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ON FLOWCHART THEORIES (I)

by

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ON FLOWCHART THEORIES (I)

By Gh. Ştefănescu

Ei numai doar durează-n vînt
Deşerte idealuri-
Cînd valuri află un mormînt,
Răsar în urmă valuri. *
(Eminescu)

Abstract. We define an equivalence relation on $\mathrm{Fl}_{\Sigma,T}$, the theory of Σ-flowcharts over a theory with iterate T, and show that the quotient structure, denoted by $\mathrm{RFl}_{\Sigma,T}$, is a theory with iterate. If T is an "almost syntactical theory with strong iterate", $\mathrm{RFl}_{\Sigma,T}$ is the free theory with strong iterate, genereted by adding Σ to T.

0. Introduction

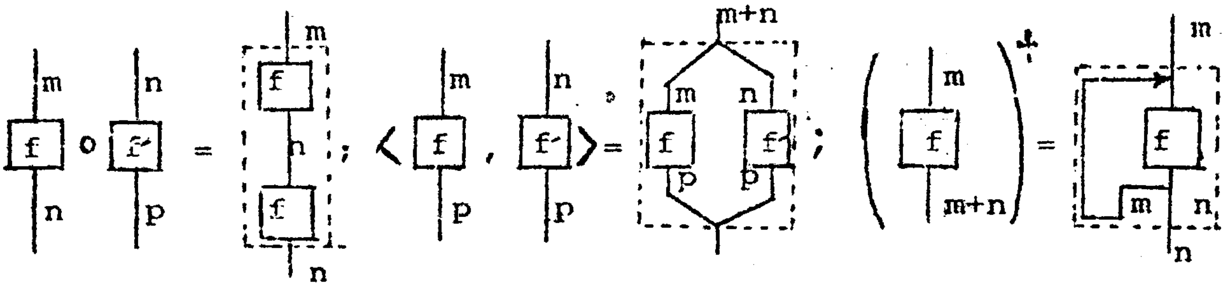
A flowchart is one of many possible notations of a computation process. This notion was strongly analysed, ten years ago, especially by C.C.Elgot [7,8], by using algebraic methods, with the aim to make it more precise from the mathematical point of view.

Making use of the ideas from [3], V.E.Căzănescu and C.Ungureanu made one more step by allowing undefined arrows and by defining, in a natural way, the iterate of a flowchart. They introduced a pure algebraic notion, called theories with iterate, which is a non-ordered generalization of rational theories [4]. Similar ideas was used by Z.Esik [9]. In [6] was given a theory of flowcharts, denoted by $\mathrm{Fl}_{\Sigma,T}$, over such a theory T, with three basic operations: composition and tupling, as in ADJ, and the new iterate. The main result of Căzănescu-Ungureanu is to give an algebraic structure (called T-module with iterate) for which $\mathrm{Fl}_{\Sigma,T}$ is the structure freely generated by Σ. On the basis of the above facts, the starting point of our paper was the following question: Why $\mathrm{Fl}_{\Sigma,T}$ is not an algebraic theory, eventually with iterate? We shall examine the axioms of algebraic theories, having in mind that the polynomial ring becomes really a ring only if, after a "syntactical" definition of sum and multiplication, we allow reductions of similar terms.

* For it is man alone, who, blind,/ Build castles in the air;/
When waves have found their grave, behind/ Waves simmer everywhere.
(translated by Leon Levitchi)

The theory of usual flowcharts is obtained by using as connection between vertices, partial functions from the initial theory with iterate $\mathfrak{PN}$, where $\mathfrak{PN}(m,n) = \{f : \{1,\dots,m\} \rightarrow \{1,\dots,n\} \mid f$ partial defined function$\}$. We shall use the picture for a flowchart $f \in Fl_{\Sigma,\mathfrak{PN}}(m,n)$, with m inputs and n outputs.

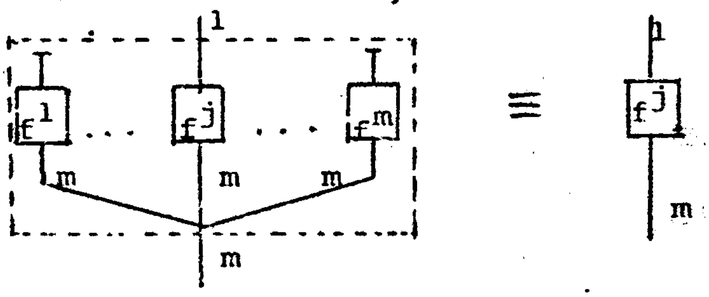
The basic operations have the following intuitive meaning,



In order that such a theory becomes an algebraic theory, it has to fulfil, among others, two axioms, in which x^m_j denote the j-th distinguished morphism from 1 to m.

a) For any $f^j \in Fl_{\Sigma,\mathfrak{PN}}(1,n)$, $j = 1,\dots,m$, it follows $x^m_j \langle f^1,\dots,f^m \rangle = f^j$.

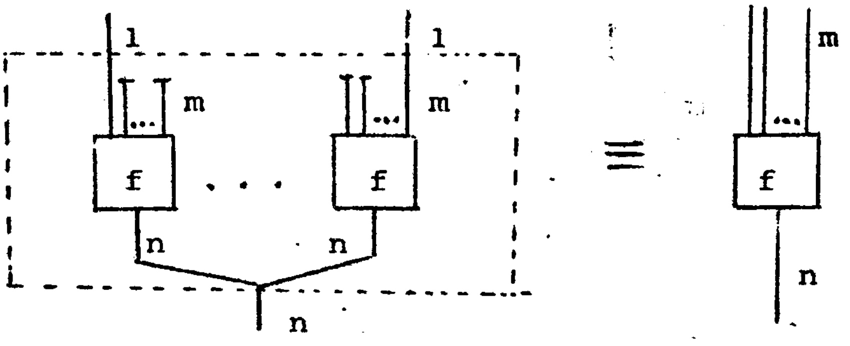
In the pictural model this needs an identity as



based on the fact that the left flowcharts f^k, $k \neq j$ are unaccessible from the input.

b) For any $f \in Fl_{\Sigma,\mathfrak{PN}}(m,n)$ it follows $\langle x^m_1 f,\dots,x^m_m f \rangle = f$.

This means



which intuitively works as we may identify all left flowcharts, being similar.

The conclusion is: "We must consider as equivalent two flowcharts if one can be obtained from the other by deleting some unaccessible vertices and by identifying some vertices with the same label and such that, after identification, they yield the same transition function". At the syntactic level this gives an elementary reduction, definable for every theory with iterate T and, based on a fact as Church-Rosser property, this leads to an equivalence relation $\cong$ even compatible with the basic operations. Hence we have a quotient structure of the reduced flowcharts $\mathrm{Fl}_{\Sigma,T}/\cong$, denoted by $\mathrm{RFl}_{\Sigma,T}$, which is itself a theory with iterate. Every flowchart has a natural unique interpretation in a theory with iterate Q, if one gives the meaning of T and Σ in Q, as was shown in [6]. The basic problem is : are two syntactical equivalent flowcharts semantic equivalent ? This fact was proved only if Q is a theory with "strong" iterate. So, we come back and ask when $\mathrm{RFl}_{.T}$ is with strong iterate, if T is so. Unfortunately, we are able to prove this only when T is "almost syntactical". Hence the main result of this paper is : $\mathrm{RFl}_{\Sigma,T}$ is the theory with strong iterate, freely generated by adding Σ to T, eachtime when $\mathrm{RFl}_{\Sigma,T}$ is with strong iterate, in particular if T is an almost syntactical theory with strong iterate. As a corollary, we point here that $\mathrm{RFl}_{\Sigma,\mathcal{PN}}$ is the theory with strong iterate, freely generated by Σ. Hence by Z.Esik result [9] and because every theory with strong iterate is an iteration theory, $\mathrm{RFl}_{\Sigma,\mathcal{PN}}$ has to be isomorphic with the theory of rational Σ-trees.

All facts are proved for many sorted theories.

Finally, I want to express my gratitude to V.E.Căzănescu for helpful discussions and for the continuous scientific and moral support.

PART I : ALGEBRAIC THEORIES WITH ITERATE

1. Notations, definitions.

As usually the free monoid generated by a sort set S will be denoted by S^*. Its typical elements are a, b, c, d, p, q. If $|a|$ is the length of the string a and $[|a|]=\{1,2,\ldots,|a|\}$, then the string $a=a_1a_2\ldots a_{|a|}$ is also considered as a function $a:[|a|]\rightarrow S$ given by the relations $a(i)=a_i$, for $i\in[|a|]$.

Let PStr_S (or PStr, when the meaning of S is clear from context) denote the S-sorted theory in which $\mathrm{PStr}_S(a,b)$ is the set of partial defined functions $x:[|a|]\rightarrow[|b|]$, such that $a=xb$, that is x preserves sorts. The typical elements of PStr are u, v, x, y, z. With 0_a we denote the unique morphism from the empty string λ to a, and with S_a^b the morphism $S_a^b=\langle 0_b+1_a, 1_b+0_a\rangle : ab\rightarrow ba$.

By restricting to total defined functions we obtain a subtheory of PStr_S denoted by Str_S, which is the initial S-sorted theory. Thus every morphism of Str_S can be considered as a morphism in an arbitrary S-sorted theory V, through the unique theory morphism $F:\mathrm{Str}_S\rightarrow V$, and we agree to omit the writing of F.

In an S sorted theory T, suppose that it is given, for every $a,b \in S^*$, an application

$$\dagger : T(a,ab) \rightarrow T(a,b)$$

called *iterate*. The following axiom

I4) $f^\dagger = yg^\dagger$ if $f \in T(a,ac)$, $g \in T(b,bc)$ and $y \in Str_S(a,b)$ are such that $f(y+1_c) = yg$,

gives other axioms by restricting y to a particular subset of functions.

I4-W) I4 only when y is a transposition S_c^d;

I4-Is) I4 only when y is a bijective function;

I4-S) I4 only when y is a surjective function;

I4-I) I4 only when y is an injective function.

1.1 Definition. A theory T is ***with iterate*** if it is given an iterate $\dagger$ for which the following axioms hold.

I1) $f\langle f^\dagger, 1_b\rangle = f^\dagger$, for $f \in T(a,ab)$;

I2) $(f(1_a+g))^\dagger = f^\dagger g$, for $f \in T(a,ab)$, $g \in T(b,c)$;

I3) $\langle f,g\rangle^\dagger = \langle f^\dagger\langle (g\langle f^\dagger, 1_{bc}\rangle)^\dagger, 1_c\rangle, (g\langle f^\dagger, 1_{bc}\rangle)^\dagger\rangle$, for $f \in T(a,abc)$, $g \in T(b,abc)$;

I4-W).

1.2 Definition. A theory with iterate is ***with strong iterate*** if I4-S) holds.

Every theory with strong iterate is an iteration theory of Esik and every iteration theory is a theory with iterate. Two examples of Esik [10] show that these three types of theories really differ. These types of theories differ only by axioms of type I4). In the case of theories with iterate and of iteration theories the used axioms of type I4) are equational, hence by a well known result such free teories exist. This is no longer true for theories with strong iterate.

