Semistrict models of connected 3-types and Tamsamani's weak 3-groupoids

Simona Paoli, Macquarie University

Main themes

- Modelling connected 3-types:

  Homotopy theory $\longrightarrow$ cat^2-groups (Loday).

  Higher category theory $\longrightarrow$ Tamsamani's weak 3-groupoids (with 1 object)

- Comparison problem.

- Semistrictification results for Tamsamani's weak 3-groupoids with 1 object.

$\mathbf{Cat}^n$-groups as homotopy models

- <u>Definition</u> $\quad \mathsf{Cat}^n(\mathsf{Gp}) = \mathsf{Cat}(\mathsf{Cat}^{n-1}(\mathsf{Gp}))$
  $\mathsf{Cat}^0(\mathsf{Gp}) = \mathsf{Gp}$

- <u>Multinerve</u> $\quad \mathcal{N} : \mathsf{Cat}^n(\mathsf{Gp}) \to [\Delta^{n^{op}}, \mathsf{Gp}]$

- <u>Classifying space</u> of $\mathcal{G} \in \mathsf{Cat}^n(\mathsf{Gp})$
  $B\mathcal{G} = B\mathcal{N}\mathcal{G}$.

- <u>Weak equivalence</u> $\quad f : \mathcal{G} \to \mathcal{G}' \quad \mathrm{mor}(\mathsf{Cat}^n(\mathsf{Gp}))$
  s.t. Bf weak homotopy equivalence.

- <u>Theorem</u>
  [Whitehead $n = 1$]
  [Loday; Bullejos-Cegarra-Duskin; Porter, $n \geq 1$]

$$\overline{B} : \frac{\mathsf{Cat}^n(\mathsf{Gp})}{\sim} \simeq \ \mathcal{H}o\binom{\text{connected}}{n+1\text{-types}} : \overline{\mathcal{P}}$$

Tamsamani's model: n=2

- Segal maps
  $\mathcal{C}$ category with finite limits, $\phi \in [\Delta^{op}, \mathcal{C}]$
  $n \geq 2 \quad \eta_n : \phi_n \to \phi_1 \times_{\phi_0} \cdots^{n} \times_{\phi_0} \phi_1.$
  fact: ϕ nerve of object of Cat$\,\mathcal{C} \Leftrightarrow$
  η_n isomorphism for all $n \geq 2$.

- Tamsamani's weak 2-nerves, $\mathcal{N}_2$.
  $\phi \in [\Delta^{2^{op}}, \text{Set}] \quad \phi_n = ([n], \text{-})$
  (i) ϕ_n nerve of category of all $n \geq 0$.
  (ii) ϕ_0 constant.
  (iii) Segal maps equivalences of categories $\forall n \geq 2$.

- Weak 2-groupoids $\mathcal{T}_2$, $\phi \in \mathcal{N}_2$ s.t.
  (i) ϕ_n nerve of groupoid, $\forall\ n \geq 0$.
  (ii) $T\phi : \Delta^{op} \to \text{Set}$ nerve of groupoid
  $(T\phi)_n = \pi_0 \phi_n$

- External equivalences of 2-nerves
  $f : \phi \to \psi \qquad \phi_1 = \coprod_{x,y \in \phi_0} \phi_{(x,y)}$
  (i) $\phi_{(x,y)} \to \psi_{(fx,fy)}$
  (ii) Tf
  equivalences of categories.

Tamsamani's model: n=3

- Tamsamani's weak 3-nerves, $\mathcal{N}_3$.

  $\phi \in [\Delta^{3^{op}}, \mathrm{Set}] \quad \phi_n = ([n], \text{-}, \text{-})$

  (i) $\phi_n \in \mathcal{N}_2 \ \forall \ n \geq 0$.

  (ii) ϕ_0 constant.

  (iii) Segal maps equivalences of 2-nerves $\forall \ n \geq 2$.

- Weak 3-groupoids $\mathcal{T}_3$, $\phi \in \mathcal{N}_3$ s.t.

