Presheaf models for concurrency

$\mathbb{P}$ category of paths, and extensions

Category of presheaves over $\mathbb{P}$:

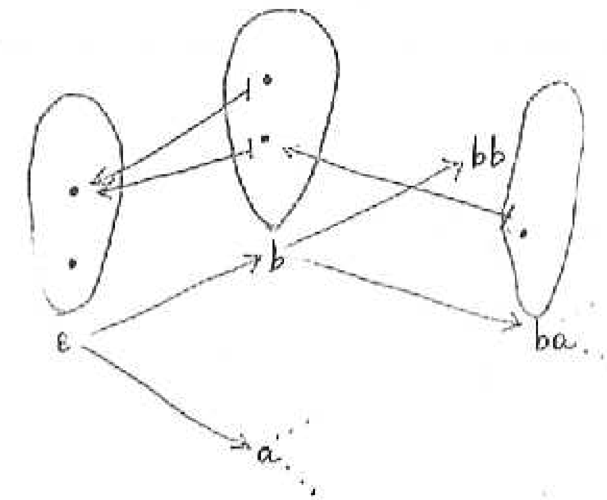
$$\widehat{\mathbb{P}} = [\mathbb{P}^{op}, Set]$$

- A presheaf as "transition system" with path shapes in $\mathbb{P}$.
  Open maps & bisimulation on presheaves.
- $\widehat{\mathbb{P}}$ as a free colimit completion
- If $F : \widehat{\mathbb{P}} \to \widehat{\mathbb{Q}}$ is colimit preserving, then F preserves open maps & bisimulation.
- The category of presheaf categories as a domain theory.

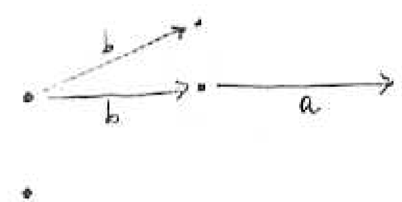
$\mathbb{P}$ – strings L^*

$\widehat{\mathbb{P}}$ – "synchronisation forests"

E.g. $L = \{a, b\}$



Cf.



Synchronisation trees over L equivalent to "rooted" presheaves over L^*,

or as presheaves over L^+.

The category of presheaves over $\mathbb{P}$:

$$\widehat{\mathbb{P}} = [\mathbb{P}^{op}, Set]$$

A presheaf X:

$X : \mathbb{P}^{op} \to Set$

$$\begin{tikzcd} X(P) & X(P') \arrow[l, "X_m"'] \\ P \arrow[r, "m"] & P' \end{tikzcd}$$

A map of presheaves (a natural transformation):

$$\begin{tikzcd} X \arrow[d, "f"] & X(P) \arrow[d, "f_P"'] & X(P') \arrow[l, "X_m"'] \arrow[d, "f_{P'}"] \\ Y & Y(P) & Y(P') \arrow[l, "Y_m"'] \\ & P \arrow[r, "m"] & P' \end{tikzcd}$$

Presheaves as "transition systems":

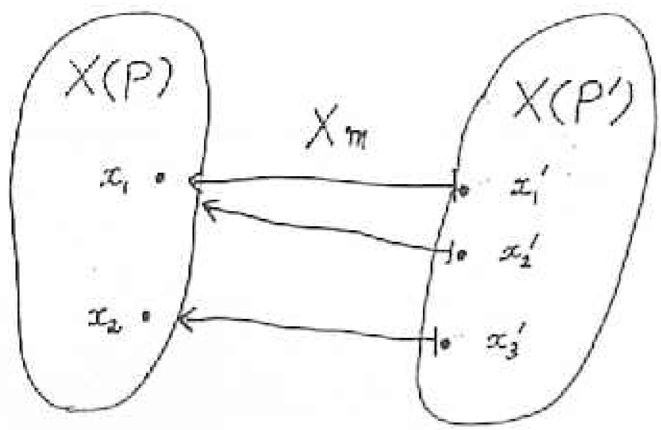
$X \in \widehat{\mathbb{P}}$ Its category of elements $el(X)$

$$\begin{tikzcd} X(P) & X(P') \\ P{:}x_1 \arrow[r, "m"] \arrow[dr, "m"'] & P'{:}x_1' \\ & P'{:}x_2' \\ P{:}x_2 \arrow[r, "m"] & P'{:}x_3' \\ P \arrow[r, "m"] & P' \end{tikzcd}$$

$X \xrightarrow{f} Y$ yields $el(X) \xrightarrow{el(f)} el(Y)$, a functor.