2. I4 holds in a theory with strong iterate.

It is known (one proof may be found in our Appendix A) that in a theory with iterate I4-Is) holds.

2.1 Lemma. *In a theory with iterate I4-I) holds.*

Proof. Every injective function $y \in Str(a,b)$ can be written as $y = (1_a+0_{a'})z$, where z is an isomorphism. The equality $f(y+1_c) = yg$, written as $f(y+1_c)(z^{-1}+1_c) = yz^{-1}zg(z^{-1}+1_c)$, shows that $f(1_a+0_{a'}+1_c) = (1_a+0_{a'})h$, where $h = zg(z^{-1}+1_c)$. Since I4 is) is satisfied in a theory with iterate, then $h^\dagger = zg^\dagger$. The last necessary fact is $f^\dagger = (1_a+0_{a'})h^\dagger$. The equality $f(1_a+0_{a'}+1_c) = (1_a+0_{a'})h$ allows us to write h as the tuple $h = \langle f(1_a+0_{a'}+1_c), h'\rangle$. By means of axiom I3) the first component of $h^\dagger$ is

$$(1_a+0_{a'})h^\dagger = (f(1_a+0_{a'}+1_c))^\dagger\langle(h'\langle(f(1_a+0_{a'}+1_c))^\dagger, 1_{a'c}\rangle)^\dagger, 1_c\rangle =$$
$$= f^\dagger(0_{a'}+1_c)\langle(h'\langle f^\dagger(0_{a'}+1_c), 1_{a'c}\rangle)^\dagger, 1_c\rangle = f^\dagger. \quad \square$$

2.2 Proposition. *In a theory with strong iterate I4) holds.*

Proof. We put together I4-I), I4-S) both valid in a theory with strong iterate. Every function $f \in Str(a,b)$ can be written as a composition of a surjective one $u \in Str(a,d)$ and of an injective one $v \in Str(d,b)$. Now the proof is finished if we can define a morphism $h \in T(d,dc)$ such that $f(u+1_c) = uh$ and $h(v+1_c) = vg$. It is natural to take the j-th component of h to be $h_j = f_k(u+1_c)$, where $k \in [|a|]$ is such that $u(k) = j$ (f_k is the k-th component of f).

1. The definition is correct as shows the following chain of implications

$$u(k) = u(k') \implies y(k) = y(k') \implies g_{y(k)} = g_{y(k')} \implies f_k(y+1_c) = f_{k'}(y+1_c) \implies$$
$$\implies f_k(u+1_c)(v+1_c) = f_{k'}(u+1_c)(v+1_c) \implies f_k(u+1_c) = f_{k'}(u+1_c),$$

where the last one is based on the fact that v, being an injective function, has a right inverse, i.e. there exists $\tilde{v}$ such that $v\tilde{v} = 1_d$.

2. The relation $f(u+1_c) = uh$ is just another writing of the definition of h.

3. The relation $h(v+1_c) = vg$ will be shown by components. If $j \in [|d|]$ and $k \in [|a|]$ are such that $u(k) = j$, then

$$h_j(v+1_c) = f_k(u+1_c)(v+1_c) = f_k(y+1_c) = g_{y(k)} = g_{v(u(k))} = g_{v(j)}. \quad \square$$

Now our aim is to look for an identity like I4) for a partial function y. In this process we need something to say what is the "domain" and the "image" of a morphism.

3. The initial theory with strong iterate.

If we can show that $PStr_S$ is the initial theory with (strong) iterate then, as for Str, every partial function of $PStr_S$ can be considered as one in an arbitrary S-sorted theory with (strong) iterate. Naturally, a morphism between two theories with (strong) iterate is a theory morphism which preserves the iterate.

3.1 Proposition. *Every ω-continous theory is a theory with strong iterate.*

Proof. In a ω-continous theory T , if $\perp_{a,c}$ denotes the least element of T(a,c) , then the iterate of $f \in T(a,ac)$ is defined by

$$f^{+} = \bigvee_{n \geqslant o} f_n$$

where $f_0 = \perp_{a,c}$ and $f_{n+1} = f\langle f_n, 1_c\rangle$,for $n \geqslant 1$. We know (see [4,5,9]) that every rational theory is with iterate, hence so is every ω-continous theory. With a proof of I4) , the proposition is concluded.

Typically, suppose $f \in T(a,ac)$, $g \in T(b,bc)$ and $y \in Str(a,b)$ are such that $f(y+1_c) = yg$. An easy induction shows that $f_n = {}^{y}\!/g_n$, for every $n \geqslant o$. Indeed, for $n = 0$ the defintion

$$f_0 = \perp_{a,c} = \langle \perp_{a_1,c}, \ldots, \perp_{a_{|a|},c}\rangle \quad \text{and} \quad g_0 = \perp_{b,c} = \langle \perp_{b_1,c}, \ldots, \perp_{b_{|b|},c}\rangle$$

allows us to see that the morphisms $y \perp_{b,c}$ and $\perp_{a,c}$ have the same components, namely

$$x_i^a y \perp_{b,c} = x_{y(i)}^b \perp_{b,c} = \perp_{a_i,c} = x_i^a \perp_{a,c} \text{ , for } i \in [|a|].$$

The inductive step is

$$yg_{n+1} = yg\langle g_n, 1_c\rangle = f(y+1_c)\langle g_n, 1_c\rangle = f\langle yg_n, 1_c\rangle = f\langle f_n, 1_c\rangle = f_{n+1}.$$

The last argument is the continuity of composition which yields

$$yg^{+} = y \bigvee_{n \geqslant o} g_n = \bigvee_{n \geqslant o} yg_n = \bigvee_{n \geqslant o} f_n = f^{+}. \qquad \square$$

Corollary. *PStr is a theory with strong iterate.* □

This corollary and the fact that PStr is the initial theory with iterate (see [6], or compute, using as the meaning of the undefined morphisms $\perp_{a,b}$ from PStr , the morphisms $1_a^{+} 0_b$ in an arbitrary theory with iterate) lead to the following theorem.

3.2 Theorem. *PStr is the initial theory with strong iterate.* □

4. Domains and Images.

Here we give, for a morphism of a theory with iterate T , something like domain and image of a partial function. A ***substring*** of a string a is a function $x \in PStr(a,a)$ included in 1_a, that is $x(j) = j$ or $x(j)$ is undefined. Then there is a natural identification of substrings of a with subsets of $[|a|]$, which is used to define inclusion, union and intersection of substrings. In fact, the subset of $[|a|]$ corresponding to x, denoted by $[|x|]$, is its definition domain.

For a morphism $f \in T(a,b)$ the set of substrings x of b such that $fx = f$ is closed under intersection. Indeed, if $fx = f$ and $fy = f$ then $fxy = f$ and $xy = x \cap y$. Similarly with the set of substrings x of a such that $xf = f$. These show the correctness of the following definition.

The ***image*** of $f \in T(a,b)$, denoted by $Im_T(f)$, is the minimal substring x of b such that $fx = f$. The **domain** of f, denoted by $Dom_T(f)$, is the minimal substring x of a such that $xf = f$.

In the particular case of $PStr$, we see that $[| Im_{PStr}(y) |]$ is the usual image of f, and similarly for domain. In addition the equalities $y = y\, Im_{Pstr}(y) = Dom_{PStr}(y)\, y$ are still valid in T. So, for $y \in PStr(\ldots)$, the inclusions

$$Im_T(y) \subseteq Im_{PStr}(y), \quad Dom_T(y) \subseteq Dom_{PStr}(y)$$

hold. Let us point also three easy observations,

1. If $Im_T(f) \subseteq y$ then $fy = f$;

2. $Im_{PStr}(x) = x$, if x is a substring;

3. $Im_T(yf) = Im_T(f)$ if y is a surjective function (for this we see that $yfx = yf$ iff $fx = f$, where the left-right implication is based on the fact that y has a left inverse z, i.e. $zy = 1$).

A substring x of $a = a'a''$ has a unique decomposition as $x = x|_{a'} + x|_{a''}$, with $x|_{a'}$, $x|_{a''}$ substrings of a', a'', respectively. We use the convention that unspecified images are computed in T.

4.1 Observations.

1) $Im(f(y+g)) \subseteq Im_{PStr}(Im(f)|_b\, y) + Im(g)$, *for* $f \in T(a,bc)$, $g \in T(c,d)$, $y \in PStr(b,e)$;

1p) $Im(fy) \subseteq Im_{PStr}(Im(f)\, y)$, *with equality if* y *is an isomorphism;*

2) $Im(\langle f,g \rangle) = Im(f) \cup Im(g)$;

3) $Im(f^{+}) \subseteq Im(f)|_b$, *for* $f \in T(a,ab)$;

4) $Im(f+g) = Im(f) + Im(g)$;

5) $Im(f(1_b + 0_d + 1_c)) = Im(f)|_b + \perp_{d,d} + Im(f)|_c$, *for* $f \in T(a,bc)$.

Proof.

1). The equality $f(y+g) = f\,\mathrm{Im}(f)(y+g) = f(\mathrm{Im}(f)|_b y + \mathrm{Im}(f)|_c g)$ and the observation $\mathrm{Im}_T(\mathrm{Im}(f)|_b\, y) \subseteq \mathrm{Im}_{PStr}(\mathrm{Im}(f)|_b y)$ show that

$$f(y+g)(\mathrm{Im}_{PStr}(\mathrm{Im}(f)|_b y) + \mathrm{Im}(f)|_c g) = f(y+g).$$

Now 1p) is a particular case of this, for $g = 1_\lambda$. The reverse inclusion, when y is an isomorphism, is a conclusion of the following implications

$$fy = fy\mathrm{Im}(fy) \implies f = fy\mathrm{Im}(fy)y^{-1} \implies \mathrm{Im}(f) \subseteq y\mathrm{Im}(fy)y^{-1} \implies$$

$$\implies y^{-1}\mathrm{Im}(f)y \subseteq \mathrm{Im}(fy) \implies \mathrm{Im}_{PStr}(\mathrm{Im}(f)y) = \mathrm{Im}_{PStr}(y^{-1}\mathrm{Im}(f)y) \subseteq \mathrm{Im}(fy).$$

2). The obvious equivalence $\langle f,g\rangle x = \langle f,g\rangle \iff fx = f$ and $gx = g$, leads to

$$x \supseteq \mathrm{Im}(\langle f,g\rangle) \iff x \supseteq \mathrm{Im}(f) \text{ and } x \supseteq \mathrm{Im}(g) \iff x \supseteq \mathrm{Im}(f) \cup \mathrm{Im}(g)$$

which yields the desired equality.

3). All we need is to apply axiom I2). Indeed, $f^\dagger(\mathrm{Im}(f)|_b) = (f(1_a+\mathrm{Im}(f)|_b))^\dagger = f^\dagger$.