  (i) $\phi_n \in \mathcal{T}_2 \ \forall \ n \geq 0$.

  (ii) $T^2\phi : \Delta^{op} \to \mathrm{Set}$ nerve of groupoid.

- Fact: external equivalences in $\mathcal{T}_2$ and $\mathcal{T}_3$

  $\equiv$ weak homotopy equivalences

- The subcategory $\mathcal{S} \subset \mathcal{T}_3$

  $\phi \in \mathcal{S}$ if $\phi \in \mathcal{T}_3$ and $\phi_0(\text{-}, \text{-}) = \{\cdot\}$.

- Theorem [Tamsamani]

$$\mathcal{T}_3/\sim^{ext} \simeq \mathcal{H}o(\text{3-types})$$

$$\mathcal{S}/\sim^{ext} \simeq \mathcal{H}o\begin{pmatrix}\text{connected} \\ \text{3-types}\end{pmatrix}$$

Summary: cat^2-gp versus $\mathcal{T}_3$.

$\mathrm{Cat}^2(\mathrm{Gp})$	$\mathcal{T}_3$
• $\mathcal{G} \in [\Delta^{2^{op}}, \mathrm{Gp}]$ $\mathcal{G}_n$ nerve of Cat (Gp) Segal maps iso.	• $\phi \in [\Delta^{3^{op}}, \mathrm{Set}]$ $\phi_n \in \mathcal{T}_2$ ϕ_0 constant, $T\phi$ iso. Segal maps equivalences
• multisimplicial inductive definition based on Gp strict structure "cubical"	• multisimplicial inductive definition based on Set weak structure "globular"

- Main issues in the comparison:

$$\text{cubical} \xrightarrow{discretization} \text{globular}$$

$$\mathrm{Gp} \xrightarrow{nerve} [\Delta^{op}, \mathrm{Set}]$$

- dealt with functors:

$$\mathrm{Cat}^2(\mathrm{Gp})/\sim \xrightarrow{disc} \mathcal{D}/\sim$$

$$\mathcal{D}/\sim \longrightarrow \mathcal{H}/\sim^{ext} \qquad \mathcal{H} \subset \mathcal{S}.$$

The discretization functor

- Key Lemma: $\mathcal{G} \in \mathrm{Cat}^2(\mathrm{Gp})$. There is $\phi \in \mathrm{Cat}^2(\mathrm{Gp})$

$$\phi_1 \times_{\phi_0} \phi_1 \xrightarrow{c} \phi_1 \underset{\sigma_0}{\overset{\partial_0, \partial_1}{\rightrightarrows}} \phi_0$$

with ϕ_0 projective in $\mathrm{Cat}(\mathrm{Gp})$ and $B\phi \simeq B\mathcal{G}$.

- Projective objects in Cat (Gp)

$d : \phi_0 \longrightarrow \phi_0^d$ weak equivalence.

ϕ_0^d discrete internal category.

section $t : \phi_0^d \longrightarrow \phi_0, \quad dt = \mathrm{id}.$

- The discrete multinerve $ds\mathcal{N}\phi \in [\Delta^{2^{op}}, \mathrm{Gp}]$

$$\cdots \phi_1 \times_{\phi_0} \phi_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \phi_1 \underset{\sigma_0 t}{\overset{d\partial_0, d\partial_1}{\rightrightarrows}} \phi_0^d$$

i) $B\, ds\mathcal{N}\phi = B\phi \simeq B\mathcal{G}$.

ii) Segal maps weak equivalences in $[\Delta^{op}, \mathrm{Gp}]$.

- Functor $\qquad disc : \mathrm{Cat}^2(\mathrm{Gp})/\sim \longrightarrow \mathcal{D}/\sim$

$\qquad disc\,[\mathcal{G}] = [ds\mathcal{N}\phi]$

$\mathcal{D} \subset [\Delta^{2^{op}}, \mathrm{Gp}]$ "internal 2-nerves".