Presheaves as "transition systems":

$X \in \widehat{\mathbb{P}}$

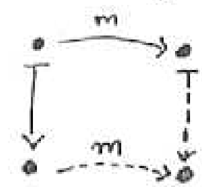


$P \xrightarrow{m} P'$

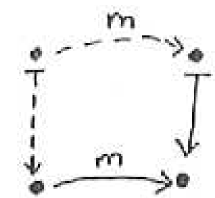
$$\begin{tikzcd} X \arrow[d, "f"'] & el(X) \arrow[d, "el(f)"'] \\ Y & el(Y) \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "m"] & Q \end{tikzcd}$$

$$\begin{tikzcd} X(P) \arrow[d, "f_P"'] & X(Q) \arrow[l, "X_m"'] \arrow[d, "f_Q"] \\ Y(P) & Y(Q) \arrow[l, "Y_m"] \end{tikzcd}$$

$el(f)$ satisfies:

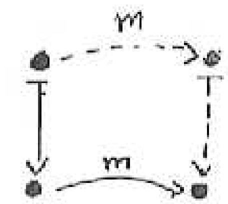


$el(f)$ a functor



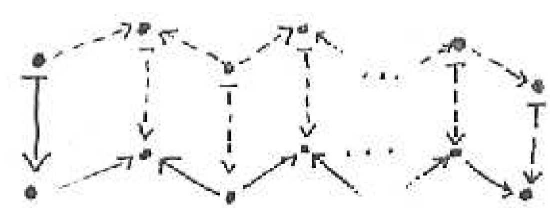
by naturality

If f is open,



"fnl. bisim"

If f is open, $el(f)$ reflects "zig-zags":



Yoneda embedding:

$$\mathbb{P} \xrightarrow{y} \widehat{\mathbb{P}}$$

$$\begin{tikzcd} P \arrow[d, "m"'] & \longmapsto & \mathbb{P}(-,P) \arrow[d, "ym"'] \\ Q & & \mathbb{P}(-,Q) \end{tikzcd} \qquad (ym)_R = m\circ -$$

y is full & faithful.

Yoneda lemma:

$$X(P) \overset{\widetilde{(-)}}{\cong} \widehat{\mathbb{P}}(y(P), X)$$

natural in P, X.

natural in P:

$$\begin{tikzcd} P \arrow[d, "m"] & X(P) \arrow[r, phantom, "\cong"] & \widehat{\mathbb{P}}(y(P), X) \\ Q & X(Q) \arrow[u, "X m"'] \arrow[r, phantom, "\cong"] & \widehat{\mathbb{P}}(y(Q), X) \arrow[u, "-\circ ym"'] \end{tikzcd}$$

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natural in X:

$$\begin{tikzcd} X \arrow[d, "f"'] & X(P) \arrow[d, "f_P"'] \arrow[r, phantom, "\cong"] & \widehat{\mathbb{P}}(y(P), X) \arrow[d, "f\circ -"] \\ Y & Y(P) \arrow[r, phantom, "\cong"] & \widehat{\mathbb{P}}(y(P), Y) \end{tikzcd}$$

Open maps in presheaves (i)

$$\mathbb{P} \xrightarrow{y} \widehat{\mathbb{P}}$$

$f : X \to Y$ is open in $\widehat{\mathbb{P}}$

iff whenever

$$\begin{tikzcd}
yP \arrow[r, "p"] \arrow[d, "ym"'] \arrow[dr, phantom, "="] & X \arrow[d, "f"] \\
yQ \arrow[r, "q"'] & Y
\end{tikzcd}$$

then $\exists r$ s.t.

$$\begin{tikzcd}
yP \arrow[r, "p"] \arrow[d, "ym"'] & X \arrow[d, "f"] \\
yQ \arrow[r, "q"'] \arrow[ur, "r"] & Y
\end{tikzcd}$$

Open maps in presheaves (ii)

$f : X \to Y$ is open in $\widehat{\mathbb{P}}$.

iff $\forall m : P \to Q$ in $\mathbb{P}$.

$$\begin{tikzcd}
X(P) \arrow[d, "f_P"'] & X(Q) \arrow[l, "Xm"'] \arrow[d, "f_Q"] \\
Y(P) & Y(Q) \arrow[l, "Ym"]
\end{tikzcd}$$

is a q.p.b.!

