FINITENESS SPACES

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CATEGORY <u>REL</u> :

OBJECTS : SETS X

MORPHISMS : $\underline{\mathrm{REL}}\,(X, Y) = \mathcal{P}(X \times Y)$

$X \otimes Y = X \multimap Y = X \,⅋\, Y = X \times Y$

AND $I = \{*\}$ (DENOTED HERE AS 1)

$X^{\perp} = X$

$X \oplus Y = X \,\&\, Y = X + Y$ (DISJOINT SUM)

$0 = \emptyset$

THESE ARE EXACTLY THE OPERATIONS PERFORMED ON <u>BASES</u> FOR THE CORRESPONDING CONSTRUCTIONS IN <u>VEC</u>. (FINITE DIMENSIONAL VECTOR SPACES)

e.g.: IF $\mathcal{B}$ IS A BASIS OF V
AND $\mathcal{C}$ ——— OF W

THEN $\mathcal{B} \times \mathcal{C}$ IS A BASIS OF $V \otimes W$

SEE $X \in \underline{REL}$ AS THE VECTOR SPACE k^X

THIS IS OK WHEN X IS FINITE.

$$k^X \otimes k^Y \simeq k^{X \times Y} \quad \text{etc}$$

IF $V = k^X$

THE DUAL V^* CAN BE IDENTIFIED TO $k^{(X^\perp)} = k^X$:

$$x \in k^X, \; x' \in k^{(X^\perp)} \leadsto \langle x, x' \rangle = \sum_{a \in X} x_a x'_a$$

BUT THIS DOES NOT MAKE SENSE WHEN X IS INFINITE.

WE <u>NEED</u> INFINITE SETS ($\sim \infty$-DIM VECTOR SPACES):

FOR THE EXPONENTIALS.

IDEA : DON'T ACCEPT ALL $x \in k^X$ AS VECTORS.

IF $\mathfrak{F}$ IS A SUBSET OF $\mathcal{P}(I)$,

DEFINE $\mathfrak{F}^\perp \subseteq \mathcal{P}(I)$ AS :

$$\mathfrak{F}^\perp = \{ u' \subseteq I \ / \ \forall u \in \mathfrak{F} \ \ u \cap u' \text{ FINITE} \}$$

ONE HAS :

$$\mathfrak{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G}^\perp \subseteq \mathfrak{F}^\perp$$

$$\mathfrak{F} \subseteq \mathfrak{F}^{\perp\perp}$$

$$\mathfrak{F}^{\perp\perp\perp} = \mathfrak{F}^\perp$$

AND :

$$u' \text{ FINITE} \Rightarrow u' \in \mathfrak{F}^\perp$$

$$u' \subseteq v' \in \mathfrak{F}^\perp \Rightarrow u' \in \mathfrak{F}^\perp$$

$$u', v' \in \mathfrak{F}^\perp \Rightarrow u' \cup v' \in \mathfrak{F}^\perp$$

REPLACE SETS X BY PAIRS $X = (|X|, \mathfrak{F}X)$

WITH $\mathfrak{F}X^{\perp\perp} = \mathfrak{F}X$

ASSOCIATE TO SUCH A PAIR THE VECTOR SPACE

$$k\langle X \rangle = \{ x \in k^{|X|} \ / \ \mathrm{Supp}(x) \in \mathfrak{F}X \}$$

(WHERE $\mathrm{Supp}(x) = \{ a \in |X| \ / \ x_a \neq 0 \}$)

IF $x \in k\langle X\rangle$ AND $x' \in k\langle X^\perp\rangle$

(WHERE $X^\perp = (|X|, \mathcal{F}X^\perp)$)

$$x_a \, x'_a \neq 0 \implies a \in \underbrace{\mathrm{Supp}(x) \cap \mathrm{Supp}(x')}_{\text{FINITE SET}}$$

SO NOW THE SUM $\langle x, x'\rangle = \sum_{a \in |X|} x_a \, x'_a$

IS WELL DEFINED.

OPERATIONS ON FINITENES SPACES

$$X^\perp = (|X|, \mathcal{F}X^\perp)$$

$$1 = (\{*\}, \{\emptyset, \{*\}\})$$

$$X \otimes Y = (|X| \times |Y|, \{u \times v \,/\, u \in \mathcal{F}X, v \in \mathcal{F}Y\}^{\perp\perp})$$

<u>FACT</u>: LET $w \subseteq |X| \times |Y|$

$$w \in \mathcal{F}(X \otimes Y) \iff \pi_1 w \in \mathcal{F}X \wedge \pi_2 w \in \mathcal{F}Y$$

$$X \multimap Y = (X \otimes Y^{\perp})^{\perp}$$

<u>FACT</u>: LET $t \subseteq |X| \times |Y|$

$t \in \mathcal{F}(X \multimap Y)$ IFF :

1) $\forall u \in \mathcal{F}X \quad t(u) = \{ b \,/\, \exists a \in u \ (a,b) \in t \} \in \mathcal{F}Y$

2) $\forall v' \in \mathcal{F}Y^{\perp} \quad t^{\perp}(v') = \{ a \,/\, \exists b \in v' \ (a,b) \in t \} \in \mathcal{F}X^{\perp}$

CONSEQUENCE :

IF $s \in \mathcal{F}(X \multimap Y)$, $t \in \mathcal{F}(Y \multimap Z)$

THEN $t \circ s \in \mathcal{F}(X \multimap Z)$

ALSO : $Id_X = \{ (a,a) \,/\, a \in |X| \} \in \mathcal{F}(X \multimap X)$.

WE HAVE DEFINED A CATEGORY OF F.S.
AND FINITARY RELATIONS (A F.R. FROM X TO Y
BEING AN ELT OF $\mathcal{F}(X \multimap Y)$) WHICH IS
SYMMETRIC MONOIDAL CLOSED (AND $*$-AUTONOMOUS).

$X \otimes Y = X \& Y = (|X| + |Y|, \{\omega \,/\, \omega \cap |X| \in \mathcal{F}X, \omega \cap |Y| \in \mathcal{F}Y\})$

$0 = \top = (\emptyset, \{\emptyset\})$

HOWEVER, INFINITE SUMS AND PRODUCTS DO NOT COINCIDE:

$$\underset{i \in I}{\&} X_i = \left(\sum_{i \in I} |X_i|, \{\omega \,/\, \forall i \;\; \omega \cap |X_i| \in \mathcal{F}X_i\}\right)$$

$$\bigoplus_{i \in I} X_i = \left(\sum_{i \in I} |X_i|, \left\{\omega \,/\, \begin{array}{l} \forall i \;\; \omega \cap |X_i| \in \mathcal{F}X_i \\ \omega_i = \emptyset \text{ FOR ALL BUT A FINITE NUMBER OF } i \end{array}\right\}\right)$$