4). When one of these morphisms is 0_c and the other is $f \in T(a,b)$ the relation holds because $\mathrm{Im}(0_c+f) = \perp_{c,c} + \mathrm{Im}(f)$ and $\mathrm{Im}(f+0_c) = \mathrm{Im}(f) + \perp_{c,c}$. In the general case, for $f \in T(a,b)$, $g \in T(c,d)$, the writing of $f+g$ as a tuple gives a proof.

$$\mathrm{Im}(f+g) = \mathrm{Im}(\langle f+0_d, 0_b+g\rangle) = \mathrm{Im}(f+0_d) \cup \mathrm{Im}(0_b+g) =$$

$$= (\mathrm{Im}(f)+\perp_{d,d}) \cup (\perp_{b,b}+\mathrm{Im}(g)) = \mathrm{Im}(f)+\mathrm{Im}(g).$$

5). The proof is based on 1p) for the isomorphism $S^b_d+1_c$.

$$\mathrm{Im}(f(1_b+0_d+1_c)) = \mathrm{Im}(f(0_d+1_{bc})(S^b_d+1_c)) = \mathrm{Im}_{PStr}(\mathrm{Im}(f(0_d+1_{bc}))(S^b_d+1_c)) =$$

$$= \mathrm{Im}_{PStr}(\mathrm{Im}(0_d+f)(S^b_d+1_c)) = \mathrm{Im}_{PStr}((\perp_{d,d}+\mathrm{Im}(f))(S^b_d+1_c)) =$$

$$= \mathrm{Im}_{PStr}(\mathrm{Im}(f)|_b+\perp_{d,d}+\mathrm{Im}(f)|_c) = \mathrm{Im}(f)|_b + \perp_{d,d} + \mathrm{Im}(f)_c. \quad \square$$

5. The extension of I4) when y is a partial function

Let us note that the axiom I4) for partial function y it is possible not to work. We improve I4) by our demand that there are the same undefined components in $yg^\dagger$ and $f^\dagger$, that is

$$\mathrm{Dom}(y)\, f^\dagger = yg^\dagger.$$

One more tricky condition is that the equations from $\mathrm{Dom}(y)\, f$ must not depend on variables that there are not in $\mathrm{Dom}(y)$, that is

$$\mathrm{Im}(\mathrm{Dom}(y)\, f) \subseteq \mathrm{Dom}(y)+1_c.$$

The special case when y is a substring is separately proved in a lemma.

5.1 Lemma. *If $f : a \to ac$ is a morphism in a theory with iterate and u is a substring of a such that $Im(uf) \subseteq u+1_c$, then*

$$uf^{\dagger} = (uf)^{\dagger}.$$

Proof. We choose an isomorphism $y : a \to a'a''$, such that $y^{-1}uv = 1_{a'}+\perp_{a'',a''}$ By using I4-Is) , we get

$$(uf)^{\dagger} = (y[(1_{a'}+\perp_{a''a''})y^{-1}f\,(y+1_c)](y^{-1}+1_c))^{\dagger} =$$
$$= y((1_{a}+\perp_{a''a''})y^{-1}f\,(y+1_c))^{\dagger} = y((1_{a'}+\perp_{a'',a''})g)^{\dagger} ,$$

where g is a notation for $y^{-1}f\,(y+1_c)$. The condition $Im(uf) \subseteq u+1_c$, written in g , looks so,

$$Im((1_{a'}+\perp_{a'',a''})g) = Im(y^{-1}uyy^{-1}f\,(y+1_c)) = Im(uf\,(y+1_c)) \subseteq$$
$$\subseteq Im_{PStr}(Im(uf)(y+1_c)) \subseteq Im_{PStr}((u+1_c)(y+1_c)) = 1_{a'}+\perp_{a'',a''}+1_c .$$

This makes the a'-component of g , denoted g' , to fulfil the condition

$$g' = g'(1_{a'}+\perp_{a'',a''}+1_c).$$

We are ready to prove the equality,

$$(1_{a'}+\perp_{a'',a''})g^{\dagger} = ((1_{a'}+\perp_{a'',a''})g)^{\dagger}.$$

If $g = \langle g',g''\rangle$, then the a'-component of $g^{\dagger}$ is

$$g'^{\dagger}\langle(g''\langle g'^{\dagger},1_{a''c}\rangle)^{\dagger}, 1_c\rangle = (g'(1_{a'}+\perp_{a'',a''}+1_c))^{\dagger}\langle(g''\langle g'^{\dagger},1_{a''c}\rangle)^{\dagger}, 1_c\rangle =$$
$$= g'^{\dagger}(\perp_{a'',a''}+1_c)\langle(g''\langle g'^{\dagger},1_{a''c}\rangle)^{\dagger}, 1_c\rangle = g'^{\dagger}\langle\perp_{a'',c},1_c\rangle ,$$

and so the left morphism is

$$(1_{a'}+\perp_{a'',a''})g^{\dagger} = \langle g'^{\dagger}\langle\perp_{a'',c},1_c\rangle, \perp_{a'',c}\rangle.$$

For the right morphism we see that $(1_{a'}+\perp_{a'',a''})g = \langle g', \perp_{a'',a'a''c}\rangle$, and so the components of its iterate are

$$g'^{\dagger}\langle(\perp_{a'',a'a''c}\langle g'^{\dagger},1_{a'',c}\rangle)^{\dagger}, 1_c\rangle = g'^{\dagger}\langle\perp_{a'',a''c}^{\dagger}, 1_c\rangle = g'^{\dagger}\langle\perp_{a'',c},1_c\rangle \quad \text{and}$$
$$\perp_{a'',c}.$$

Now we come back to f and finish the proof.

$$(uf)^{\dagger} = y((1_{a'}+\perp_{a'',a''})g)^{\dagger} = y(1_{a'}+\perp_{a'',a''})g^{\dagger} = y(1_{a'}+\perp_{a'',a''})y^{-1}f^{\dagger} = uf^{\dagger}. \quad \square$$

5.2 **Proposition.** *In a theory with strong iterate, if* $f : a \to ac$, $g : b \to bc$ *are such that* $Dom(y)\, f\, (y+1_c) = yg$ *, for one* $y \in PStr(a,b)$ *, and* f *fulfils the condition* $Im(Dom(y)\, f) \subseteq Dom(y) + 1_c$ *then*

$$Dom(y)\, f^{\dagger} = yg^{\dagger}.$$

Proof. A first step is given by the above lemma, namely we have

$$Dom(y)\, f^{\dagger} = (Dom(y)\, f)^{\dagger}.$$

The proof goes on with a choice of an isomorphism $z : a \to a'a''$ such that $y = z(v + \perp_{a'',\lambda})$, with $v \in Str(a',b)$. This transform $Dom(y)\, f$ to a canonical form h with the a''-component $\perp$,

$$h = z^{-1}(Dom(y)\, f)(z+1_c) = (z^{-1}Dom(y))\, f(z+1_c) = \langle z^{-1}|_{a''}, \perp_{a'',a}\rangle f(z+1_c) =$$
$$= \langle h', \perp_{a'',a'a''c}\rangle.$$

Now we show the following identity

$$(v+\perp_{a'',\lambda})g = z^{-1}yg = z^{-1}Dom(y)\, f(z+1_c)(v+\perp_{a'',\lambda}+1_c) = h(v+\perp_{a'',\lambda}+1_c).$$

To a particular fact from this, namely to the equality of its a'-component,

$$vg = h'(v+\perp_{a'',\lambda}+1_c) = h'(1_{a'}+\perp_{a'',\lambda}+1_c)(v+1_c)$$

apply 14) using 2.2 . Therefore, $vg^{\dagger} = h'^{\dagger}(\perp_{a'',\lambda}+1_c) = h'^{\dagger}\langle\perp_{a'',c}, 1_c\rangle$. An easy computation of $h^{\dagger}$, gives $h^{\dagger} = \langle h'^{\dagger}\langle\perp_{a'',c}, 1_c\rangle, \perp_{a'',c}\rangle$, hence,

$$h^{\dagger} = \langle vg^{\dagger}, \perp_{a'',c}\rangle = (v+\perp_{a'',\lambda})\langle g\ , 0_c\rangle = (v+\perp_{a'',\lambda})g\ .$$

The conclusion that the proposition holds is now obvious.

$$Dom(y)\, f^{\dagger} = (Dom(y)\, f)^{\dagger} = zh^{\dagger} = z(v+\perp_{a'',\lambda})g^{\dagger} = yg^{\dagger}. \qquad \square$$

6. Some properties of a theory with iterate

We are giving for the begining three properties of a theory with iterate T .

6.1 Observations.

1) $\langle\langle f, g(0_a+1_{bc})\rangle^{\dagger}, 1_c\rangle = \langle f^{\dagger}, 1_{bc}\rangle\langle g^{\dagger}, 1_c\rangle$, *for* $f \in T(a,abc)$, $g \in T(b,abc)$;

2) $\langle f(1_a+0_b+1_c), g(0_a+1_{bc})\rangle^{\dagger} = \langle f^{\dagger}, g^{\dagger}\rangle$, *for* $f \in T(a,ac)$, $g \in T(b,bc)$;

3) $(g(S^b_a+1_c))^{\dagger}\langle(f(S^b_a+1_c)\langle(g(S^b_a+1_c))^{\dagger}, 1_{bc}\rangle)^{\dagger}, 1_c\rangle = (g\langle f^{\dagger}, 1_{bc}\rangle)^{\dagger}$, *for* $f \in T(a,abc)$, $g \in T(b,abc)$.

Proof.

1) The second component of $\langle f, g(0_a+1_{bc})\rangle^\dagger$ is $(g(0_a+1_{bc})\langle f^\dagger,1_{bc}\rangle)^\dagger = g^\dagger$ and the first one is $f^\dagger\langle g^\dagger,1_c\rangle$. Therefore

$$\langle\langle f, g(0_a+1_{bc})\rangle^\dagger, 1_c\rangle = \langle\langle f^\dagger\langle g^\dagger,1_c\rangle, g^\dagger\rangle, 1_c\rangle =$$

$$= \langle f^\dagger\langle g^\dagger,1_c\rangle, \langle g^\dagger,1_c\rangle\rangle = \langle f^\dagger,1_{bc}\rangle\langle g^\dagger,1_c\rangle.$$

2) The second component of $\langle f(1_a+0_b+1_c), g(0_a+1_{bc})\rangle^\dagger$ is

$$(g(0_a+1_{bc})\langle (f(1_a+0_b+1_c))^\dagger, 1_{bc}\rangle)^\dagger = g^\dagger$$

and the first one is

$$(f(1_a+0_b+1_c))^\dagger\langle g^\dagger,1_c\rangle = f^\dagger(0_b+1_c)\langle g^\dagger,1_c\rangle = f^\dagger.$$

3) The axiom I4-W) applied to $\langle f,g\rangle(S^b_a+1_c) = S^b_a\langle g(S^b_a+1_c), f(S^b_a+1_c)\rangle$, gives $\langle f,g\rangle^\dagger = S^b_a\langle g(S^b_a+1_c), f(S^b_a+1_c)\rangle^\dagger$. Based on I3) the equality of the second component of $\langle f,g\rangle^\dagger$ with the first of $\langle g(S^b_a+1_c), f(S^b_a+1_c)\rangle^\dagger$ gives the desired identity. □

This part dedicated to theories with iterate is nearly finished. One more fact is another axiomatic system. We shall use the following notations with the hope that , in fact ambiguous, this notation will be clear evertime when it will be used.

$$x_b^{abc} = 0_a+1_b+0_c$$

$$f^{\dagger b} = (f\langle x_{a'}^{abca'c'}, x_b^{abca'c'}, x_{c'}^{abca'c'}\rangle)^\dagger \text{ , for } f\in T(abc,a'bc').$$

6.2 Proposition. *In a theory T , if the axioms I1), I2), I4-W) hold, then the axiom I3) is equivalent with the following couple of axioms,*

V1) $(f^{\dagger a})^{\dagger b} = f^{\dagger ab}$ *, for* $f\in T(ab,abc)$;

V2) $\langle f,g\rangle^{\dagger a} = \langle x_a^{ac},g\rangle\langle f^\dagger,1_c\rangle$ *, for* $f\in T(a,ac)$, $g\in T(b,ac)$.

