Math 2030, Matrix Theory and Linear Algebra I, Fall 2011

Final Exam, December 13, 2011

FIRST NAME: ____________________ **LAST NAME:** ____________________ **STUDENT ID:** __________

SIGNATURE: ____

Part I: True or false questions

Decide whether each statement is true or false. If it is false, give a reason. (2 points each)

Problem 1. If A is diagonalizable then A^{-1} is diagonalizable.

Problem 2. For a linear system $Ax = b$ where the entries of A are real numbers and A is 17×17, it's possible for the system to have exactly seventeen solutions.

Problem 3. The sum of two elementary matrices of the same size is an elementary matrix.

Problem 4. If A is an $n \times n$-matrix then $\det(kA) = k \det(A)$.

Problem 5. If A is a 2×2 square matrix with integer entries, then $\det A$ is an integer.

Problem 6. If A is an $n \times n$-matrix, then $\det(AA^T A) = (\det A)^3$.

Problem 7. Similar matrices have the same eigenvalues.

Problem 8. For all non-zero vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, one has $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$.

Problem 9. If $\mathbf{v}_1, \mathbf{v}_2$ are two non-zero vectors in $\mathbb{R}^3$, span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin.

Problem 10. A linearly independent set of vectors in $\mathbb{R}^n$ has at least n elements.

Problem 11. Let A and B be 2×2-matrices. If $AB = 0$ then $A = 0$ or $B = 0$.

Problem 12. If A is an invertible $n \times n$-matrix, then the equation $Ax = b$ is consistent for each $b \in \mathbb{R}^n$.

Problem 13. Let A be an $n \times n$ triangular matrix with n distinct eigenvalues. Then the determinant of A is equal to the product of its eigenvalues.

Problem 14. Every $n \times n$-matrix A with real entries has at least one real eigenvalue.

Problem 15. If A is $n \times n$ and diagonalizable, then A has n distinct eigenvalues.

Problem 16. The vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ are linearly independent over $\mathbb{Z}_5$.

Part II: Multiple choice questions

Please **circle the letter** (a), (b), (c), or (d) corresponding to the correct answer for each question.
(2 points each)

Problem 1. For all non-zero vectors $\mathbf{v}$ in $\mathbb{R}^n$, the non-zero vector $\mathbf{u}$ is orthogonal to:

(a) $\text{proj}_{\mathbf{v}}(\mathbf{u})$ (b) $\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})$ (c) $\mathbf{v} + \text{proj}_{\mathbf{u}}(\mathbf{v})$ (d) $\text{proj}_{\mathbf{u}}(\mathbf{v})$

Problem 2. Which of the following expresses the fact that the vectors $\mathbf{u}$ and $\mathbf{v}$ have the same length?

(a) $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v}$ (b) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$ (c) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ (d) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$

Problem 3. The distance between the two planes $2x - y + z = 1$ and $-4x + 2y - 2z = 1$ is

(a) $\dfrac{3}{2\sqrt{6}}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{9}{24}$ (d) $\dfrac{3}{\sqrt{6}}$

Problem 4. The system

$$\begin{array}{rcrcrcl} 5x & - & y & + & z & = & 0 \\ 4x & - & 3y & + & 7z & = & 0 \end{array}$$

has

(a) only a trivial solution

(b) the unique solution $x = 4, y = 31, z = 11$

(c) no solution

(d) an infinite number of solutions

Problem 5. Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation?

(a) $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$

(b) $f\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{\sqrt{x^2 + y^2}}\begin{bmatrix} x \\ y \end{bmatrix}$

(c) $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$

(d) $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 + y^2 \\ 2xy \end{bmatrix}$

Problem 6. If $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ r \\ s \end{bmatrix}$ is a linear combination of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ -1 \end{bmatrix}$ then r and s must be, respectively,

(a) $3, 1$ (b) $2, 0$ (c) $1, 3$ (d) $0, -2$

Problem 7. If A and B are $n \times n$ symmetric matrices, which of the following is not necessarily symmetric?

(a) $-2B^T$ (b) $A + B$ (c) AB (d) $A^T A$

Problem 8. If A and B are $n \times n$-matrices and if $\det A = 2$, $\det B = 3$, then $\det(AB^{-1}) =$

(a) $(-1)^n \dfrac{2}{3}$ (b) $\dfrac{2}{3}$ (c) $(-1)^n 6$ (d) 6

Problem 9. Assume that a certain 5×5-matrix has two eigenvalues, and that the eigenspace corresponding to one of them is 3-dimensional. What must the dimension of the eigenspace of the second eigenvalue be if the matrix is diagonalizable?