First semistrictification result.

- <u>The subcategory $\mathcal{H} \subset \mathcal{S}$.</u>

  $\phi \in \mathcal{S}$ and Segal maps $\phi_n \to \phi_1 \times \cdots^{n} \times \phi_1$ iso.

  Objects of $\mathcal{H}$ are "semistrict".

- <u>Theorem</u> [P.] Commutative diagram

$$\begin{array}{ccc} \mathsf{Cat}^2(\mathsf{Gp})/\!\sim & \xrightarrow{\;F\;} & \mathcal{H}/\!\sim^{ext} \\ & {\scriptstyle B}\searrow \quad \swarrow{\scriptstyle B} & \\ & \mathcal{H}o\begin{pmatrix}\text{connected}\\ \text{3-types}\end{pmatrix} & \end{array}$$

  where $F : \mathsf{Cat}^2(\mathsf{Gp})/\sim \xrightarrow{disc} \mathcal{D}/\!\sim \xrightarrow{R} \mathcal{H}/\!\sim^{ext}$.

  Let $\mathcal{H}o_{\mathcal{S}}(\mathcal{H}) \subset \mathcal{S}/\!\sim^{ext}$ full subcategory with objects in $\mathcal{H}$. Then

$$\frac{\mathsf{Cat}^2(\mathsf{Gp})}{\sim} \simeq \mathcal{H}o_{\mathcal{S}}(\mathcal{H}).$$

- <u>Corollary</u>: Every object of $\mathcal{S}$ is equivalent to an object of $\mathcal{H}$ through a zig-zag of external equivalences.

- <u>Remark</u>: $\mathcal{H} \subset \mathsf{Mon}(\mathcal{T}_2, \times)$.

Second semistrictification result.

- <u>The subcategory $\mathcal{K} \subset \mathcal{S}$</u>.
  $\phi \in \mathcal{S}$ and ϕ_n strict 2-groupoid $\forall\ n \geq 0$.
  Objects of $\mathcal{K}$ are semistrict but $\mathcal{K} \neq \mathcal{H}$.

- <u>Theorem</u>[P.] Commutative diagram

$$
\begin{array}{ccc}
\mathcal{S}/{\sim^{ext}} & \xrightarrow{\overline{St}} & \mathcal{K}/{\sim^{ext}} \\
 & {\scriptstyle B}\searrow \quad \swarrow{\scriptstyle B} & \\
 & \mathcal{H}o\begin{pmatrix}\text{connected} \\ \text{3-types}\end{pmatrix} &
\end{array}
$$

Let $\mathcal{H}o_{\mathcal{S}}(\mathcal{K}) \subset \mathcal{S}/{\sim^{ext}}$ full subcategory with objects in $\mathcal{K}$. Then

$$\mathcal{S}/{\sim^{ext}} \simeq \mathcal{H}o_{\mathcal{S}}(\mathcal{K})$$

<u>idea of proof</u>:

$St : \mathcal{T}_2 \xrightarrow{G} Bigpd \xrightarrow{st} 2\text{-}gpd \xrightarrow{\nu} \mathcal{T}_2^{st}$

$\psi \in \mathcal{S}, \qquad (\overline{St}\,\psi)_n = St\,\psi_n.$

$(\overline{St}\,\psi)_n = St\,\psi_n \simeq St\,(\psi_1 \times \overset{n}{\cdots} \times \psi_1) \simeq$

$\simeq St\,\psi_1 \times \cdots \times St\,\psi_1 = (\overline{St}\,\psi)_1 \times \cdots \times (\overline{St}\,\psi)_1$

hence $\overline{St}\,\psi \in \mathcal{K}$.

The comparison with Gray groupoids.

- <u>Gray groupoids.</u>
  Gray =(2-cat, $\otimes_{gray}$).
  Gray-enriched category with invertible cells.