Proof. Our proof begins with the implication V2) ==> V2') , where V2') is

V2') $\langle f,g\rangle^{\dagger b} = \langle f,x_b^{bc}\rangle\langle g^\dagger,1_c\rangle$, for $f\in T(a,bc)$, $g\in T(b,bc)$.

Indeed, using I4-W) in the second equality, we have

$$S^a_b\langle f,g\rangle^{\dagger b} = S^a_b\langle f(0_a+1_{bc}), g(0_a+1_{bc})\rangle^\dagger = \langle g(1_b+0_a+1_c), f(1_b+0_a+1_c)\rangle^\dagger =$$

$$= \langle g,f\rangle^{\dagger b} = \langle x_b^{bc},f\rangle\langle g^\dagger,1_c\rangle.$$

Now suppose that V2) holds, hence also V2') . Then we can write V1) more precisely, i.e. for $f = \langle g,h\rangle\in T(ab,abc)$,

$$\langle g,h\rangle^\dagger = (\langle g,h\rangle^{\dagger a})^{\dagger b} = (\langle x_a^{abc},h\rangle\langle g^\dagger,1_{bc}\rangle)^{\dagger b} = \langle g^\dagger, h\langle g^\dagger,1_{bc}\rangle\rangle^{\dagger b} =$$

$$= \langle g^{\dagger}, x_b^{bc}\rangle \langle (h\langle g^{\dagger}, 1_{bc}\rangle)^{\dagger}, 1_c\rangle = \langle g^{\dagger}\langle (h\langle g^{\dagger}, 1_{bc}\rangle)^{\dagger}, 1_c\rangle, \; (h\langle g^{\dagger}, 1_{bc}\rangle)^{\dagger}\rangle.$$

So V1) is identical with I3). Thus the only necessary implication is I3) ==> V2). The following computation shows its validity.

$$\begin{aligned}\langle f,g\rangle^{\dagger a} &= \langle f(1_a+0_b+1_c), g(1_a+0_b+1_c)\rangle^{\dagger} = \\ &= \langle f^{\dagger}(0_b+1_c)\langle \dots, 1_c\rangle, \; (g(1_a+0_b+1_c)\langle f^{\dagger}(0_b+1_c), 1_{bc}\rangle)^{\dagger}\rangle = \\ &= \langle f^{\dagger}, (g\langle f^{\dagger}(0_b+1_c), 0_b+1_c\rangle)^{\dagger}\rangle = \langle f^{\dagger}, (g\langle f^{\dagger}, 1_c\rangle(0_b+1_c))^{\dagger}\rangle = \\ &= \langle f^{\dagger}, g\langle f^{\dagger}, 1_c\rangle\rangle = \langle x_a^{ac}, g\rangle\langle f^{\dagger}, 1_c\rangle. \quad \square\end{aligned}$$

PART II : FLOWCHART THEORIES

7. Flowchart operations

If one trys to construct a flowchart theory, he needs a labeling set Σ, for internal vertices, and something to connect them. For this we use an S-sorted theory with iterate T. The particular interesting case of connection with, possible undefined, arrows is that of $PStr_S$. In all that follows we suppose Σ is endowed with two functions

$$r_{in}, r_{out} : \Sigma \to S^* ,$$

where $r_{in}(\sigma)$ gives the number and the sorts of inputs into the "statement box" represented by σ, and similarly $r_{out}(\sigma)$, for its outputs. The monoid extensions of them are denoted by $r^*_{in}, r^*_{out} : \Sigma^* \to S^*$.

7.1 Definition. A *Σ-flowchart over T*, with input a and output b (remember $a,b \in S^*$) is a triple (i,t,e), where:

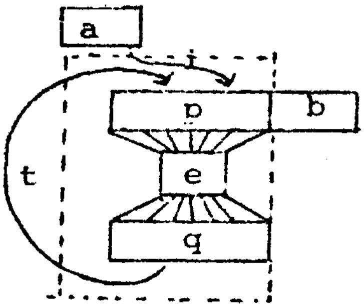
$i \in T(a, r^*_{in}(e)b)$ - is its input morfism;
$t \in T(r^*_{out}(e), r^*_{in}(e)b)$ - is its transition morfism;
$e \in \Sigma^*$ - is the string of labels of the ordered set of internal vertices.

The set of Σ-flowchart over T, between a and b, will be denoted by $Fl_{\Sigma,T}(a,b)$. Its typical elements are f, f', ... and their corresponding components are (i,t,e), (i',t',e'), ... For every internal vertex $j \in [|e|]$, denote by t_j its transition component, i.e. $t_j = x^{r^*_{out}(e)}_{r_{out}(e_j)}\, t$. For $r^*_{in}(e), r^*_{in}(e'),\dots$ use typically p, p', ... and identically with q for r_{out}. The last convention given here is to denote by

$$y_{in} : PStr_S(r^*_{in}(e), r^*_{in}(e')) \quad \text{and} \quad y_{out} : PStr_S(r^*_{out}(e), r^*_{out}(e'))$$

the extensions of $y \in PStr_\Sigma(e,e')$ to the inputs and outputs, respectively. For example, the quite difficult writing of y_{in} is $y_{in} = \langle z_1,\dots,z_{|e|}\rangle$ where, putting $p_j = r_{in}(e_j)$, $p'_j = r_{in}(e'_j)$, z_j are

$$z_j = \begin{cases} 0_{p'_1 \dots p'_{k-1}} + 1_{p_j} + 0_{p'_{k+1} \dots p'_{|e'|}} & \text{if } y(j) = k \text{ (hence } p_j = p'_k), \\ \perp_{p'_j, p'} & \text{if } y(j) \text{ is undefined.} \end{cases}$$



Now the basic operations on flowcharts have the following exact definition.

The *composition* of $f : a \rightarrow b$ with $f' : b \rightarrow c$ is

$$f\,f' = (i(1_p+i'),\ \langle t(1_p+i'),\ t'(0_p+1_{p'c})\rangle,\ ee').$$

The *tupling* of $i : a \rightarrow c$ with $f' : b \rightarrow c$, making use of the notations $x = 1_p+0_{p'}+1_c$ and $x' = 0_p+1_{p'c}$ is

$$\langle f,f'\rangle = (\langle ix, i'x'\rangle,\ \langle tx, t'x'\rangle,\ ee'),$$

The *iterate* of $f : a \rightarrow ab$ is

$$f^\dagger = (i^{\dagger a},\ t\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle,\ e).$$

8. Flowchart accessibility

In a flowchart $f : a \rightarrow b$, the set of substrings $z \subseteq 1_e$, such that their extensions z_{in}, z_{out} fulfil the condition

$$\text{ac)} \quad Im(i) \cup Im(z_{out}t) \subseteq z_{in}+1_b\ ,$$

is closed under intersection. It is more clearly, if we write this condition by components using 4.3-2, i.e.

$$\text{ac-i)} \quad Im(i) \subseteq z_{in}+1_b$$
$$\text{ac-t)} \quad Im(t_j) \subseteq z_{in}+1_b\ \text{,for } j \in [\![z]\!].$$

This makes consistent the following definition.

8.1 Definition. The ***accessible part*** of f is the minimal substring of e, denoted by $Ac(f)$, with the property ac).

In order to help the intuition we point here that z fulfils ac), as a set, if each vertex, which is effective reachable by the input morfism, is in z and, for every vertex from z it contains all vertices which are effective reachable by its transition morphism.

Some properties, connected with the behaviour of accessible part when one makes a basic operation, more or less intuitively clear, are proved here, with the help of 4.3 .

8.2 Proposition.

1) $Ac(f\,f') \subseteq Ac(f) + Ac(f')$;

2) $Ac(\langle f,f'\rangle) = Ac(f) + Ac(f')$;

3) $Ac(f^{\dagger}) \subseteq Ac(f)$.

Proof.

1). For $f : a \to b$, $f' : b \to c$ the substring $Ac(f)+Ac(f')$ fulfils ac) in $f\,f'$, more precisely ac-i) and ac-t). Apply 4.3-1) to prove ac-i).

$$Im(i(1_p+i')) \subseteq Im(i)|_p + Im(i') \subseteq Ac(f)_{in} + Ac(f')_{in} + 1_c.$$

For ac-t), if $j \in [\![Ac(f)]\!]$, the proof is as before and if $j \in [\![Ac(f')]\!]$ as follows

$$Im(t'_j(0_p+1_{p'c})) = Im(0_p+t'_j) = \perp_{p,p}+Im(t'_j) \subseteq Ac(f)_{in} + Ac(f')_{in} + 1_c.$$

2). In a similar manner it may be shown that if $f : a \to c$ and $f' : b \to c$ then the substring $Ac(f)+Ac(f')$ fulfils ac) in $\langle f,f'\rangle$. In order to conclude that it really holds an equality one has to show that $z|_e$ fulfils ac) in f, and $z|_{e'}$ fulfils ac) in f', where $z = Ac(\langle f,f'\rangle)$. Splitted in parts the condition ac-i),

$$Im(\langle i(1_p+0_{p'}+1_c),\ i'(0_p+1_{p'c})\rangle) \subseteq z_{in}+1_c,$$

gives the validity of ac-i) for $z|_e$ in f and for $z|_{e'}$ in f'. Indeed, by 4.1-5)

$$Im(i(1_p+0_{p'}+1_c)) = Im(i)|_p+\perp_{p',p'}+Im(i)|_c \quad \text{and}$$

$$Im(i'(0_p+1_{p'c})) = \perp_{p,p}+Im(i').$$

Similarly ac-t) for z in $\langle f,f'\rangle$, i.e.

$$Im(t_j(1_p+0_{p'}+1_c)) \subseteq z_{in}+1_c, \text{ if } j \in [\![z|_e]\!] \text{ and}$$

$$Im(t'_j(0_p+1_{p'c})) \subseteq z_{in}+1_c \text{ if } j \in [\![z|_{e'}]\!],$$

proves the validity of ac-t) for $z|_e$ in f and for $z|_{e'}$ in f'.