EXAMPLE. $N = \bigoplus_{i \in \mathbb{N}} 1$

$|N| = \mathbb{N}$ $\quad u \in \mathcal{F}N$ IFF u FINITE

$N^\perp = \underset{i \in \mathbb{N}}{\&} \perp \qquad (\perp = 1)$

$|N^\perp| = \mathbb{N} \qquad \mathcal{F}N^\perp = \mathcal{P}(\mathbb{N})$.

$$|!X| = \underbrace{\mathcal{M}_{FIN}(|X|)}$$

ALL FINITE MULTISETS OF ELEMENTS OF $|X|$.

$$\mathcal{F}(!X) = \{ \mathcal{M}_{FIN}(u) \;/\; u \in \mathcal{F}X \}^{\perp\perp}$$

<u>FACT</u>: LET $U \subseteq |!X| = \mathcal{M}_{FIN}(|X|)$.

$$U \in \mathcal{F}(!X) \iff \bigcup_{\mu \in U} \mathrm{Supp}(\mu) \in \mathcal{F}X$$

EXAMPLE: $\quad X = !\mathbb{N}$

$$U \in \mathcal{F}X \iff \bigcup_{\mu \in U} \mathrm{Supp}(\mu) \text{ FINITE}$$

$$U = \{ [\,], [0], [0,0], [0,0,0], \ldots \} \in \mathcal{F}X$$

SINCE $\displaystyle \bigcup_{\mu \in U} \mathrm{Supp}(\mu) = \{0\}$

$$U = \{ [0], [1], [2], \ldots \} \notin \mathcal{F}X.$$

$U' = \{[0], [0,1], [0,0,2], [0,0,0,3], \dots\} \in \mathcal{F}X^\perp$

SINCE $\{\mu \in U' \mid \mathrm{Supp}\,\mu \subseteq u\}$ FINITE $\forall u \in \mathcal{F}N$.

FUNDAMENTAL ISO:

$$!\top \simeq 1$$

$$!(X \& Y) \simeq\ !X \otimes !Y$$

$$\begin{aligned} |!(X\&Y)| &= \mathcal{M}_{fin}(|X| + |Y|) \\ &\simeq \mathcal{M}_{fin}(|X|) \times \mathcal{M}_{fin}(|Y|) \\ &= |!X| \times |!Y| = |!X \otimes !Y| \end{aligned}$$

AND THIS BIJECTION IS AN ISO OF FINITENESS SPACES.

ASSOCIATED VECTOR SPACES

$$k\langle X\rangle = \{ x \in k^{|X|} \ / \ \mathrm{Supp}(x) \in \mathcal{F}X \}$$

$x, y \in k\langle X\rangle$

$$\mathrm{Supp}(x+y) \subseteq \mathrm{Supp}(x) \cup \mathrm{Supp}(y) \in \mathcal{F}X$$

SINCE $\mathcal{F}X = \mathcal{F}X^{\perp\perp}$

LINEAR TOPOLOGY ON $k\langle X\rangle$:

FOR $u' \in \mathcal{F}X^{\perp}$, DEFINE

$$V_X(u') = \{ x \in k\langle X\rangle \ / \ \mathrm{Supp}(x) \cap u' = \emptyset \}$$

$$0 \in V_X(u')$$

$$V_X(u' \cup v') = V_X(u') \cap V_X(v')$$

SAY THAT $U \subseteq k\langle X\rangle$ IS OPEN IF:

$$\forall x \in U \ \exists u' \in \mathcal{F}X^{\perp} \quad \underbrace{x + V_X(u')}_{\{x+y \ / \ y \in V_X(u')\}} \subseteq U$$

FACTS:

- ADDITION $k\langle X\rangle \times k\langle X\rangle \longrightarrow k\langle X\rangle$

  AND SCAL. MULT. $k \times k\langle X\rangle \longrightarrow k\langle X\rangle$

  ARE CONTINUOUS (k WITH DISCRETE TOPOLOGY)

- $k\langle X\rangle$ IS HAUSDORFF

  $x \neq y \Rightarrow \exists \underset{v'}{u'} \in \mathcal{F}X^{\perp} \quad x + V_X(u') \cap y + V_X(v') = \emptyset$

  (take $a \in |X|$ st. $x_a \neq y_a$

  AND SET $u' = v' = \{a\}$)

- $k\langle X\rangle$ IS COMPLETE

  $x(n) \in k\langle X\rangle \qquad n \in \mathbb{N}$

  CAUCHY IN THE SENSE THAT:

  $\forall u' \in \mathcal{F}X^{\perp} \exists p \in \mathbb{N} \ \forall n, m \geq p \quad x(n) - x(m) \in V_X(u')$

  THEN $x(n)$ CONVERGES

  ($\exists x \in k\langle X\rangle \ \forall u' \in \mathcal{F}X^{\perp}$

  $\exists n \ \forall p \geq n \quad x(p) - x \in V_X(u')$

  I.E.: $x(p)$ AND x

  COINCIDE ON u')

MORPHISMS : LINEAR AND CONTINUOUS MAPS

$$\mathcal{L}(X,Y) = \{ f : k\langle X\rangle \to k\langle Y\rangle \ / \ f \text{ LIN. AND CONT.} \}$$

FACT : $\mathcal{L}(X,Y)$ AND $k\langle X \multimap Y\rangle$ ARE ISOMORPHIC k-VECTOR SPACES.

IF $A \in k\langle X \multimap Y\rangle$ (SO $A \in k^{|X|\times|Y|}$, SEEN AS A MATRIX)

DEFINE

$$\hat{A} : k\langle X\rangle \longrightarrow k^{|Y|}$$
$$x \longmapsto A.x$$

$$(A.x)_b = \sum_{a\in|X|} A_{ab}\, x_a$$

SINCE $A \in k\langle X \multimap Y\rangle$, $\mathrm{Supp}\, A \in \mathcal{F}(X \multimap Y)$

SO $\{ a \in |X| \ / \ A_{ab} \neq 0 \} \in \mathcal{F} X^{\perp}$

HENCE THIS SUM IS FINITE.

FOR THE SAME REASON, $A.x \in k\langle Y\rangle$.

CONVERSELY, TO $f \in \mathcal{L}(X, Y)$, ASSOCIATE ITS MATRIX $M(f) \in k^{|X| \times |Y|}$

$$M(f)_{a,b} = f(e_a)_b$$

($e_a \in k\langle X \rangle$ DEFINED BY $(e_a)_b = \begin{cases} 1 & \text{IF } a = b \\ 0 & \text{OTHERWISE} \end{cases}$)

IT TURNS OUT THAT $M(f) \in k\langle X \multimap Y \rangle$.

ONE HAS TO SHOW: $\mathrm{Supp}\, M(f) \in \mathcal{F}(X \multimap Y)$

$$= \mathcal{F}(X \otimes Y^\perp)^\perp$$

$$= \{ u \times v' \,/\, u \in \mathcal{F}X, v' \in \mathcal{F}Y^\perp \}^\perp$$

LET $u \in \mathcal{F}X$, $v' \in \mathcal{F}Y^\perp$, ONE HAS TO SHOW:

$w = \mathrm{Supp}\, M(f) \cap (u \times v')$ FINITE.