- <u>Theorem</u> [Joyal - Tierney, Leroy]
  $\mathcal{H}o(3-\text{types}) \simeq Gray\text{-}gpd/\!\sim$
  $\mathcal{H}o(\text{conn. 3-types}) \simeq (Gray\text{-}gpd)_0/\!\sim$.

- <u>Theorem</u> [P.] Commutative diagram

$$
\begin{array}{ccccc}
\mathcal{H}o_{\mathcal{S}}(\mathcal{H}) & \xrightarrow{S} & (Gray\text{-}gpd)_0/\!\sim & \xleftarrow{T} & \mathcal{H}o_{\mathcal{S}}(\mathcal{K}) \\
& {\scriptstyle B}\searrow & \downarrow{\scriptstyle B} & \swarrow{\scriptstyle B} & \\
& & \mathcal{H}o\begin{pmatrix}\text{connected}\\ \text{3-types}\end{pmatrix} & &
\end{array}
$$

<u>idea of proof</u>:

\- Monoidal functor

$(\mathcal{T}_2, \times) \xrightarrow{G} (Bigpd, \times) \xrightarrow{st} (2\text{-}gpd, \otimes_{gray})$

$\phi \in \mathcal{H} \subset \text{Mon}\,(\mathcal{T}_2, \times) \Rightarrow st\,G\,\phi \in (Gray\text{-}gpd)_0$

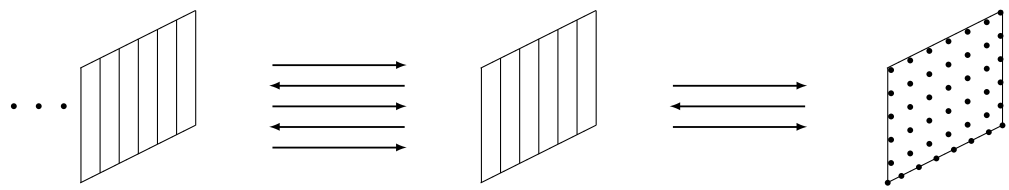
Let $S(\phi) = st\,Bic\,\phi$.

\- Every object of $\mathcal{K}$ is equivalent to one of $\overline{St}\,\mathcal{H}$.

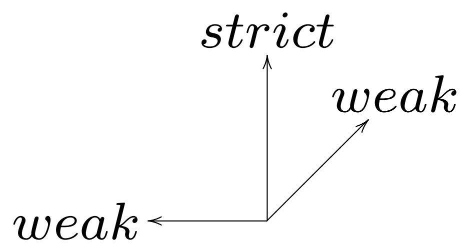
$T[\psi] = T[\overline{St}\,\phi] = [st\,G\,\phi]$.

Conclusion: modelling connected 3-types using Tamsamani's model.

- Tamsamani's weak 3-groupoids, $\mathcal{S}$



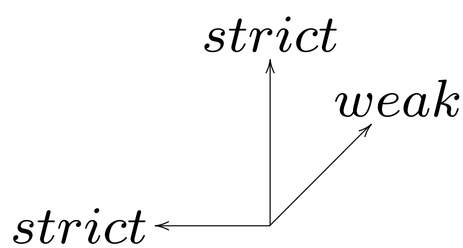
$$\mathcal{S}/\!\sim^{ext} \simeq \mathcal{H}o\begin{pmatrix}\text{connected}\\ \text{3-types}\end{pmatrix}$$



- Semistrict cases.

a) $\mathcal{H} \subset \mathcal{S}$

$$\mathcal{H}o_{\mathcal{S}}(\mathcal{H}) \simeq \mathcal{H}o\begin{pmatrix}\text{connected}\\ \text{3-types}\end{pmatrix}$$
$$\mathcal{H} \subset \text{Mon}\,(\mathcal{T}_2, \times).$$



b) $\mathcal{K} \subset \mathcal{S}$

$$\mathcal{H}o_{\mathcal{S}}(\mathcal{K}) \simeq \mathcal{H}o\begin{pmatrix}\text{connected}\\ \text{3-types}\end{pmatrix}$$