3). We shall show that Ac(f) fulfils ac) in f . An easy proof of ac-i) is

$$\mathrm{Im}(i^{+a})|_p = \mathrm{Im}((i(S^a_p+1_b))^+)|_p \subseteq \mathrm{Im}(i(S^a_p+1_b))|_p = \mathrm{Im}(i)|_p \subseteq \mathrm{Ac}(f)_{in} .$$

On the other hand, if $j \in [\![\mathrm{Ac}(f)]\!]$, with the convention that $\mathrm{Im}(t_j)|_p = u$, $\mathrm{Im}(i^{+a})|_p = v$ the following computation

$$t_j \langle x^{pb}_p, i^{+a}, x^{pb}_b\rangle ((u \cup v) + 1_b) = t_j \langle (u \cup v) + 0_b, \ i^{+a}((u \cup v + 1_b), \ 0_p + 1_b\rangle =$$
$$= t_j((u \cup v) + 1_{ab}) \langle x^{pb}_p, i^{+a}, x^{pb}_b\rangle = t_j\langle x^{pb}_p, i^{+a}, x^{pb}_b\rangle$$

leads to the desired conclusion, i.e.

$$\mathrm{Im}(t_j\langle x^{pb}_p, i^{+a}, x^{pb}_b\rangle) \subseteq (u \cup v) + 1_b \subseteq \mathrm{Ac}(f)_{in} + 1_b . \qquad \square$$

9. Syntactic equivalent flowcharts

Our introductory words lead to the conclusion that we have to consider as equivalent two fowcharts if one of them may be obtained from the other by deleting some unaccessible vertices and by identifying some vertices, with the same label and such that, after identification, they yield the same tramsition component. Here we are able to say precisely what this means.

9.1 Definition. We say that the surjective partial function $y \in \mathrm{PStr}_\Sigma(e,e')$ *reduces* the flowchart $f : a \to b$ to $f' : a \to b$, and write $f \xmapsto[y]{} f'$, if the following conditions hold.

wac) $\mathrm{Ac}(f) \subseteq \mathrm{Dom}(y)$;
co-i) $i' = i(y_{in}+1_b)$;
co-t) $\mathrm{Dom}(y_{out})\, t(y_{in}+1_b) = y_{out} t'$.

Remarks.

1. The condition wac) says that y is total defined on the accessible part of f and is obviously valid if Dom(y) fulfils ac) in f.

2. The condition co) = co-i) + co-t) correlates the connection of f to that of f'. In particular, co-t) , written by components as

$$\text{co-tc)} \quad t_j(y_{in}+1_b) = t'_{y(j)} \quad \text{for } j \in [\![\mathrm{Dom}(y)]\!] ,$$
only

shows that y has to identify two vertices if they yield the same transition component, namely

$$\text{co-t')} \quad t_j(y_{in}+1_b) = t_k(y_{in}+1_b) \ , \text{ if } y(j) = y(k).$$

3. When they are given only a flowchart f and a surjective partial function

$y \in PStr_{\Sigma}(e,e')$ such that wac) and co-t') hold, one can find a unique flowchart f', to which y reduces f. In fact, for f' the input is $i' = i(y_{in}+1_b)$, and for $k \in [|e'|]$, the k-th component of its transition is $t'_k = t_j(y_{in}+1_b)$, where $j \in [|e|]$ is such that $y(j) = k$.

4. If the partial function $y \in PStr_{\Sigma}(e,e')$ is total defined, then wac) is obviously valid, and when y is injective co-t') is obvious.

5. Some examples of reductions are given in the Appendix B.

The basic question is now to see what is happening when one makes more and more possible reductions. The first remark is that there are some trivial reductions, given by isomorphisms. Such a reduction only permutes the writing order of vertices. If $f \xmapsto[y]{} f'$ and y is an isomorphism, then we say that f is ***isomorphic*** with f', and write $f \simeq f'$. Another reduction is ***effective***, i.e. really reduces the number of vertices. A flowchart is said to be ***reduced*** (or ***minimal***) if it has no effective reductions. Now it seems to be clear that every flowchart has a finite chain of reductions to a minimal one. A more difficult problem is to show that different chains give isomorphic minimal flowcharts. This problem is similar to that solved by Church-Rosser theorem in the case of λ-expressions.

Remember that $\llbracket x \rrbracket$ is the subset of $[|a|]$ corresponding to the substring x of a. Therefore, if $y \in PStr(a,b)$, then $y(\llbracket x \rrbracket)$ is the usual image of the set $\llbracket x \rrbracket$ by means of y.

9.2 Lemma. *If $f \xmapsto[y]{} f'$ then $Im_{PStr}(Ac(f)y)$ fulfils the condition ac) in f'. As its corresponding set is $y(\llbracket Ac(f) \rrbracket)$ we have*

$$\llbracket Ac(f') \rrbracket \subseteq y(\llbracket Ac(f) \rrbracket).$$

Proof. The similar proofs of ac-i) and ac-t) are based on 4.3-1p) , which allows us to use ac) for $Ac(f)$ in $f : a \to b$.

$$Im(i') = Im(i(y_{in}+1_b)) \subseteq Im_{PStr}(Im(i)(y_{in}+1_b)) \subseteq$$

$$\subseteq Im_{PStr}((Ac(f)_{in}+1_b)(y_{in}+1_b)) = (Im_{PStr}(Ac(f)y))_{in} + 1_b ,$$

If $j \in y(\llbracket Ac(f) \rrbracket)$, that is $j = y(k)$ with $k \in \llbracket Ac(f) \rrbracket$, then

$$Im(t_j) = Im(t_k(y_{in}+1_b)) \subseteq \ldots \subseteq (Im_{PStr}(Ac(f)y))_{in} + 1_b . \qquad \square$$

The reduction is a reflexive relation. It is not always a transitive one. However, the following result holds.

9.3 Observation. *If $f \xmapsto[y]{} f'$, $f' \xmapsto[y']{} f''$ and $Im_{PStr}(Ac(f)y) \subseteq Dom(y')$, then for every $u \subseteq 1_e$ which fulfils ac) in f, one has $f \xmapsto[uyy']{} f''$. In particular $f \xmapsto[yy']{} f''$*

Proof. The function yy' is surjective. From the supplementary condition, written as $y(\llbracket Ac(f)\rrbracket) \subsetneq \llbracket Dom(y')\rrbracket$, and $\llbracket Ac(f)\rrbracket \subseteq \llbracket Dom(y)\rrbracket$, $Ac(f) \subseteq u$ it follows that uyy' fulfils wac) in f. An easy computation shows the validity of co-i) for uyy', where as usually b is the cosource of f.

$$i'' = i'(y'_{in}+1_b) = i(y_{in}+1_b)(y'_{in}+1_b) = i(u_{in}+1_b)(y_{in}y'_{in}+1_b) = i((uyy')_{in}+1_b).$$

No more difficult is to see that co-tc) holds. Indeed, for $j\in\llbracket Dom(uyy')\rrbracket$, namely $j\in\llbracket Dom(uy)\rrbracket$ and $y(j)\in\llbracket Dom(y')\rrbracket$, remark that

$$t''_{(uyy')(j)} = t''_{y'(y(j))} = t'_{y(j)}(y'_{in}+1_b) = t_j(y_{in}+1_b)(y'_{in}+1_b) =$$
$$= t_j(u_{in}+1_b)(y_{in}y'_{in}+1_b) = t_j((uyy')_{in}+i_b). \qquad \square$$

9.4 Lemma. *If* $f \xrightarrow[y]{} f''$, $f \xrightarrow[y']{} f'$ *and there exists* $y''\in PStr_{\Sigma}(e',e'')$ *such that* $y = y'y''$ *then* $f' \xrightarrow[y'']{} f''$.

Proof. From the equality $y = y'y''$ it follows that y'' is, as y , surjective. With 9.2), y'' fulfils wac). Indeed,

$$\llbracket Dom(y'')\rrbracket = y'(\llbracket Dom(y'y'')\rrbracket) = y'(\llbracket Dom(y)\rrbracket) \supseteq y'(\llbracket Ac(f)\rrbracket) \supseteq \llbracket Ac(f')\rrbracket .$$

The inputs of f' and f'' are correlated by y'' ,

$$i'(y''_{in}+1_b) = i(y'_{in}y''_{in}+1_b) = i(y_{in}+1_b) = i'' ,$$

where b is the cosource of f . For the last necessary fact, i.e. co-tc), let $j\in\llbracket Dom(y'')\rrbracket$, namely $j = y'(k)$ with $k\in\llbracket Dom(y'y'')\rrbracket = \llbracket Dom(y)\rrbracket$. The following computation finishes the proof,

$$t_j(y''_{in}+1_b) = t_k(y'_{in}y''_{in}+1_b) = t_k(y_{in}+1_b) = t''_{y(k)} = t''_{y''(j)} . \qquad \square$$

The basic fact of this paragraph, something as Church-Rosser property [12,13], is contained in the following lemma.

9.5 Main lemma. *If* $f \xrightarrow[y']{} f'$ *and* $f \xrightarrow[y'']{} f''$ *then there exists* $\bar{f}$ *such that* $f' \mapsto \bar{f}$ *and* $f'' \mapsto \bar{f}$.

A) Proof of 9.5 when y', y'' are function (hence total defined).

Let $\sim$ be the least equivalence relation on $[|e|]\times[|e|]$ which contains $\sim'$ and $\sim''$, where $\sim' = Ker(y') = \{(j,k) \mid j,k\in[|e|] \text{ and } y'(j) = y'(k)\}$ and $\sim'' = Ker(y'')$. For $\sim$ use the following constructive definition

$$j \sim k \iff \begin{cases} \text{there is a sequence of elements from } [|e|],\ j=n_1,\dots,n_m=k \text{ such that} \\ \text{for every } 1\leqslant p\leqslant m \text{ or } n_p \sim' n_{p+1} \text{ , or } n_p \sim'' n_{p+1}. \end{cases}$$

This allows us to see that $\sim$, as $\sim'$, $\sim''$, may not identify elements of different sorts. Therefore $\sim$ has a representation as $\sim = Ker(y)$, for one surjective $y \in Str_\Sigma(e,\tilde{e})$. Before trying to show that y yields a reduction, i.e. fulfils co-t') , let us denote by z', z'' the functions $z' \in Str_\Sigma(e',\tilde{e})$, $z'' \in Str_\Sigma(e'',\tilde{e})$ such that $y'z' = y = y''z''$. Use co-t') for y' in the following way

$$n_p \sim' n_{p+1} \implies t_{n_p}(y'_{in}+1_b) = t_{n_{p+1}}(y'_{in}+1_b) \implies$$

$$\implies t_{n_p}(y'_{in}z'_{in}+1_b) = t_{n_{p+1}}(y'_{in}z'_{in}+1_b) \implies t_{n_p}(y_{in}+1_b) = t_{n_{p+1}}(y_{in}+1_b)$$

where as usually b is the cosource of f. Similarly for $\sim''$. These two types of implications and a glance at the definition of $\sim$, yield the necessary implication, i.e.

$$j \sim k \implies t_j(y_{in}+1_b) = t_k(y_{in}+1_b).$$

The remark already given shows that there is a flowchart $\tilde{f}$ such that $f \underset{y}{\longmapsto} \tilde{f}$. By lemma 9.4 , z', z'' have to be reductions. □

The reduction of the general case to this one is based on the following lemma.