BUT f CONTINUOUS, SO $\exists u' \in \mathcal{F}X^\perp$ S.T.
$f(V_X(u')) \subseteq V_Y(v')$

$a \notin u' \Leftrightarrow e_a \in V_X(u') \Rightarrow f(e_a) \in V_Y(v') \Leftrightarrow \mathrm{Supp}\, f(e_a) \cap v' = \emptyset$

SO $w = \bigcup_{a \in u \cap u'} (\{a\} \times w(a))$

BUT:
- $u \cap u'$ FINITE
- $\forall a \quad w(a) \subseteq \mathrm{Supp}\, f(e_a) \cap v'$ FINITE

SO W IS FINITE.

$$\begin{array}{ccc} \mathcal{L}(X,Y) & \simeq & k\langle X \multimap Y\rangle \\ f & \longmapsto & M(f) \\ \hat{A} & \longleftarrow\!\shortmid & A \end{array}$$

COMPOSITION OF LIN. CONT. MAPS CORRESPONDS TO PRODUCT OF MATRICES:

$$A \in k\langle X \multimap Y\rangle \qquad B \in k\langle Y \multimap Z\rangle$$

$$BA \in k\langle X \multimap Z\rangle$$

FOR $c \in |Z|$ AND $a \in |X|$, $BA_{a,c} = \sum_{b \in |Y|} A_{ab} B_{bc}$

(THE SUM IS ALWAYS FINITE BECAUSE

$$\mathrm{Supp}\, A \in \mathcal{F}\langle X \multimap Y\rangle \qquad \mathrm{Supp}\, B \in \mathcal{F}\langle Y \multimap Z\rangle$$

$$\{a\} \in \mathcal{F}X \qquad \{c\} \in \mathcal{F}Z^{\perp} \quad)$$

$k\langle X \multimap Y\rangle$ CARRIES ITS OWN TOPOLOGY.

WHAT IS THE CORRESPONDING TOPOLOGY OF $\mathcal{L}(X,Y)$?

ANSWER : TAKE ALL SUBSETS $\mathcal{W}(K,V) \subseteq \mathcal{L}(X,Y)$

$$\mathcal{W}(K,V) = \{ f \in \mathcal{L}(X,Y) \ / \ f(K) \subseteq V \}$$

WHERE $K \subseteq k\langle X\rangle$ COMPACT

$V \subseteq k\langle Y\rangle$ NEIGHBORHOOD OF 0

AS BASIC NEIGHBORHOODS OF 0.

<u>TENSOR PRODUCT</u>

$$|X \otimes Y| = |X| \times |Y|$$

$$\mathcal{F}(X \otimes Y) = \{ u \times v \ / \ u \in \mathcal{F}X, \ v \in \mathcal{F}Y \}^{\perp\perp}$$

$$= \{ w \subseteq |X| \times |Y| \ / \ \pi_1 w \in \mathcal{F}X, \ \pi_2 w \in \mathcal{F}Y \}.$$

IF $x \in k\langle X\rangle$, $y \in k\langle Y\rangle$

THEN $x \otimes y \in k^{|X| \times |Y|}$

DEFINED BY $(x \otimes y)_{a,b} = x_a y_b$

SATISFIES $\mathrm{Supp}\,(x \otimes y) \subseteq \mathrm{Supp}\, x \times \mathrm{Supp}\, y$

SO $x \otimes y \in k\langle X \otimes Y\rangle$.

THE MAP $\quad \tau: k\langle X\rangle \times k\langle Y\rangle \to k\langle X\otimes Y\rangle$

$$(x,y) \longmapsto x\otimes y$$

IS BILINEAR, BUT NOT CONTINUOUS IN GENERAL.

SAY THAT A BILINEAR MAP $f: k\langle X\rangle \times k\langle Y\rangle \to k\langle Z\rangle$ IS HYPOCONTINUOUS IF, FOR ALL NEIGHBORHOOD W OF 0 IN $k\langle Z\rangle$:

1) FOR ALL $K \subseteq k\langle X\rangle$ COMPACT, THERE IS A NEIGH. V OF 0 IN $k\langle Y\rangle$ S.T.
$$f(K\times V)\subseteq W$$

2) FOR ALL $L\subseteq k\langle Y\rangle$ COMPACT, THERE IS A NEIGH. U OF 0 IN $k\langle X\rangle$ S.T.
$$f(U\times L)\subseteq W$$

FACT : • τ IS HYPOCONTINUOUS

• FOR ALL $f: k\langle X\rangle \times k\langle Y\rangle \to k\langle Z\rangle$ BILINEAR AND HYPOCONTINUOUS

THERE IS EXACTLY ONE $\tilde{f} \in \mathcal{L}(X \otimes Y, Z)$ S.T.

$$\begin{tikzcd}
k\langle X \rangle \times k\langle Y \rangle \arrow[rr, "f"] \arrow[dr, "\tau"'] & & k\langle Z \rangle \\
& k\langle X \otimes Y \rangle \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

$\otimes$ IS FUNCTORIAL :

$f \in \mathcal{L}(X, X')$ $\qquad g \in \mathcal{L}(Y, Y')$

THEN THE MAP $\quad h : k\langle X \rangle \times k\langle Y \rangle \to k\langle X' \otimes Y' \rangle$
$\qquad (x, y) \mapsto \tau(f(x), g(y))$

IS HYPOCONTINUOUS (BECAUSE τ IS AND f, g ARE LIN. CONT.).

SO THERE IS EXACTLY ONE $f \otimes g = \tilde{h}$ S.T.

$$\begin{tikzcd}
k\langle X \rangle \times k\langle Y \rangle \arrow[r, "f \times g"] \arrow[d, "\tau"'] & k\langle X' \rangle \times k\langle Y' \rangle \arrow[d, "\tau"] \\
k\langle X \otimes Y \rangle \arrow[r, "f \otimes g"] & k\langle X' \otimes Y' \rangle
\end{tikzcd}$$

$f \otimes g \in \mathcal{L}(X \otimes Y, X' \otimes Y')$

CORRESPONDING OPERATION ON MATRICES:

$$A \in k\langle X \multimap X' \rangle \qquad B \in k\langle Y \multimap Y' \rangle$$

$$A \otimes B \in k\langle X \otimes Y \multimap X' \otimes Y' \rangle$$

$$(A \otimes B)_{(a,b),(a',b')} = A_{a,a'} \; B_{b,b'}$$

$$\widehat{A \otimes B} = \widehat{A} \otimes \widehat{B}$$

<u>FACT</u> : THE CATEGORY OF F.S. AND CONT. LIN. MAPS IS $*$-AUTONOMOUS

$k\langle X^{\perp}\rangle$ IS THE TOPOLOGICAL DUAL OF $k\langle X\rangle$

(SPACE OF ALL CONT. LIN. FUNCTIONS FROM $k\langle X\rangle$ TO k)

