MATH 2135, LINEAR ALGEBRA, Winter 2013

Handout 5: Problems
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Problem 1. Calculate the determinants of the following matrices (a) over the field $\mathbb{R}$ of real numbers, (b) over the field $\mathbb{Z}_5$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 3 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Problem 2. Consider the vector space $\mathbf{P}_2(t)$ of polynomials of degree at most 2. Consider the linear function $f : \mathbf{P}_2(t) \to \mathbf{P}_2(t)$ defined by: $f(p(t)) = p(t+2)$. For example,

$$f(t^2 + 2t + 1) = (t+2)^2 + 2(t+2) + 1 = t^2 + 4t + 4 + 2t + 4 + 1 = t^2 + 6t + 9.$$

Consider the basis $S = \{1, t, t^2\}$ of $\mathbf{P}_2(t)$. Find the matrix representations of f with respect to the basis S. What is the determinant of f?

Problem 3. Consider the formula for the determinant of an $n \times n$-matrix A:

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Using this formula, prove that if a matrix A has two identical rows, then $\det A = 0$. (Hint: to see what is going on, first try this for $n = 3$).

Problem 4. (a) Find the characteristic polynomial, eigenvalues, and eigenvectors of the following matrices. (b) Determine the algebraic multiplicity and geometric multiplicity of each eigenvalue. (c) Which of the matrices can be diagonalized? Diagonalize them.

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -2 & 1 \\ 0 & -4 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 5. Find a unitary matrix U and a diagonal matrix D such that $U^* A U = D$.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Problem 6. A matrix B is called a *square root* of a matrix A if $B^2 = A$. Find four different square roots of A. (Hint: work relative to a basis in which A is diagonal, then convert the answer to the original basis).

$$A = \begin{pmatrix} -2 & -6 \\ 3 & 7 \end{pmatrix}$$

(Remember that it is easy to double-check your answer).

Problem 7. Find the matrix A whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 0$, with respective eigenvectors

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

(Remember that it is easy to double-check your answer).

Problem 8. Consider the sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ obtained by the following rule: $a_0 = 1$, $a_1 = 1$, and for all $n \geqslant 1$, $a_{n+1} = a_n + 2a_{n-1}$. Therefore, the sequence starts with 1 and 1, and the next number is calculated as the sum of the current number plus twice the previous number. The first few elements of this sequence are:

$$1, 1, 3, 5, 11, 21, 43, \ldots$$

The goal of this exercise is to find a direct formula for the nth element of this sequence.

Let $v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ denote the vector consisting of the nth and the $n+1$st elements of the sequence. Then we have the following relationship:

$$v_0 = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ a_n + 2a_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} v_{n-1},$$

for all $n \geqslant 1$. We therefore obtain the following formula for v_n:

$$v_n = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that this formula involves raising the matrix $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ to the nth power.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find an invertible P and a diagonal matrix D such that $D = P^{-1}AP$.

(c) Prove that $A^n = PD^nP^{-1}$, for all n.

(d) Using (c) and (b), give an explicit formula for A^n.

(e) Give an explicit formula for v_n.

(f) Give an explicit formula for a_n.

(g) Check your formula by using it to compute the first few elements of the sequence.