9.6 **Lemma.** *Suppose that* $f \underset{z}{\longmapsto} \bar{f}$ *is a reduction of* f *by an injective* $z \in PStr_\Sigma(e,\bar{e})$ *with* $Dom(z) = Ac(f)$. *For every reduction* $f \underset{y}{\longmapsto} f'$, *if* $f' \underset{z'}{\longmapsto} \bar{f}'$ *is a reduction of* f' *by an injective* $z' \in PStr_\Sigma(e',\bar{e}')$ *with* $[\![Dom(z')]\!] = y([\![Ac(f)]\!])$, *then* $\bar{f} \underset{\bar{y}}{\longmapsto} \bar{f}'$, *where* $\bar{y}$ *is a total defined function.*

Proof. There exists a surjective function $\bar{y} \in PStr_\Sigma(\bar{e},\bar{e}')$ induced by y , such that $z\bar{y} = yz'$ (the restriction of y to corresponding domain and codomain, ordered according to $\bar{e}$, $\bar{e}'$). By 9.3 , yz' is a reduction and now by 9.4 , $\bar{y}$ is a reduction.

B) **Proof of 9.5.** In the following diagram

$$\begin{tikzcd}
 & f' \arrow[rr, mapsto, "z'"] & & \bar{f}' \arrow[ddr, mapsto, "u'"] & \\
 & 1 & & & \\
f \arrow[uur, mapsto, "y'"] \arrow[rr, mapsto, "z"] \arrow[ddr, mapsto, "y''"'] & & \bar{f} \arrow[uur, mapsto, "\bar{y}'"] \arrow[ddr, mapsto, "\bar{y}''"] & 3 & \tilde{f} \\
 & 2 & & & \\
 & f'' \arrow[rr, mapsto, "z''"] & & \bar{f}'' \arrow[uur, mapsto, "u''"'] &
\end{tikzcd}$$

the drawing of the parallelograms 1, 2 are based on 9.6 and that of the parallelogram 3 , on case A). The two steps reductions $f' \underset{z'}{\longmapsto} \bar{f}' \underset{u'}{\longmapsto} \tilde{f}$ and $f'' \underset{z''}{\longmapsto} \bar{f}'' \underset{u''}{\longmapsto} \tilde{f}$ may be done even in one step, as shows 9.3 (u', u'' are total defined). □

It is clear that the isomorphism relation is an equivalence. With $\overset{*}{\longmapsto}$, the standard notation for the transitive closure of $\longmapsto$, the following proposition gives a positive answer to the starting question.

9.7 Proposition. *If $f \overset{*}{\longmapsto} f^1$, $f \overset{*}{\longmapsto} f^2$ and f^1 , f^2 are minimal flowcharts, then f^1 and f^2 are isomorphic.*

Proof. Clearly, we have to apply the main lemma in an inductive way. Suppose $f = f^{10} \longmapsto f^{11} \longmapsto \dots \longmapsto f^{1m} = f_1$ and $f = f^{20} \longmapsto f^{21} \longmapsto \dots \longmapsto f^{2n} = f^2$, with $m,n \geq 0$. The inductive variable is $m+n$. If $m = 0$ (or $n = 0$) it is obvious, because $f = f^1$ ($f = f^2$) is minimal and all reductions are made by isomorphisms. If $m \geq 1$ and $n \geq 1$ apply the main lemma, in order to obtain the $\bar{f}^{\cdots}$ morphisms from the following diagram,

$$\begin{tikzcd}
 & f^{11} \arrow[r, mapsto] \arrow[dd, mapsto] & f^{12} \arrow[r, mapsto] \arrow[d, mapsto] & \dots \arrow[r, mapsto] & f^{1(m-1)} \arrow[r, mapsto] \arrow[d, mapsto] & f^{1m} \arrow[d, mapsto] \\
 & & \bar{f}^{12} \arrow[r, mapsto] & \dots \arrow[r, mapsto] & \bar{f}^{1(m-1)} \arrow[r, mapsto] & \bar{f}^{1m} \\
f \arrow[uur, mapsto] \arrow[ddr, mapsto] & \bar{f} \arrow[ur, mapsto] \arrow[dr, mapsto] & & & & \\
 & & \bar{f}^{22} \arrow[r, mapsto] & \dots \arrow[r, mapsto] & \bar{f}^{2(n-1)} \arrow[r, mapsto] & \bar{f}^{2n} \\
 & f^{21} \arrow[r, mapsto] \arrow[uu, mapsto] & f^{22} \arrow[r, mapsto] \arrow[u, mapsto] & \dots \arrow[r, mapsto] & f^{2(n-1)} \arrow[r, mapsto] \arrow[u, mapsto] & f^{2n}.
\end{tikzcd}$$

Remark that $\bar{f}^{1m}$, as reduction of $f^{1m} = f^1$, is minimal and isomorphic with f^1. Similarly for $\bar{f}^{2n}$. By the inductive hypothesis $\bar{f}^{1m} \simeq \bar{f}^{2n}$, hence $f^1 \simeq f^2$. □

A minimal flowchart f' such that $f \overset{*}{\longmapsto} f'$ is called a ***total reduction*** of f (by 9.7, two total reductions of f are isomorphic). We say that two flowcharts are ***equivalent*** , and write $f \equiv f'$, if they have a common total reduction.

Finally, we remark that $\equiv$ is indeed an equivalence relation.

10. The theory of reduced flowcharts

The aim of this paragraph is to introduce operations on quotient sets

$$RFl_{\Sigma,T}(a,b) = Fl_{\Sigma,T}(a,b)/{\equiv_{a,b}} ,$$

where $\equiv_{a,b}$ is the restriction of $\equiv$ to $Fl_{\Sigma,T}(a,b)$, in order to yield a theory with iterate. These operations are induced by those of $Fl_{\Sigma,T}$. In order to show that this makes sense we have to prove the compatibility of $\equiv$ with the operations of $Fl_{\Sigma,T}$.

10.1 Lemma. *The reduction ($\longmapsto$) is compatible with composition, tupling and iterate.*

Poof. 1) Composition. If $f : a \rightarrow b$ and $f' : b \rightarrow c$ have the reductions $f \underset{y}{\longmapsto} \bar{f}$ and $f' \underset{y'}{\longmapsto} \tilde{f}$ we shall prove that $f\, f' \underset{y+y'}{\longmapsto} \bar{f}\, \tilde{f}$. The partial function $y+y'$ is surjective and by 8.2-1) fulfils wac) in $f\, f'$. An easy computation shows that co) holds.

$$i(1_p+i')\,(y_{in}+y'_{in}+1_c) = i(y_{in}+1_b)\,(1_{\bar{p}}+i'(y'_{in}+1_c)) = \bar{i}\,(1_{\bar{p}}+\tilde{i}\,).$$

$$\mathrm{Dom}(y_{out}+y'_{out})\ \langle t(1_p+i'),\ t'(0_p+1_{p'b})\rangle\ (y_{in}+y'_{in}+1_c) =$$

$$= \langle \mathrm{Dom}(y_{out})\ t(y_{in}+1_b)(1_{\bar{p}}+i'(y'_{in}+1_c)),\ \mathrm{Dom}(y'_{out})\ t'(y'_{in}+1_c)(0_{\bar{p}}+1_{\tilde{p}c})\rangle =$$

$$(y_{out}+y'_{out}) \rightarrow = \langle \bar{t}(1_{\bar{p}}+\tilde{i}\,),\ \tilde{t}(0_{\bar{p}}+1_{\tilde{p}c})\rangle.$$

2) Tupling. In a similar manner, using 8.2-2), one can show that, if $f : a \rightarrow c$, $f' : b \rightarrow c$, $f \underset{y}{\longmapsto} \bar{f}$ and $f' \underset{y'}{\longmapsto} \tilde{f}$, then $\langle f,f'\rangle \underset{y+y'}{\longmapsto} \langle \bar{f},\tilde{f}\rangle$.

3) Iterate. Let $f \underset{y}{\longmapsto} f'$ be a reduction of $f : a \rightarrow ab$. We claim that $f^{\dagger} \underset{y}{\longmapsto} f'^{\dagger}$. The third point of 8.2 shows the validity of wac). For co) use the following computation.

$$i^{\dagger a}(y_{in}+1_b) = (i(S^a_p+1_b))^{\dagger}(y_{in}+1_b) = (i(S^a_p+1_b)(1_p+y_{in}+1_b))^{\dagger} =$$

$$= (i(y_{in}+1_{ab})(S^a_{p'}+1_b))^{\dagger} = (i'(S^a_{p'}+1_b))^{\dagger} = i'^{\dagger a}.$$

$$\mathrm{Dom}(y_{out})\ t\langle x^{pb}_p,\ i^{\dagger a},\ x^{pb}_b\rangle(y_{in}+1_b) = \mathrm{Dom}(y_{out})\ t\langle y_{in}+0_b,\ i^{\dagger a}(y_{in}+1_b),\ 0_{p'}+1_b\rangle =$$

$$= \mathrm{Dom}(y_{out})\ t(y_{in}+1_a+1_b)\ \langle x^{p'b}_{p'},\ i'^{\dagger a},\ x^{p'b}_b\rangle = y_{out}\ t'\langle x^{p'b}_{p'},\ i'^{\dagger a},\ x^{p'b}_b\rangle.\ \square$$

10.2 Proposition. *The equivalence relation* $\equiv$ *is compatible with composition, tupling and iterate.*

Proof. By a usual trick we shall equlize with identities the number of elementary reductions ($\longmapsto$) which appear in $f \overset{*}{\longmapsto} \bar{f}$ and $f' \overset{*}{\longmapsto} \tilde{f}$. This extends the compatibility with composition and tupling from $\longmapsto$ to $\overset{*}{\longmapsto}$. The compatibility of $\overset{*}{\longmapsto}$ with iterate is an easy consequence of 10.1 .