$X \& Y$ IS THE CATEGORICAL PRODUCT OF X AND Y

$X \oplus Y$ ———————— CO-———— ————

REMEMBER THAT $X \& Y = X \oplus Y$

$$k\langle X \& Y\rangle = k\langle X \oplus Y\rangle \cong \underbrace{k\langle X\rangle \times k\langle Y\rangle}_{\text{WITH THE PRODUCT TOPOLOGY.}}$$

IF $f \in \mathcal{L}(X, Z)$ AND $g \in \mathcal{L}(Y, Z)$

THERE IS A "CO-PAIRING" MAP

$$h \in \mathcal{L}(X \oplus Y, Z)$$

I.E. $$h : k\langle X \rangle \times k\langle Y \rangle \longrightarrow k\langle Z \rangle$$

$$h(x, y) = f(x) + g(y)$$

EXPONENTIALS AND POWER SERIES

REMINDER: IN <u>REL</u> WE HAVE A FUNCTOR

$$! : \underline{REL} \to \underline{REL}$$

$$!X = \mathcal{M}_{fin}(X)$$

AND IF $\quad R \subseteq X \times Y$

$$!R \subseteq !X \times !Y$$

$$!R = \{ ([a_1, \dots, a_n], [b_1, \dots, b_n]) \;/\; \forall i \; (a_i, b_i) \in R \}$$

$!$ IS A COTRIPLE (CO-MONAD)

$$\varepsilon \cancel{\delta}_X \subseteq !X \times !!X \qquad \delta_X \subseteq !X \times X$$

$$\delta_X = \{ ([a], a) \;/\; a \in X \}$$

$$\varepsilon_X = \{ (\mu_1 + \cdots + \mu_n, [\mu_1, \dots, \mu_n]) \;/\; \mu_i \in !X \}$$

WE HAVE SEEN HOW TO DEFINE $!X$ WHEN X IS A FINITENESS SPACE:

$$|!X| = \mathcal{M}_{fin}(|X|)$$

$$\mathcal{F}(!X) = \{ U \subseteq |!X| \;/\; \bigcup_{\mu \in U} Supp(\mu) \in \mathcal{F}X \}$$

DERELICTION AND PROMOTION ARE FINITARY: (EXERCISE)

$$\delta_{|X|} \in \mathcal{F}(!X \multimap X)$$

$$\varepsilon_{|X|} \in \mathcal{F}(!X \multimap !!X)$$

THE CO-KLEISLI CONSTRUCTION IN <u>REL</u>:

BUILD A CATEGORY $\mathcal{C}$

- OBJECTS ARE THOSE OF <u>REL</u> (SETS)
- $\mathcal{C}(X,Y) = \underline{REL}(!X, Y)$

$Id_X \in \mathcal{C}(X,X) = \underline{REL}(!X,X)$

WE TAKE $Id_X = \delta_X$

IF $R \in \mathcal{C}(X,Y)$ AND $S \in \mathcal{C}(Y,Z)$

$R \in \underline{REL}(!X,Y)$ $S \in \underline{REL}(!Y,Z)$

$!R \in \underline{REL}(!!X, !Y)$

$S \circ !R \in \underline{REL}(!!X, Z)$

$S \bullet R = S \circ !R \circ \varepsilon_X \in \underline{REL}(!X, Z) = \mathcal{C}(X,Z)$

THE EQUATIONS SATISFIED BY δ AND ε IMPLY THAT THIS DEFINES A CATEGORY

MOREOVER, DUE TO THE ISO $!(X \& Y) \simeq !X \otimes !Y$

$\mathscr{C}$ IS A CCC.

FIRST, ONE HAS TO CHECK THAT

$X \& Y$ IS THE CATEGORICAL PRODUCT OF

X AND Y IN $\mathscr{C}$ (NOT ONLY IN $\underline{REL}$)

$$
\begin{aligned}
\text{THEN: } \mathscr{C}(X \& Y, Z) &= \underline{REL}(\,!(X \& Y), Z) \\
&\simeq \underline{REL}(\,!X \otimes !Y, Z) \\
&\simeq \underline{REL}(\,!X, !Y \multimap Z)
\end{aligned}
$$

SO $Y \Rightarrow Z = !Y \multimap Z$ IS THE OBJECT

OF MORPHISMS FROM Y TO Z (Z^Y) IN $\mathscr{C}$.

LET X BE A FINITENESS SPACE

IF $x \in k\langle X\rangle$ AND $\mu \in |!X|$, SET

$$x^{\mu} = \prod_{a \in |X|} x_a^{\mu(a)} \in k$$

($\mu(a)$ = # OF OCCURRENCES OF a IN μ)

REMARK THAT $x^{\mu} \neq 0 \Rightarrow \mathrm{Supp}\,\mu \subseteq \mathrm{Supp}\,x$

SET $x^! = (x^{\mu})_{\mu \in |!X|} \in k^{|!X|}$

THEN

$$\begin{aligned} \mathrm{Supp}\,x^! &= \{\mu \in |!X| \;/\; x^{\mu} \neq 0\} \\ &\subseteq \{\mu \in |!X| \;/\; \mathrm{Supp}\,\mu \subseteq \mathrm{Supp}\,x\} \\ &= \mathcal{M}_{fi}(\mathrm{Supp}\,x) \in \mathcal{F}(!X) \end{aligned}$$

SINCE $\mathrm{Supp}\,x \in \mathcal{F}X$

SO $x^! \in k\langle !X\rangle$

GIVEN $A \in k\langle !X \to Y\rangle$, WE CAN DEFINE

$$\tilde{A} : k\langle X\rangle \longrightarrow k\langle Y\rangle$$
$$x \longmapsto A \cdot x^!$$

$$\tilde{A}(x) \in k\langle Y\rangle$$

$$\tilde{A}(x)_b = \sum_{\mu \in |!X|} A_{\mu,b}\,(x^!)_\mu$$
$$= \sum_{\mu \in |!X|} A_{\mu,b}\, x^\mu$$

A IS A POWER SERIES FROM X TO Y

(BUT COMPUTING $\tilde{A}(x)_b$ INVOLVES ONLY A FINITE SUMMATION)

<u>EXAMPLES</u> IN ALL CASES, $Y = 1$ $(|Y| = \{*\})$

1) $X = 1$ $\quad |!X| = \{[\,], [*], [*,*], \dots\}$
$\simeq \mathbb{N}$

IF $U \subseteq |!X|$, $\mathrm{Supp}\, U \subseteq \{*\} \in \mathfrak{F}X$

SO $\mathfrak{F}(!X) = \mathcal{P}(\mathbb{N})$.