(a) 5 (b) 3 (c) 4 (d) 2

Problem 10. A vector in the null space of $A = \begin{bmatrix} 1 & 1 & -2 & -1 \\ -1 & 4 & -3 & 1 \\ 0 & 7 & 1 & -8 \end{bmatrix}$ is:

(a) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$

Problem 11. If A is a non-zero 4×7-matrix, then possible values for nullity(A) are:

(a) $6, 5, 4, 3, 2$ (b) $6, 5, 4, 3$ (c) $7, 6, 5, 4, 3$ (d) $4, 3, 2, 1$

Problem 12. Consider the basis $\mathcal{B} = \left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$, and consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Determine the coefficients c_1, c_2, c_3 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. Which of the following is true?

(a) $c_1 = 0$ (b) $c_1 = 2$ (c) $c_1 = 1$ (d) $c_1 = -1$

Problem 13. Which of the following vectors is in the column space of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Problem 14. Let $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ x_1 - x_2 + x_3 \end{bmatrix}$ and $S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - 2y_2 \\ y_1 + y_2 \end{bmatrix}$. Then $[S \circ T] =$

(a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 3 & -3 \\ 2 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}$

Problem 15. Let $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. While of the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$?

(a) $\{\mathbf{v} \mid \mathbf{v} + \mathbf{n} = \mathbf{0}\}$

(b) $\{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{n} = 0\}$

(c) $\{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{v} = 1\}$

(d) $\{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{n} = 1\}$

Problem 16. Which of the following is a basis of $\mathbb{R}^3$?

(a) $\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}\}$

(b) $\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\}$

(c) $\{\begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}\}$

(d) $\{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$

Part III: Detailed answer questions

(6 points each)

Problem 1. (a) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$. Find a unit vector in the direction of $-\mathbf{u}$.

(b) The planes $2x + y - z = 2$ and $x + y + z = 3$ intersect in a line. Find the vector form equation of this line.

(c) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Describe all vectors $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 1$. Is this a vector subspace of $\mathbb{R}^3$?

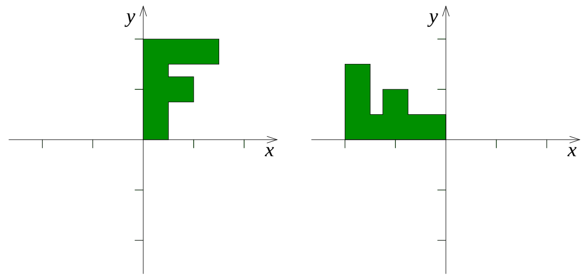
Problem 2. Find the distance from the point $(2, 2, 2)$ to the plane p with equation $x + y - z = 0$.

Problem 3. Solve the following system of equations using Gauss-Jordan elimination.

$$\begin{array}{rcrcrcr} -2x & + & y & + & z & = & 4 \\ x & - & 2y & + & z & = & 1 \\ x & + & y & - & 2z & = & -5 \end{array}$$

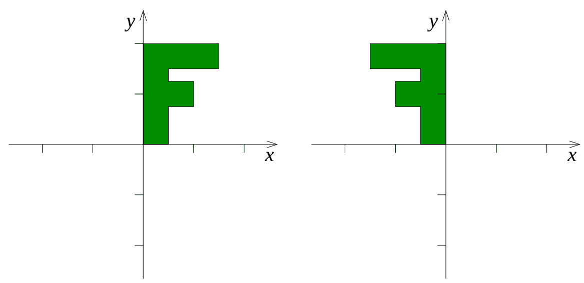
Problem 4. For each of the following two linear functions $T : \mathbb{R}^2 \to \mathbb{R}^2$, find $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$, and give the standard matrix $[T]$ of T.

(a) T is a rotation by 90 degrees, as shown in the illustration:



Answer: $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) =$ ☐ $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) =$ ☐ $[T] =$ ☐

(b) T is a reflection about the y-axis:



Answer: $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) =$ ☐ $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) =$ ☐ $[T] =$ ☐

Problem 5. Find the inverse of $A = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ if it exists.

Problem 6. Compute the determinant of A.

$$A = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & 0 & 2 & 0 \\ 2 & -2 & 1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

Problem 7. Find bases for $\text{col}(A)$ and $\text{null}(A)$ if

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 2 & 5 & -1 \end{bmatrix}$$

Problem 8. Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 5 & 4 & -4 \\ -8 & -7 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

Extra page for rough work.