$$
\begin{tikzcd}
p \in \mathrm{Nat}(yP, X) \arrow[d, "f \circ -"'] & \mathrm{Nat}(yQ, X) \arrow[l, "- \circ ym"'] \arrow[d, "f \circ -"] \\
\mathrm{Nat}(yP, Y) & \mathrm{Nat}(yQ, Y) \ni q \arrow[l, "- \circ ym"]
\end{tikzcd}
$$

Bisimulation on presheaves

$$
\begin{tikzcd}
& R \arrow[dl, "\varnothing"'] \arrow[dr, "\varnothing"] & \\
X_1 & & X_2
\end{tikzcd}
$$

surjective open maps

Alternative definition

$$\mathbb{P}_\perp \xrightarrow{y_0} \widehat{\mathbb{P}} \quad \text{strict Yoneda}$$

$$\perp \mapsto \varnothing \quad \text{empty presheaf}$$

$$P \mapsto y(P) = \mathbb{P}(-, P)$$

A map is $\mathbb{P}_\perp$-open iff

it is surjective $\mathbb{P}$-open.

A colimit of $\mathbb{I} \xrightarrow{D} \mathcal{E}$ is a colimiting cone

$$\begin{tikzcd} & \bullet & \\ & \mathrm{colim}(\mathbb{I} \xrightarrow{D} \mathcal{E}) \arrow[u, dashed] & \\ D(I) \arrow[rr, "Dm"] \arrow[ur, "c_I"] \arrow[uur, bend left] & & D(J) \arrow[ul, "c_J"'] \arrow[uul, bend right] \end{tikzcd}$$

When $\mathcal{E} = \underline{Set}$, $\qquad I \xrightarrow{m} J$

$$\mathrm{colim}(\mathbb{I} \xrightarrow{D} Set) = \dot{\bigcup_{I \in \mathbb{I}}} D(I) \big/ \sim$$

$$c_I(u) = \{I : u\}_\sim$$

where

$\sim$ is least equiv. reln. s.t.

$$I:u \sim J:v \text{ if}$$

$$\exists m : I \to J. \quad (Dm)(u) = v$$

So $I:u \sim J:v$ iff

$$\begin{tikzcd}[column sep=small] u \bullet \arrow[r, mapsto, "Dm_1"] & \bullet & \bullet \arrow[l, mapsto, "Dm_2"'] \arrow[r, mapsto] & \bullet & \cdots & \bullet & \bullet\, v \arrow[l, mapsto, "Dm_k"'] \\ I \arrow[r, "m_1"] & I_1 & I_2 \arrow[l, "m_2"'] \arrow[r] & \cdot & \cdots & I_{k-1} & J \arrow[l, "m_k"'] \end{tikzcd}$$

"Density"

A presheaf is a colimit of representables:

$$X \cong \mathrm{colim}\,(el(X) \xrightarrow{\pi_X} \mathbb{P} \xrightarrow{y} \widehat{\mathbb{P}})$$

colimiting cone:

$$\begin{tikzcd} & X & \\ y(P) \arrow[ur, "\tilde{x}"] \arrow[rr, "ym"'] & & y(Q) \arrow[ul, "\tilde{y}"'] \end{tikzcd}$$

$$P:x \xrightarrow{m} Q:y$$

A preservation property for open maps.

$$\begin{tikzcd} \mathbb{P} \arrow[r, "y"] \arrow[d, "F"'] & \widehat{\mathbb{P}} \arrow[dl, "L"] \\ \widehat{\mathbb{Q}} & \end{tikzcd}$$

$L =_{def} Lan_y F$ "left Kan extn. of F along y"

[! ($\cong$) colim pres^g]

Defn. of L on objects:

$$L(X) = \mathrm{colim}\left(el(X) \xrightarrow{\pi_X} \mathbb{P} \xrightarrow{F} \widehat{\mathbb{Q}}\right)$$

On morphisms:

$$\begin{tikzcd} X \arrow[d, "h"'] & \mapsto & L(X) \arrow[d, "L(h)"] \\ Y & & L(Y) \end{tikzcd}$$

the ! mediating morphism from the colimit $L(X)$ to $L(Y)$:

$$\begin{tikzcd} el(X) \arrow[d, "el(h)"'] \arrow[dr, "F\circ\pi_X"] & \\ el(Y) \arrow[r, "F\circ\pi_Y"'] & \widehat{\mathbb{Q}} \end{tikzcd}$$

$(P,x) \mapsto (P, h_P(x))$

Thm. h is $\mathbb{P}$-open $\Rightarrow L(h)$ is $\mathbb{Q}$-open.