IF $x \in k\langle X\rangle \simeq k$, $\quad x^! = (x^n)_{n\in\mathbb{N}}$

$!X \multimap Y \simeq (!X)^\perp$

$|(!X)^\perp| = \mathbb{N}$ AND $\mathcal{F}((!X)^\perp) = \mathcal{P}_{fin}(\mathbb{N})$

SO $A \in k\langle (!X)^\perp\rangle$ IS A FAMILY $A = (A_m)_{m\in\mathbb{N}}$

WITH $A_m = 0$ FOR ALL BUT A FINITE NUMBER OF m.

$$\tilde{A}(x) = \sum_{n\in\mathbb{N}} A_n x^n$$

A IS A POLYNOMIAL $\quad k\langle (!1)^\perp\rangle \simeq k[\xi]$

TAKE NOW $X = N$ $\quad (|X| = \mathbb{N},\ \mathcal{F}X = \mathcal{P}_{fin}(\mathbb{N}))$

SEE THE ELEMENTS OF $\mathbb{N}$ AS FORMAL INDETERMINATES

$\xi_0, \xi_1, \xi_2, \dots$

SEE THE ELEMENTS OF $\mathcal{M}_f(\mathbb{N}) = |!X| = |(!X)^\perp|$ AS FINITE FORMAL PRODUCTS OF INDETERMINATES, E.G.

$\mu = [0, 2, 2, 2, 7, 7] \rightsquigarrow \xi_0 \xi_2^3 \xi_7^2$ DENOTED AS ξ^μ

A.K.A. "PRIMITIVE MONOMIALS"

AN ELEMENT $A \in k\langle\langle (!X)^{\perp} \rangle\rangle$ CAN BE SEEN AS A FORMAL LINEAR COMBINATION

$$\sum_{\mu \in |!X|} A_\mu \xi^\mu$$

SUBJECT TO $\mathrm{Supp}\, A \in \mathfrak{F}((!X)^{\perp})$

I.E. FOR ANY GIVEN <u>FINITE</u> $u \subseteq \mathbb{N}$,

THERE ARE ONLY FINITELY MANY μ's

SUCH THAT $A_\mu \neq 0$ AND $\mathrm{Supp}\,\mu \subseteq u$

SO THAT WHEN $x \in k\langle X \rangle = k^{(\mathbb{N})}$

THE SUM $\sum_{\mu \in |!X|} A_\mu x^\mu$ IS FINITE.

EXAMPLE OF SUCH AN A:

$$A = \sum_{m \in \mathbb{N}} \xi_m^2$$

$$A = \sum_{m \in \mathbb{N}} \xi_0^m \xi_m = \xi_0 + \xi_0 \xi_1 + \xi_0^2 \xi_2 + \xi_0^3 \xi_3 + \cdots$$

EXAMPLE (CONTINUED)

DEFINE $\mathrm{Bool} = 1 \oplus 1$

$$|\mathrm{Bool}| = \{ tt, ff \}$$

$$k\langle \mathrm{Bool} \rangle \simeq k^2$$

THE SPACE OF ALL FORMAL LINEAR COMBINATIONS

$$\alpha\, tt + \beta\, ff \qquad \alpha, \beta \in k$$

X ARBITRARY F. S.

$$\mathrm{If} : \mathrm{Bool} \multimap !X \multimap !X \multimap X$$
$$\mathrm{Bool} \otimes !X \otimes !X \multimap X$$

$\mathrm{If} \in k\langle \mathrm{Bool} \otimes !X \otimes !X \multimap X \rangle$, GIVEN BY

$$\mathrm{If}_{b,\lambda,\rho,a} = \begin{cases} 1 & \text{IF } b = tt, \lambda = [a], \rho = [\,] \\ 1 & \text{IF } b = ff, \lambda = [\,], \rho = [a] \\ 0 & \text{OTHERWISE} \end{cases}$$

WE CAN COMPOSE ~~THE~~ If WITH DERELICTION (FORGET THAT IT IS LINEAR IN Bool)

$$\mathrm{If} : !\mathrm{Bool} \otimes !X \otimes !X \multimap X$$
$$!(\mathrm{Bool} \,\&\, X \,\&\, X) \multimap X$$

$$\widetilde{\mathrm{If}}(\alpha\, tt + \beta\, ff, x, y) = \alpha x + \beta y$$

A PARANOÏD PROGRAM $\quad P : Bool \to X \to X \to X$

$$P(\gamma, x, y) = If(\gamma, If(\gamma, x, y), If(\gamma, y, x))$$

THE CORRESPONDING POWER SERIES IN

$$k\langle\, !(Bool \,\&\, X \,\&\, X) \multimap X \,\rangle$$

IS

$$\begin{aligned} & P(\alpha tt + \beta ff, x, y) \\ &= \alpha(\alpha x + \beta y) + \beta(\alpha y + \beta x) \\ &= (\alpha^2 + \beta^2)x + 2\alpha\beta y \end{aligned}$$

! MUST BE A FUNCTOR.

$$A \in k\langle X \multimap Y\rangle$$

$$!A \in k\langle !X \multimap !Y\rangle$$

LET $c \in k\langle (!Y)^{\perp}\rangle \cong k\langle !Y \multimap \perp\rangle$

$$\tilde{c} : k\langle Y\rangle \to k$$

IF $y \in k\langle Y\rangle$, $\tilde{c}(y) = \langle c, y^{!}\rangle$

LET $x \in k\langle X\rangle$ $A.x \in k\langle Y\rangle$

$$\tilde{c}(A.x) = \langle c, (A.x)^{!}\rangle$$

ON THE OTHER HAND, WE CAN COMPOSE

$$!X \xrightarrow{!A} !Y \xrightarrow{c} \perp$$

$$c\,!A \in k\langle (!X)^{\perp}\rangle$$

AND APPLY THIS POWER SERIES TO x

$$\widetilde{c\,!A}\,(x) = \langle c\,!A, x^{!}\rangle$$

$$= \langle c, !A.x^{!}\rangle$$

THE BASIC REQUIREMENT ON $!A$ IS THAT THE 2 THINGS COINCIDE:

$$\forall x \in k\langle X\rangle \quad (A.x)^{!} = !A.x^{!}$$

AFTER SOME COMPUTATIONS, ONE OBTAINS:

$$(!A)_{\lambda,\rho} = \sum_{\sigma \in L(\lambda,\rho)} \begin{bmatrix} \rho \\ \sigma \end{bmatrix} A^{\sigma}$$

$$\lambda \in |!X| = \mathcal{M}_{fin}(|X|) \qquad \rho \in |!Y|$$

$$L(\lambda,\rho) = \Big\{ \sigma \in \mathcal{M}_{fin}(|X| \times |Y|) \;/\; \forall a \in |X| \;\; \sum_{b \in |Y|} \sigma(a,b) = \lambda(a) \quad \forall b \in |Y| \;\; \sum_{a \in |X|} \sigma(a,b) = \rho(b) \Big\}$$

$$\begin{bmatrix} \rho \\ \sigma \end{bmatrix} = \frac{\prod_{b \in |Y|} \rho(b)!}{\prod_{\substack{a \in |X| \\ b \in |Y|}} \sigma(a,b)!} \in \mathbb{N}$$

$$A^{\sigma} = \prod_{\substack{a \in |X| \\ b \in |Y|}} A_{a,b}^{\sigma(a,b)}$$

REMARK:

$$\mathrm{Supp}(!A) \subseteq \; ! \; \mathrm{Supp}(A) = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \;/\; \forall i \;\; (a_i,b_i) \in \mathrm{Supp}\, A\}$$

FUNCTOR ! IN REL.

