad \Gamma, A[u/x] \vdash \varphi}{\Gamma \vdash \varphi}(\exists E)$$

where the rules $(\forall I)$ and $(\exists E)$ are subject to the condition that u is not free in the conclusion of the rule, i.e., in Γ, $\forall .A$, and φ.

8 Practice problems

You don't have to do all these. They are just for practice.

Problem 1 Prove the following in natural deduction:

(a) $Q \to \forall x\, P(x) \equiv \forall x\,(Q \to P(x))$ — assume that x does not occur in Q.

(b) $\neg \exists x\, P(x) \equiv \forall y\, \neg P(y)$.

(c) $\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\,(P(x) \wedge Q(x))$.

(d) $\forall x\, P(x) \vee \forall x\, Q(x) \vdash \forall x\,(P(x) \vee Q(x))$.

(e) $\exists x\, \forall y\, P(x,y) \vdash \forall y\, \exists x\, P(x,y)$.

(f) $\exists x\, \forall y\, P(x,y) \vdash \exists z\, P(z,z)$.

(g) $\exists x\, P(x) \vee \exists x\, Q(x) \equiv \exists x(P(x) \vee Q(x))$.

(h) $\exists x(P(x) \wedge Q(x)) \vdash \exists x\, P(x) \wedge \exists x\, Q(x)$.

(i) $\exists x\, P(x,x) \vdash \exists y\, \exists z\, P(y,z)$.

(j) $\forall x\,(A(x) \to B(x)) \vdash \exists x\, \neg B(x) \to \exists x\, \neg A(x)$.

(k) $\neg \exists x\,(A(x) \wedge B(x)) \equiv \forall x\,(A(x) \to \neg B(x))$.

(l) $\exists x\, \forall y\, P(x,y,x) \vdash \exists x\, \forall y\, \exists z\, P(x,y,z)$.

(m) $\vdash \forall x(P(x) \to \exists y\, P(y))$.

(n) $\vdash \forall x(\forall y\, P(y) \to P(x))$.

(o) $\forall x\, P(x) \vdash \exists x\, P(x)$.

(p) $\forall x(A(x) \to B(x)), \forall y(B(y) \to C(y)) \vdash \forall z(A(z) \to C(z))$.

(q) $\exists x\, A(x), \forall x\,(A(x) \to B(x)) \vdash \exists x(A(x) \wedge B(x))$.

(r) $\forall x\, A(x), \exists x\,(A(x) \to B(x)) \vdash \exists x(A(x) \wedge B(x))$.

(s) $\neg \exists x(A(x) \vee B(x)) \equiv \forall x \neg A(x) \wedge \forall x \neg B(x)$.

(t) $\exists x\, P(x) \to \forall y\, Q(y) \equiv \forall x \forall y(P(x) \to Q(y))$.

Problem 2 Prove the following by natural deduction. Note: each of these problems requires the $\neg\neg$-elimination rule.

(u) $Q \to \exists x\, P(x) \equiv \exists x\,(Q \to P(x))$ — assume that x does not occur in Q.

(v) $\neg \forall x\, P(x) \equiv \exists y\, \neg P(y)$.

(w) $\exists x\,(A(x) \wedge B(x)) \equiv \neg \forall x\,(A(x) \to \neg B(x))$.

(x) $\vdash \exists x(\exists y\, P(y) \to P(x))$.

(y) $\neg \forall x(A(x) \wedge B(x)) \equiv \exists x \neg A(x) \vee \exists x \neg B(x)$.

(z) $\forall x\, P(x) \to \exists y\, Q(y) \equiv \exists x \exists y(P(x) \to Q(y))$.

Problem 3 In Problem 1 (d), (e), (f), (h), (i), (j), (l), (o), (p), (q), (r), prove that the converse direction does not hold by giving a counterexample, i.e., a structure where it is false.