We shall now prove the compatibility of the relation $\equiv$ with composition. If $f \equiv \bar{f} : a \rightarrow b$ and $f' \equiv \bar{f}' : b \rightarrow c$, then there exist two minimal flowcharts $\tilde{f}$ anf $\tilde{f}'$ such that $f \overset{*}{\longmapsto} \tilde{f} \overset{*}{\longleftarrow\!\!\dashv} \bar{f}$ and $f' \overset{*}{\longmapsto} \tilde{f}' \overset{*}{\longleftarrow\!\!\dashv} \bar{f}'$. The above note shows that $f\, f' \overset{*}{\longmapsto} f\, f' \overset{*}{\longleftarrow\!\!\dashv} f\, f'$, hence $f\, f' \equiv f\, f'$. The compatibility of $\equiv$ with tupling and iterate are left to the reader. $\square$

This proposition allows us to introduce, in a consistent way, operations in $RFl_{\Sigma,T}$.

Definition. The result of an operation (*composition, tupling or iterate*) in $RFl_{\Sigma,T}$ is the class of corresponding operation in $Fl_{\Sigma,T}$, computed with some representation of its arguments.

10.3 Theorem. *$RFI_{\Sigma,T}$ is a theory with iterate.*

Proof. $FI_{\Sigma,T}$ is a category with associative tupling (see [6], or compute directly). The distinguished morphisms are

$$x^a_k = (x^a_k, 0_a, \lambda) \text{ , for } k \in [|a|],$$

where λ denotes the empty string. Hence $\langle x^a_1, \dots, x^a_{|a|}\rangle = 1_a$.

In order to see that $RFI_{\Sigma,T}$ is an algebraic theory we have to prove the validity of two axioms.

a) $$x^a_k \langle f^1, \dots, f^{|a|}\rangle \equiv f^k \text{ ,}$$

for $k \in [|a|]$, where $a = a_1 \dots a_{|a|}$ and $f^j = (i^j, t^j, e^j) : a_j \to b$, for $j \in [|a|]$. In fact we have to prove formally what we intuitively know, i.e.

$$x^a_k \langle f^1, \dots, f^{|a|}\rangle \xmapsto[y]{} f^k \text{ , where } y = \langle \bot_{e^1,e^k}, \dots, 1_{e^k}, \dots, \bot_{e^{|a|},e^k}\rangle.$$

Remark that y is a surjective, partial function. The notation $x^j = x^{p^1 \dots p^{|a|}}_{p^j} + 1_b$, (remember $p = r^*_{in}(e)$) allows us to write the left expresion as

$$x^a_k \langle f^1, \dots, f^{|a|}\rangle = (i^k x^k, \langle t^1 x^1, \dots, t^{|a|} x^{|a|}\rangle, e^1 \dots e^{|a|}).$$

At the begining we show that Dom(y) fulfils wac) in $x^a_k \langle f^1, \dots, f^{|a|}\rangle$, splitted in ac-i) and ac-t). Indeed,

$$Im(i_k x^k) \subseteq Im(x^k) \subseteq Dom(y)_{in} + 1_b$$

and similarly for $[] Dom(y) [] \ni y = |e^1 \dots e^{j-1}| + m$, with $m \in [|e^j|]$, because the j-th component of the transition morphism is the m-th component of $t^k x^k$.

For co) remark that $x^k(y_{in} + 1_b) = 1_{p^k b}$, therefore

$$i^k x^k (y_{in} + 1_b) = i^k \qquad \text{, and}$$

$$Dom(y_{out}) \langle t^1 x^1, \dots, t^{|a|} x^{|a|}\rangle (y_n + 1_b) =$$

$$= \langle \bot_{r^*_{out}(e^1), p^k b}, \dots, t^k x^k (y_{in} + 1_b), \dots, \bot_{r^*_{out}(e^{|a|}), p^k b}\rangle = y_{out}\, t_k.$$

b) $$f \equiv \langle x^a_1 f, \dots, x^a_{|a|} f\rangle \text{ ,}$$

for $f : a \to b$. With its intuitive meaning in mind we look for a proof of $\langle x^a_1 f, \dots, x^a_{|a|} f\rangle \xmapsto[y]{} f$, where, making use of the notation $w^{(n)} = w \dots w$ by n times, y is given by $y = \langle 1_e, \dots, 1_e\rangle : e^{(|a|)} \to e$. This is a surjective, total function. We only have to show co). A direct computation of the right expresion gives

$$\langle x^a_1 f, \dots, x^a_{|a|} f\rangle = (\langle x^a_1 ix^1, \dots, x^a_{|a|} ix^{|a|}\rangle, \langle tx^1, \dots, tx^{|a|}\rangle, e^{(|a|)})$$

where $x^k = 0_{p(k-1)} + 1_p + 0_{p(|a|-k)} + 1_b$. Remark that $y_{in} = \langle 1_p,\ldots,1_p\rangle$. So,

$$\langle x^a_1\, ix^1,\ldots,x^a_{|a|}\, ix^{|a|}\rangle (\langle 1_p,\ldots,1_p\rangle + 1_b) = \langle x^a_1\, i,\ldots,x^a_{|a|}\, i\rangle = i \quad \text{, and}$$

$$\langle tx^1,\ldots,tx^{|a|}\rangle (\langle 1_p,\ldots,1_p\rangle + 1_b) = \langle t,\ldots,t\rangle = y_{out}\, t.$$

The second step is to show that in $RFI_{\Sigma,T}$ the axioms of iterate hoid, in the equivalent form of 6.2 , i.e. I1), I2), V1), V2), I4-W). In the Appendix C we show how look some axioms written for partial iterates.

I1) $\quad f\langle f^{\dagger}, 1_b\rangle \equiv f^{\dagger}$

for $f : a \rightarrow ab$. More precisely, $f\langle f^{\dagger},1_b\rangle \xmapsto[y]{} f^{\dagger}$, where $y = \langle 1_e, 1_e\rangle : ee \rightarrow e$. Of course y is a surjective, total function. By computing the right hand side we obiain the follwing representation,

$$f\langle f^{\dagger}, 1_b\rangle = (\ i(1_p + \langle i^{\dagger a}, x^{pb}_b\rangle),\ \langle t(1_p + \langle i^{\dagger a}, x^{pb}_b\rangle),\ t\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle(0_p + 1_{pb})\rangle,\ ee).$$

Remark that y fulfils co). Indeed,

$$i(1_p + \langle i^{\dagger a}, x^{pb}_b\rangle)(\langle 1_p, 1_p\rangle + 1_b) = i\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle = i(S^a_p + 1_b)\langle i^{\dagger a}, 1_{pb}\rangle =$$
$$= i(S^a_p + 1_b)\langle (i(S^a_p + 1_b))^{\dagger}, 1_{pb}\rangle = i^{\dagger a} \ ;$$

$$\langle t(1_p + \langle i^{\dagger a}, x^{pb}_b\rangle),\ t\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle(0_p + 1_{pb})\rangle (\langle 1_p, 1_p\rangle + 1_b) =$$
$$= \langle t\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle,\ t\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle\rangle = y_{out}\, t\langle x^{pb}_p, i^{\dagger a}, x^{pb}_b\rangle.$$

I2) holds in $FI_{\Sigma,T}$ (see [6], or compute), hence also in $RFI_{\Sigma,T}$.

Before trying to prove V1), V2) we remark that the partia' iterate for $f : abc \rightarrow a'bc'$ can be computed with the following formula

$$f^{\dagger b} = (i^{\dagger b},\ t\langle x^{pa'c'}_{pa'},\ x^{abc}_b\, i^{\dagger b},\ x^{pa'c'}_{c'}\rangle,\ e)\ .$$

V1) $\quad (f^{\dagger a})^{\dagger b} = f^{\dagger}$

for $f : ab \rightarrow abc$, holds even in $FI_{\Sigma,T}$. Indeed,

$$(f^{\dagger a})^{\dagger b} = (i^{\dagger a},\ t\langle x^{pbc}_p,\ x^{ab}_a\, i^{\dagger a},\ x^{pbc}_{bc}\rangle,\ e)^{\dagger b} =$$
$$= ((i^{\dagger a})^{\dagger b},\ t\langle x^{pbc}_p,\ x^{ab}_a\, i^{\dagger a},\ x^{pbc}_{bc}\rangle \langle x^{pc}_p,\ x^{ab}_b(i^{\dagger a})^{\dagger b},\ x^{pc}_c\rangle,\ e) =$$
$$= (i^{\dagger ab},\ t\langle x^{pc}_p,\ x^{ab}_a i^{\dagger a} \langle x^{pc}_p,\ x^{ab}_b(i^{\dagger a})^{\cancel{\dagger b},\ \cancel{x^{pc}_c\rangle}},\ x^{ab}_b(i^{\dagger a})^{\dagger b},\ x^{pc}_c\rangle,\ e) =$$
$$= (i^{\dagger ab},\ t\,\langle x^{pc}_p,\ x^{ab}_a(i^{\dagger a})^{\dagger b},\ x^{ab}_b(i^{\dagger a})^{\dagger b},\ x^{pc}_c\rangle,\ e) =$$
$$= (i^{\dagger ab},\ t\,\langle x^{pc}_p,\ (i^{\dagger a})^{\dagger b},\ x^{pc}_c\rangle,\ e) = f^{\dagger ab} = f^{\dagger}\ .$$

$$\text{V2)} \qquad \langle f,f'\rangle^{\dagger a} \equiv \langle x_a^{ac}, f'\rangle\langle f^{\dagger}, 1_c\rangle,$$

for $f : a \rightarrow ac$ and $f' : b \rightarrow ac$. We shall compute in turn the left and the right side.

$$\langle f,f'\rangle^{\dagger a} = (\langle i(1_p+0_{p'}+1_{ac}), i'(0_p+1_{p'ac})\rangle, \langle t(1_p+0_{p'}+1_{ac}), t'(0_p+1_{p'ac})\rangle, ee')^{\dagger a} =$$
$$= (g, \langle t(1_p+0_{p'}+1_{ac}), t'(0_p+1_{p'ac})\rangle \langle x_{pp'}^{pp'c}, x_a^{ab} g, x_c^{pp'c}\rangle, ee')$$

where

$$g = (\langle i(1_p+0_{p'}+1_{ac}), i'(0_p+1_{p'ac})\rangle)^{\dagger a} =$$
$$= \langle x_a^{pp'ac}, i'(0_p+1_{p'ac})\rangle \langle x_{pp'}^{pp'c}, (i(1_p+0_{p'}+1_{ac}))^{\dagger a}, x_c^{pp'c}\rangle =$$
$$= \langle i^{\dagger a}(1_p+0_{p'}+1_c), \; i'\langle x_{p'}^{pp'c}, i^{\dagger a}(1_p+0_{p'}+1_{ac}), x_c^{pp'c}\rangle\rangle .$$

Hence the transition component is

$$\langle t\langle x_p^{pp'c}, i^{\dagger a}(1_p+0_{p'}+1_{ac}), x_c^{pp'c}\rangle, \; t'\langle x_{p'}^{pp'c}, i^{\dagger a}(1_p+0_{p'}+1_{ac}), x_c^{pp'c}\rangle\rangle .$$