SO $\mathrm{Supp}(!A) \in \mathcal{F}(!X \multimap !Y)$, IE $!A \in k\langle !X \multimap !Y \rangle$.

WEAKENING AND CONTRACTION

$\top$, THE TERMINAL OBJECT, NEUTRAL ELEMENT OF &

$|\top| = \emptyset \qquad \mathcal{F}\top = \{\emptyset\}$

$0 \in \mathbf{k}\langle X \multimap \top\rangle$ (UNIQUE)

$w_X = !0 \in \mathbf{k}\langle !X \multimap !\top\rangle$ BUT $!\top = 1$

CORRESPONDS TO WEAKENING $\quad (w_X)_{\mu,*} = \begin{cases} 1 & \text{IF } \mu = [\,] \\ 0 & \text{OTHERWISE} \end{cases}$

$Y \xrightarrow{f} Z$

$!X \otimes Y \xrightarrow{w_X \otimes Y} 1 \otimes Y \xrightarrow{f} Z \qquad (1 \otimes Y = Y)$

$\Delta_X \in \mathbf{k}\langle X \multimap X \& X\rangle$ (DIAGONAL

$(\Delta_X)_{a,(i,b)} = \begin{cases} 1 & \text{IF } a = b \\ 0 & \text{OTHERWISE} \end{cases}$,

$\Delta_X \cdot x = (x, x)$

WE GET $\quad !\Delta_X \in \mathbf{k}\langle !X \multimap !(X \& X)\rangle$

BUT $\quad !(X \& X) \simeq !X \otimes !X \quad$ (AS FIN. SPACES)

FINALLY WE GET $\quad c_X \in \mathbf{k}\langle !X \multimap !X \otimes !X\rangle$

EXERCISE : SHOW THAT

$$(c_X)_{\mu,(\alpha,\rho)} = \begin{cases} 1 & \text{IF } \mu = \alpha + \rho \\ 0 & \text{OTHERWISE} \end{cases}$$

CORRESPONDS TO CONTRACTION

$$!X \otimes Y \xrightarrow{c_X \otimes Y} !X \otimes !X \otimes Y \xrightarrow{f} Z$$

CO-TRIPLE STRUCTURE

JUST SEE $\delta_X \in \mathcal{F}(!X \multimap X)$ AND $\varepsilon_X \in \mathcal{F}(!X \multimap !!X)$ AS MATRICES WITH 0 AND 1 COEFF,

$$(\delta_X)_{\mu,a} = \begin{cases} 1 & \text{IF } \mu = [a] \\ 0 & \text{OTHERWISE} \end{cases}$$

$$(\varepsilon_X)_{\mu,[\mu_1,\dots,\mu_n]} = \begin{cases} 1 & \text{IF } \mu = \mu_1 + \dots + \mu_n \\ 0 & \text{OTHERWISE} \end{cases}$$

FACT : THE COMPOSITION IN $\mathscr{C}$ IS THE USUAL COMPOSITION OF POWER SERIES

$$A \in k\langle !X \multimap Y \rangle$$

$$B \in k\langle !Y \multimap Z \rangle$$

$$C \in k\langle !X \multimap Z \rangle$$

$$C: \quad !X \xrightarrow{\varepsilon_X} !!X \xrightarrow{!A} !Y \xrightarrow{B} Z$$

$$C = B \; !A \; \varepsilon_X$$

SATISFIES $\tilde{C}(x) = \tilde{B}(\tilde{A}(x))$

AND OF COURSE $\tilde{\delta}_X(x) = x$

ADDITIONAL STRUCTURE

WHAT FOLLOWS HAS NO SYNTACTIC COUNTERPART IN LINEAR LOGIC.

$\partial^0_X \in k\langle X \multimap\, !X\rangle$ THE "DERIVATION AT 0" MAP

$$(\partial^0_X)_{a,\mu} = \begin{cases} 1 & \text{IF } \mu = [a] \\ 0 & \text{OTHERWISE} \end{cases}$$

$A \in k\langle !X \multimap Y\rangle$ BEING GIVEN,

$A\,\partial^0_X \in k\langle X \multimap Y\rangle$

$$(A\,\partial^0_X)_{a,b} = A_{[a],b}$$

SO, IF $x \in k\langle X\rangle$, $A\,\partial^0_X \cdot x$

CONTAINS ALL <u>LINEAR</u> TERMS OF $\tilde{A}(x)$

$$\tilde{A}(x) = \tilde{A}(0) + A\,\partial^0_X \cdot x + \tilde{C}(x)$$

WHERE $\tilde{C} \in k\langle !X \multimap Y\rangle$ SATISFIES:

$$\tilde{C}_{\mu,b} = 0 \quad \text{AS SOON AS } \#\mu \leq 1$$

$A\,\partial^0_X$ IS THE DERIVATIVE OF A AT 0.

SO WE CAN COMPUTE THE DERIVATIVE OF

$$\cancel{A\in}\quad f \in \mathcal{L}(!X, Y) = \mathcal{C}(X, Y)$$

AT POINT 0, AND IT IS AN ELEMENT $f'(0)$

OF $\qquad \mathcal{L}(X, Y)$

AND IF WE WANT TO COMPUTE $f'(x)$ FOR

AN ARBITRARY $x \in k\langle X \rangle$?

SINCE $\& = \oplus$, WE HAVE A CO-DIAGONAL

$$a_X \in k\langle X \& X \multimap X \rangle$$

WHICH IS JUST ADDITION : $a_X(x, y) = x + y$

$$m_X = !a_X \in k\langle\, !(X \& X) \multimap !X \rangle$$

THAT IS (UP TO ISO) $\quad m_X \in k\langle !X \otimes !X \multimap !X \rangle$

EXERCISE :
$$(m_X)_{d,\rho,\mu} = \begin{cases} \binom{\mu}{d} & \text{IF } d+\rho=\mu \\ 0 & \text{OTHERWISE} \end{cases}$$

WHERE $\qquad \binom{\mu}{d} = \prod_{a\in|X|} \dfrac{\mu(a)!}{d(a)!\,\rho(a)!} \in \mathbb{N} \quad (d+\rho=\mu)$

IS THE GENERALIZED BINOMIAL COEFFICIENT.

m_X IS CHARACHTERIZED BY

$$m_X(x^!, y^!) = (x+y)^!$$

REMARK $\quad m_X \in k\langle !X \otimes !X, \multimap !X\rangle$

CAN BE SEEN AS BILINEAR HYPOCONT. MAP ON $k\langle !X\rangle \times k\langle !X\rangle$, AND DEFINES AN ASSOCIATIVE COMMUTATIVE MULTIPLICATION.