Cor. Colim. pres^g functors $\widehat{\mathbb{P}} \to \widehat{\mathbb{Q}}$ preserve open maps.

$$(LY)(Q) = \biguplus_{(P,y) \in el(Y)} (FP)(Q) \Big/ \sim$$

where $\sim$ is the least equiv. reln. s.t.

$$(P,y):u \sim (P',y'):v$$

if $\exists m\ (P,y) \xrightarrow{m} (P',y')$ in $el(Y)$

& $(Fm)_Q(u) = v$

ie.

$(P,y):u \sim (P',y'):v$

iff

$$u \cdot \xrightarrow{(Fm_1)_Q} \cdot \xleftarrow{(Fm_2)_Q} \cdot \;\cdots\; \cdot \rightarrow \cdot \xleftarrow{(Fm_k)_Q} \cdot v$$

$$(P,y) \xrightarrow{m_1} \cdot \xleftarrow{m_2} \cdot \;\cdots\; \cdot \rightarrow \cdot \xleftarrow{m_k} (P',y')$$

$$\begin{tikzcd}
X \arrow[d, "h"'] & (P,x):(FP)(Q) \ni (P,x):u \arrow[d, mapsto, "Lh \text{ on representatives:}"'] \\
Y & (P,h_p(x)):(FP)(Q) \ni (P,h_p(x)):u
\end{tikzcd}$$

$$Q \xrightarrow{n} Q'$$

LY(n) on representatives:

$$\begin{tikzcd}
(P,y):(FP)(Q) & (P,y):(FP)(Q') \\
(P,y):(FP(n))(w) \arrow[u, phantom, sloped, "\in"] & (P,y):w \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto]
\end{tikzcd}$$

Assuming $h : X \to Y$, prove $Lh : LX \to LY$.

To prove: the quasi p.b. property

$$Q \xrightarrow{n} Q'$$

$$\begin{tikzcd}
(P,x):u \arrow[ddd, mapsto, bend right] & \sim\ (P',x'):(FP'(n))(w) \arrow[ddd, mapsto] & & (P',x'):w \arrow[ll, mapsto] \arrow[ddd, mapsto, bend left] \\
& LX(Q) \arrow[d, "(Lh)_Q"] & LX(Q') \arrow[l, "LX(n)"'] \arrow[d, "(Lh)_{Q'}"] & \\
& LY(Q) & LY(Q') \arrow[l, "LY(n)"] & \\
(P,h_p(x)):u & \sim\ (P',y'):(FP'(n))(w) & & (P',y'):w \arrow[ll, mapsto]
\end{tikzcd}$$

with $h_{p'}(x') = y'$.

Example

$\widehat{L^+}$ = synchronisation trees over L.

Pom_L^+ = nonempty finite pomsets over L.

Event structures over L
embed full-and-faithfully in $\widehat{Pom_L^+}$.

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, "F"] \arrow[d, "y_{\mathbb{P}}"'] & & \mathbb{Q} \arrow[d, "y_{\mathbb{Q}}"] \\
\widehat{\mathbb{P}} \arrow[rr, shift left, "F_!"] \arrow[rr, phantom, "\perp"] & & \widehat{\mathbb{Q}} \arrow[ll, shift left, "F^*"]
\end{tikzcd}$$

$F_!$ and F^* are colimit preserving so
preserve open maps.
When e.g. $F : L^+ \hookrightarrow Pom_L^+$.

Showing preservation of colimits
Suppose $F : \mathcal{A} \to \mathcal{B}$ sends
initial objects to initial objects. Then,
F preserves $\mathbb{I}$-colimits iff there is

$$F(\mathrm{colim}\, D) \cong \mathrm{colim}(F \circ D)$$

natural in $D \in [\mathbb{I}, \mathcal{A}]$.

Refs.

- Notes on Category Theory
- A Calculus for Categories + M. Caccamo.

Categories of non-det. processes "$2 \mapsto Sets$"

$\mathbb{P}$ p.o. (or category) of comp. paths

Eg. L^+

A non-det. process of type $\mathbb{P}$ is a functor $X : \mathbb{P}^{op} \longrightarrow Set$

generalised char. fn.

Non-det. processes of type $\mathbb{P}$ form a category

$$\widehat{\mathbb{P}} = [\mathbb{P}^{op}, Set] \quad \text{"presheaves over } \mathbb{P}\text{"}$$

Eg. • When $\mathbb{P} = L^+$, $\widehat{\mathbb{P}}$ iso. cat. of synchronisation trees.