UNIT: $\quad \mathbb{1} \in k\langle !X\rangle \qquad \mathbb{1}_\mu = \begin{cases} 1 & \text{IF } \mu = [\,] \\ 0 & \text{OTHERWISE.} \end{cases}$

GIVEAN $\quad A \in k\langle !X \multimap Y\rangle$

$$\begin{aligned} B = A\, m_X &\in k\langle !X \otimes !X \multimap Y\rangle \\ &\simeq k\langle !(X \& X) \multimap Y\rangle \end{aligned}$$

$$\breve{B}(x,y) = \tilde{A}(x+y)$$

TO COMPUTE THE DERIVATIVE OF A AT $x \in k\langle X\rangle$ COMPUTE THE DERIVATIVE OF $\tilde{A}(x + \_)$ AT POINT 0.

CATEGORICALLY: $\qquad$ DEFINE $\partial_X : !X \otimes X \multimap !X$ AS

$$!X \otimes X \xrightarrow{!X \otimes \partial_X^0} !X \otimes !X \xrightarrow{m_X} !X$$

IF $\quad A : !X \multimap Y$

$\qquad A\,\partial_X : !X \otimes X \multimap Y$

USING MONOIDAL CLOSEDNESS, WE GET

$$A' : !X \multimap (X \multimap Y)$$

IN OTHER WORDS, STARTING FROM $f \in \mathscr{C}(X, Y)$

WE ARRIVE TO $f' \in \mathscr{C}(X, X \multimap Y)$

THIS CAN BE ITERATED

$$f^{(n)} \in \mathscr{C}(X, \underbrace{X \multimap X \multimap \cdots \multimap X}_{n} \multimap Y)$$

$$\text{IF} \quad f^{(n)} \in \mathscr{C}(X, X^{\otimes n} \multimap Y)$$

$f^{(n)}(x)$ IS A SYMMETRIC n-LINEAR MAP

TAYLOR FORMULA:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) . x^{\otimes n}$$

THIS SERIES CONVERGES IN $k\langle Y \rangle$.

SYNTACTIC OUTCOME : DIFFERENTIAL λ-CALCULUS
AND PROOF-NETS
(WITH LAURENT REGNIER)

TERMS :

* Variables x
* ORDINARY APPLICATION $(s)t$
* ABSTRACTION $\lambda x\, t$
* DIFFERENTIAL APPLICATION $D_i\, t \cdot u$

TYPING :
$$\frac{\Gamma \vdash t : A_1 \to \cdots \to A_i \to B \qquad \Gamma \vdash u : A_i}{\Gamma \vdash \underbrace{D_i t . u}_{} : A_1 \to \cdots \to A_i \to B}$$

IDEA : THE DERIVATIVE OF t W.R.T. ITS i-TH ARGUMENT, $D_i\, t$, SHOULD BE OF TYPE $A_1 \to \cdots \to A_i \to (A_i \multimap B)$

TO AVOID INTRODUCING LINEAR APPLICATION, WE PROVIDE $D_i\, t$ WITH ITS LINEAR ARGUMENT $u : A_i$

$D_i\, t$ CAN BE RECOVERED AS
$\lambda x\; D_i t . x$
IF NEEDED.

* LINEAR COMBINATION (COEFF IN k)

$$\sum \alpha_i t_i \qquad (\text{ALSO } 0)$$

TYPING: $\dfrac{}{\Gamma \vdash 0 : A}$ $\qquad \dfrac{\Gamma \vdash t_i : A}{\Gamma \vdash \sum \alpha_i t_i : A}$

EQUATIONS (CAN BE AVOIDED CONSIDERING CANONICAL FORMS)

* ALL CONSTRUCTIONS ARE LINEAR,

E.G. $D_i\left(\sum_p \alpha_p t_p\right) \cdot \left(\sum_q \beta_q u_q\right) = \sum_{p,q} \alpha_p \beta_q D_i t_p \cdot u_q$

WITH ONLY ONE EXCEPTION:

WE DO NOT HAVE $(s)\left(\sum \alpha_i t_i\right) = \sum_i \alpha_i (s) t_i$

* $D_i (s)t \cdot u = (D_{i+1} s \cdot u) t$
* $D_{i+1} (\lambda x\, s) \cdot u = \lambda x (D_i s \cdot u)$
* $D_i (D_j t \cdot u) \cdot v = D_j (D_i t \cdot v) \cdot u$

$s : A_1 \to A_2 \to B \qquad t : A_1 \qquad u : A_2$

$(s) t : A_2 \to B \qquad D_1 (s) t \cdot u : A_2 \to B$

$D_2 s \cdot u : A_1 \to A_2 \to B \qquad \overset{=}{(D_2 s \cdot u) t} : A_2 \to B$

REDUCTION RULES

* β - REDUCTION, AS USUAL

$$(\lambda x\ s)\,t \to s[t/x]$$

* DIFFERENTIAL REDUCTION

$$D_1(\lambda x\ s)\cdot u \to \lambda x\left(\frac{\partial s}{\partial x}\cdot u\right)$$

WHERE " $\frac{\partial s}{\partial x}\cdot u$ " IS AN OPERATION ANALOGOUS TO SUBSTITUTION, TO BE DEFINED BY INDUCTION ON s.

$\frac{\partial s}{\partial x}\cdot u$ SIMILAR TO $s[u/x]$

WITH ONE ESSENTIAL DIFFERENCE:

$\frac{\partial s}{\partial x}\cdot u$ IS LINEAR IN u

$s[u/x]$ IS NOT, IN GENERAL.

DEFINITION OF $\frac{\partial s}{\partial x} . u$:

* $s = y$ $\qquad \frac{\partial y}{\partial x} . u = \begin{cases} u & \text{IF } y = x \\ 0 & \text{OTHERWISE} \end{cases}$

WHEN $y \neq x$, THIS CORRESPONDS TO DERIVATING A CONSTANT.

* $$\frac{\partial (dy\, t)}{\partial x} . u = dy \left(\frac{\partial t}{\partial x} . u \right)$$

* $$\frac{\partial D_i t . v}{\partial x} . u = D_i \left(\frac{\partial t}{\partial x} . u \right) . v + D_i t . \left(\frac{\partial v}{\partial x} . u \right)$$

BECAUSE $D_i t . v$ IS LINEAR IN t AND IN v (BILINEAR IN (t, v)).

* OF COURSE $\frac{\partial}{\partial x} \left(\sum \alpha_i t_i \right) . u = \sum \alpha_i \frac{\partial t_i}{\partial x} . u$

* $$\frac{\partial (s) t}{\partial x} . u = \left(\frac{\partial s}{\partial x} . u \right) t + \left(D_1 s . \left(\frac{\partial t}{\partial x} . u \right) \right) t$$

IN $(s) t$, s IS IN LINEAR POSITION

t IS NOT IN LINEAR POSITION

$x : C \vdash s : A \to B \quad , \quad t : A \quad \Gamma \vdash u : C$

$x : C \vdash (s)\, t : B$

[GENERAL FACT : WHEN $\Gamma, x : A \vdash t : B$
AND $\Gamma \vdash u : A$
ONE HAS $\Gamma, x : A \vdash \frac{\partial t}{\partial x} \cdot u : B$]

$$x : C \vdash \frac{\partial (s) t}{\partial x} \cdot u : B$$

$$x : C \vdash \frac{\partial t}{\partial x} \cdot u : A$$

$$x : C \vdash D_1 s \cdot \left(\frac{\partial t}{\partial x} \cdot u\right) : A \to B$$

$$x : C \vdash \left(D_1 s \cdot \left(\frac{\partial t}{\partial x} \cdot u\right)\right) t : B$$

$\frac{\partial t}{\partial x} \cdot u$ IS LINEAR IN u

$(s)\left(\frac{\partial t}{\partial x} \cdot u\right) : B$ IS NOT LINEAR IN u

ONE HAS FIRST TO "MAKE s LINEAR IN ITS ARGUMENT" AND THAT'S THE ROLE OF D_1

$D_1 s \cdot \left(\frac{\partial t}{\partial x} \cdot u\right)$ LINEAR IN u.

$\frac{\partial t}{\partial x} \cdot u$ IS LINEAR SUBSTITUTION OF x BY u IN t, I.E. SUBSTITUTION OF <u>EXACTLY ONE</u> OCCURRENCE OF x BY u.

BUT IN GENERAL x HAS SEVERAL OCCURRENCES, AND SO THERE ARE SEVERAL POSSIBILITIES: THAT'S WHY WE HAVE SUMS OF TERMS.

EXAMPLE : IN $(x)((x)y)$, x HAS ~~TH~~ TWO OCCURRENCES

LINEAR NOT LINEAR

$$\frac{\partial (x)(x)y}{\partial x} \cdot u = (u)(x)y + \left(D_1 x \cdot \left(\frac{\partial (x)y}{\partial x} \cdot u\right)\right)(x)y$$

$$= (u)(x)y + (D_1 x \cdot (u)y)(x)y$$

THE PROBLEM IS THAT WE DON'T KNOW YET ~~WHAT~~ WHAT WILL BE THE VALUE OF x: IT CAN BE A LINEAR FUNCTION, AND THEN THE 2nd OCC. OF x WILL BE LINEAR. BUT IF THE VALUE OF x IS NOT A LINEAR FN., THE SECOND OCCURRENCE WILL NOT BE LINEAR.

EXAMPLE : A REDUCTION IN DIFF. d-CALCULUS

$(dx\ D_1 x . x)\ dx\,(x)x$

$$\to\quad D_1(dx\,(x)x)\ .\ dx\,(x)x \qquad (\beta)$$

$$\to\quad dx\left(\frac{\partial (x)x}{\partial x} . dy\,(y)y\right) \qquad (\beta\text{-DIFF})$$

$$=\ dx\left(\left(\frac{\partial x}{\partial x} . dy\,(y)y\right)x + \left(D_1 x . \left(\frac{\partial x}{\partial x} . dy\,(y)y\right)\right)x\right)$$

$$=\ dx\left((dy\,(y)y)\,x + (D_1 x . dy\,(y)y)\,x\right)$$

$$\to\quad dx\,(x)x\ +\ dx\,(D_1 x . dy\,(y)y)\,x \qquad (\beta)$$

ITERATED DERIVATIVES AND TAYLOR FORMULA

$$t : A \to B \qquad u_1, \dots, u_n : A$$

$$D_1(\cdot\cdot(D_1\ (D_1 t . u_1) . u_2) \cdots) . u_n\ =\ D_1^n t . (u_1, \dots, u_n)$$

$$= D_1^n t . (u_{\sigma 1}, \dots, u_{\sigma n})$$

$$\sigma \in \mathfrak{S}_n .$$

IF $u : A$, $\quad D_1^n t . u^n = D_1^n t . (\underbrace{u, \dots, u}_{n})$

$$D_1^n t . u^n : A \to B, \quad \text{SO} \quad (D_1^n t . u^n)0 : B$$

THE FORMAL SUM $\sum \frac{1}{n!}(D_1^n t . u^n)\,0$ (NOT PART OF THE PRESENT SYNTAX) IS THE TAYLOR EXPANSION OF $(t)u$

MEANING OF THE TAYLOR FORMULA:

LET t, u BE ORDINARY λ-TERMS, AND

ASSUME THAT $(t)u \sim_\beta *$ (A VARIABLE)

TAYLOR EXPANSION OF THE APPLICATION $(t)u$:

$$\sum_n \frac{1}{n!}(D_1^n t . u^n)0$$

FACT: THERE IS EXACTLY ONE $n \in \mathbb{N}$ S.T.

$(D_1^n t . u^n) 0 \not\sim 0$ (IN THE DIFF. λ-CALCULUS)

AND FOR THIS n, $(D_1^n t . u^n) 0 \sim n! . *$

INITIALIZATION: $n := 0$

(1) BECAUSE $(t)u \sim_\beta *$, THERE ARE 2 POSSIBILITIES:

- EITHER $t \sim_\beta \lambda x\, *$ EXIT
- OR $t \sim_\beta \lambda x\ (x) s_1 \cdots s_k = \lambda x\ (x) \bar{s}$ }

THEN $(\lambda x\ (x) \bar{s})\, u \sim_\beta *$

$(u)\, \bar{s}[u/x] \sim_\beta *$

BUT

$$(\lambda x\ (u)\bar{s}\ )u \ \rightsquigarrow_\beta\ (u)\bar{s}[u/x]$$

SO $(\lambda x\ (u)\bar{s}\ )u \ \rightsquigarrow_\beta *$

SET $t = \lambda x\ (u)\bar{s}$ AND GOTO (1)

$m := m+1$

THIS PROGRAMS TERMINATES AND OUTPUTS THE VALUE OF m S.T. $(D_1^m t . u^m)0 \sim m! *$

EXAMPLE : $t = \lambda f\ (f)\underbrace{\lambda x\ (f)\lambda d\ x}_{h}$ $\quad u = \lambda x\ (x)(x)*$

$(\lambda f\ (f)h\ )u$

1. $(\lambda f\ (u)h\ )u$

$\rightarrow_\beta\ (\lambda f\ \ (h)(h)*\ )u$

$=\ (\lambda f\ \ (\lambda x\ \ (f)\lambda d\ x)(h)*\ )u$

$\rightarrow_\beta\ (\lambda f\ \ (f)\lambda d\ (h)*\ )u$

2. $(\lambda f\ \ (u)\lambda d\ (h)*)u$

$\rightarrow_\beta\ (\lambda f\ \ (\lambda d\ (h)*)(\lambda d\ (h)*)*\ )u$

$\rightarrow_\beta\ (\lambda f\ \ (h)*)u$

$\rightarrow_\beta\ (\lambda f\ \ (f)\lambda d\ *)u$

3. $(\lambda f\ \ (u)\lambda d\ *)u$

$\rightarrow_\beta\ (\lambda f\ \ (\lambda d\ *)(\lambda d\ *)*)u$

$\rightarrow_\beta\ (\lambda f\ *)u$ $\quad$ END $\quad m=3$