• When $\mathbb{P}$ = cat. of finite pomsets, $\widehat{\mathbb{P}}$ embeds event structures fully & faithfully.

Each type $\mathbb{P}$ associated with $\mathbb{P}$-bisimulation from open maps.

"$2 \mapsto Set$"

Lin

maps $\mathbb{P} \xrightarrow[\text{lin}]{} \mathbb{Q}$ are

colimit preserving functors $\widehat{\mathbb{P}} \longrightarrow \widehat{\mathbb{Q}}$

corr. to functors $\mathbb{P} \longrightarrow \widehat{\mathbb{Q}}$

corr. to profunctors $\mathbb{P} \times \mathbb{Q}^{op} \longrightarrow Set$

Maps of *Lin* preserve open maps & bisim.

[GL Cattani + gw]

But are too restrictive...

Cts

maps $\mathbb{P} \xrightarrow[\text{Cts}]{} \mathbb{Q}$ are
filtered-colimit preserving functors $\widehat{\mathbb{P}} \to \widehat{\mathbb{Q}}$

corr. to maps $!\mathbb{P} \xrightarrow[\text{Lin}]{} \mathbb{Q}$

where $!\mathbb{P}$ is finite colimit completion of $\mathbb{P}$.

- Cts supports a den. sem. of HOPLA its trans. semantics based on decomposition of presheaves over $!\mathbb{P}$. Adequacy: For $t : !\mathbb{O}$

  $[\![t]\!](\bot) \cong$ set of derivations of $t \xrightarrow{\bullet}$.

Proof based on logical "relations" in which sets of realizers replace truth values.

- Now are several alternative exponentials some of which give (surjective) open map preservation.

[M. Nygaard + gw LICS 02]

Aff

maps $\mathbb{P} \xrightarrow[\text{Aff}]{} \mathbb{Q}$ are
connected-colimit preserving functors $\widehat{\mathbb{P}} \to \widehat{\mathbb{Q}}$

corr. to linear maps $\mathbb{P}_\bot \xrightarrow[\text{Lin}]{} \mathbb{Q}$

(as $\mathbb{P}_\bot$, $\underset{\text{strict Yoneda}}{j} : \mathbb{P}_\bot \to \widehat{\mathbb{P}}$ is conn.-colimit completion of $\mathbb{P}_\bot$)

- Maps of Aff preserve surj. open maps & bisimulation.
- Aff supports semantics of "affine HOPLA" with automatic congruence of bisimulation; supports an operational semantics based on decomposition of presheaves over $\mathbb{P}_\bot$ at 1st order. [M. Nygaard + gw LICS 02]
- An event-structure semantics of "affine HOPLA" gives representation of presheaf semantics at first order & exposes $\otimes$ as a parallel composition.
- Aff supports semantics of n.d. dataflow.
  [Hildebrandt, Panangaden + gw CONCUR 97, Nygaard + gw LICS 0.]
- Aff supports 'true concurrency' event structure semantics of CCS & related languages [Cattani + gw CSL 97].

Name generation

$\mathbb{I}$ finite sets of names with injections

$Cts_2^{\mathbb{I}}$ trace model of π-Calculus [Hennessy 96]

$Aff_{set}^{\mathbb{I}}$ model of π-Calculus + bisimulation [Cattani + Stark + gw 97]

- Denotational semantics suggest a 'metalanguage' based on
  - new name abstractions (of type $\delta\mathbb{P}$)
  - "dynamic" prefixed sums (eg. $\mathcal{N}.\mathbb{P}$ so $\mathcal{N}.\mathbb{P}(s) = s \times \mathbb{P}(s)_\perp$)
  - recursion & n.d. sums
  - λ-calculus (But do $Cts_2^{\mathbb{I}}$, $Cts_{set}^{\mathbb{I}}$ have fn. spaces $\mathbb{P} \to \mathbb{Q}$?)

[an op. sem. + F. Zappa Nardelli]

Hot spots

- Operational understanding of $\mathcal{L}in$ with $(\ )_\perp$, $!$ and variations for which need syntax (= language + opl. sem.)

Three lines of attack:

- From process languages [M. Nygaard + gw]
- From (linear) logic [P. Baillot + gw]
- Representations (eg. as event structures) [M. Nygaard + gw]

- Name generation & higher-order processes. Existence of function spaces? [F. Zappa Nardelli + gw]