On the other hand because $1_{p'}+\langle i^{\dagger a}, x_c^{pc}\rangle = \langle x_{p'}^{p'pc}, i^{\dagger a}(0_{p'}+1_{pc}), x_c^{p'pc}\rangle$, the right side is

$$\langle x_a^{ac}, f'\rangle\langle f^{\dagger}, 1_c\rangle = (\langle x_a^{p'ac}, i'\rangle, t', e')(\langle i^{\dagger a}, x_c^{pc}\rangle, t\langle x_p^{pc}, i^{\dagger a}, x_c^{pc}\rangle, e) =$$
$$= (\langle i^{\dagger a}(0_{p'}+1_{pc}), \; i'\langle x_{p'}^{p'pc}, i^{\dagger a}(0_{p'}+1_{pc}), x_c^{p'pc}\rangle\rangle,$$
$$\langle t'\langle x_{p'}^{p'pc}, i^{\dagger a}(0_{p'}+1_{pc}), x_c^{p'pc}\rangle, \; t\langle x_p^{p'pc}, i^{\dagger a}(0_{p'}+1_{pc}), x_c^{p'pc}\rangle\rangle, \; e'e).$$

Now it is clear that the isomorphism $S_e^{e'} : ee' \rightarrow e'e$ reduces the left to the right flowchart. As V.E.Căzănescu remark, V2') holds even in $\mathrm{Fl}_{\Sigma,T}$ and I3) may be replaced with V1), V2'), as well.

$$\text{I4-Is)} \qquad y(y^{-1}f(y+1_c))^{\dagger} = f^{\dagger}$$

for any $f : a \rightarrow ac$ and any isomorphism $y : a \rightarrow b$. A direct computation gives

$$y(y^{-1}f(y+1_c))^{\dagger} = y\,(y^{-1}i\,(1_p+y+1_c), t(1_p+y+1_c), e)^{\dagger} =$$
$$= y\,(g, t(1_p+y+1_c)\langle x_p^{pc}, g, x_c^{pc}\rangle, e)$$

where

$$g = (y^{-1}\, i(1_p+y+1_c))^{\dagger b} = (y^{-1}\, i(1_p+y+1_c)(S_p^b+1_c))^{\dagger} =$$
$$= (y^{-1}[i(S_p^a+1_c)](y+1_c))^{\dagger} = y^{-1}(i(S_p^a+1_c))^{\dagger} = y^{-1}\, i^{\dagger a} .$$

Therefore

$$y(y^{-1}f(y+1_c))^{\dagger} = y\,(y^{-1}\, i^{\dagger a}, \; t(1_p+y+1_c)\langle x_p^{pc}, \cancel{y^{-1}\, i^{\dagger a}}, x_c^{pc}\rangle, \; e) =$$
$$= (yy^{-1}\, i^{\dagger a}, \; t\,\langle x_p^{pc}, yy^{-1}\, i^{\dagger a}, x_c^{pc}\rangle, e) = f^{\dagger} . \qquad \square$$

11. When is RFI $_{\Sigma,T}$ a theory with strong iterate ?

For futher reasons (an answer of following question: When two syntactic equivalent flowchart are semantic equivalent ?) we ask when RFI $_{\Sigma,T}$ is a theory with strong iterate. Obviously, T must be a theory with strong iterate. But, unfortunately, we are able to show that RFI$_{\Sigma,T}$ is with strong iterate only when T , in addition, fulfils the condition

AS) $Im_T(f\ y) = Im_{PStr}(Im_T(f)\ y)$, for every $y \in PStr(\ldots)$,

and then call it an *almost syntactical theory.* Firstly a lemma.

11.1 Lemma. *If T is an almost syntactical theory ~~with~~ iterate, then the reduction is even a transitive relation, i.e.* $\overset{*}{\longmapsto} = \longmapsto$.

Proof. By 9.2, 9.3 the only obstruction for the equality $\overset{*}{\longmapsto} = \longmapsto$ is $y(\llbracket Ac(f) \rrbracket) \subseteq \llbracket Ac(f') \rrbracket$, if $f \underset{y}{\longmapsto} f'$ and $f : a \to b$. We shall prove this in its equivalent form $y^{-1}(\llbracket Ac(f') \rrbracket) \supsetneq \llbracket Ac(f) \rrbracket$,keeping in mind that $Ac(f)$ is the minimal substring of e which fulfils ac) in f. Using AS) in

$$Im(i(y_{in}+1_b)) = Im(i') \subsetneq Ac(f')_{in}+1_b$$

one has an inclusion

$$(y_{in}+1_b)(\llbracket Im(i) \rrbracket) \subseteq \llbracket Ac(f')_{in}+1_b \rrbracket$$

equivalent with ac-i),

$$\llbracket Im(i) \rrbracket \subseteq (y^{-1}(\llbracket Ac(f') \rrbracket)_{in}+1_b .$$

For ac-t) if $j \in y^{-1}(\llbracket Ac(f') \rrbracket)$, that is $y(j) \in \llbracket Ac(f') \rrbracket$, as before

$$Im(t_j(y_{in}+1_b) = Im(t'_{y(j)}) \subsetneq Ac(f')_{in}+1_b$$

leads to the desirated inclusion. □

11.2 Theorem. *If T is an almost syntactical theory with strong iterate, then RFI $_{\Sigma,T}$ is with strong iterate.*

Proof. We have only to show I4-S)

$f^{+} \equiv y(f')^{+}$ if $f : a \to ac$, $f' : b \to bc$ and $y \in Str(a,b)$ is a surjective function such that $f(y+1_c) \equiv y\ f'$.

Let us suppose that f, f' are minimal flowcharts. Then y f' is also a minimal flowchart. Indeed, if it is not so and $u \in PStr(b,a)$ is a left inverse of y , i.e. $uy = 1_b$, then every effective reduction $yf' \underset{z'}{\longmapsto} f''$ are still effective reduction for $f' = uyf' \underset{z'}{\longmapsto} uf''$, but f' has no effective reductions. Hence the equivalence

$f(y+1_c) \equiv yf'$ is in fact a reduction $f(y+1_c) \xmapsto[z]{} yf$. This reduction is also good for $f^{\dagger}$, i.e. $f^{\dagger} \xmapsto[z]{} y(f')^{\dagger}$. The first remark is that z fulfils ac), making effective use of the condition AS). Indeed

$$\mathrm{Im}(i^{\dagger a})|_{|p} \subseteq \mathrm{Im}(i)|_{p} \underset{AS)}{===} \mathrm{Im}(i(1_p+y+1_c))|_{p} \subseteq \mathrm{Dom}(z_{in})$$

and for $j \in [\![\mathrm{Dom}(z)]\!]$,

$$\mathrm{Im}(t_j\langle x_p^{pc}, i^{\dagger a}, x_c^{pc}\rangle)|_{p} \subseteq \mathrm{Im}(t_j)|_{|p} \cup \mathrm{Im}(i^{\dagger a})|_{p} \underset{AS)}{===}$$

$$= \mathrm{Im}(t_j(1_p+y+1_c))|_{p} \cup \mathrm{Im}(i^{\dagger a})|_{p} \subseteq \mathrm{Dom}(z_{in}).$$

The reduction $f(y+1_c) \xmapsto[z]{} yf'$, shows that

$$i(1_p+y+1_c)(z_{in}+1_{bc}) = yi' \qquad \text{and}$$

$$\mathrm{Dom}(z_{out})\, t(1_p+y+1_c)(z_{in}+1_{bc}) = z_{out}\, t' \;.$$

Let apply I4-S) to the first eqality written as

$$[i(S_p^a+1_c)(1_a+z_{in}+1_c)]\,(y+1_{p'c}) = y\,[i'(S_{p'}^b+1_c)].$$

This gives co-i).

$$i^{\dagger a}(z_{in}+1_c) = y(i')^{\dagger b} \;.$$

For the transition component, the computation looks as follows.

$$\mathrm{Dom}(z_{out})\, t\langle x_p^{pc}, i^{\dagger a}, x_c^{pc}\rangle\, (z_{in}+1_c) = \mathrm{Dom}(z_{out})\, t\langle z_{in}+0_c, i^{\dagger a}(z_{in}+1_c), x_c^{p'c}\rangle =$$

$$= \mathrm{Dom}(z_{out})\, t(z_{in}+1_{ac})\langle x_{p'}^{p'c}, y(i')^{\dagger b}, x_c^{p'c}\rangle =$$

$$= \mathrm{Dom}(z_{out})\, t(z_{in}+1_{ac})(1_{p'}+y+1_c)\,\langle x_{p'}^{p'c}, (i')^{\dagger b}, x_c^{p'c}\rangle =$$

$$= z_{out}\, t'\langle x_{p'}^{p'c}, (i')^{\dagger b}, x_c^{p'c}\rangle \;. \qquad \square$$

12. Semantic equivalence (the main result)

We are now attaining the main question. If one prescribes a morphism $\varphi_T : T \to Q$ and a *rank-preserving* function $\varphi_{\Sigma} : \Sigma \to Q$, i.e. $\varphi_{\Sigma}(\sigma) \in Q(r_{in}(\sigma), r_{out}(\sigma))$, then *every flowchart of $Fl_{\Sigma,T}$ has a natural semantic interpretation in a theory with iterate* Q, defined by

$$\varphi^{*}(i,t,e) = \varphi_T(i)\,\langle\!\langle\, \varphi_{\Sigma}^{*}(e)\, \varphi_T(t))^{\dagger}, 1_b\rangle$$

for $(i,t,e) \in Fl_{\Sigma,T}(a,b)$, where $\varphi_{\Sigma}^{*} : (\Sigma^{*},\cdot) \to (Q,+)$ is the unique monoid extension of φ_{Σ}. Have two syntactic equivalent flowchart the same interpretation? This is the point where we need Q to be with strong iterate.

12.1 Proposition. *If T is with iterate, then two syntactic equivalent flowchart of $Fl_{\Sigma,T}$ are semantic equivalent in every Q with strong iterate. Formally, for every $(\varphi_\Sigma, \varphi_T)$, the above extension $\varphi^\#$ fulfils*

$$f \equiv f' \implies \varphi^\#(f) = \varphi^\#(f').$$

Proof. It is enough to prove this for elementary reductions, i.e. for $f, f' : a \to b$

$$f \xmapsto[y]{} f' \implies \varphi^\#(f) = \varphi^\#(f').$$

Let us suppose that $Dom(y)$ fulfils ac) in f. We try to apply the most general 14) made in 5.2 . The passing from e to e' may be done with the following formula

$$Dom(y_{in})\, \varphi_\Sigma^*(e) y_{out} = y_{in}\, \varphi_\Sigma^*(e').$$

Making use of the remark $Im_Q(Dom(y_{in})\, \varphi_\Sigma^*(e)) \subseteq Dom(y_{out})$, one can obtain a first relation

$$Dom(y_{in})\, \varphi_\Sigma^*(e)\, \varphi_T(t)(y_{in}+1_b) = Dom(y_{in})\, \varphi_\Sigma^*(e)\, Dom(y_{out})\, \varphi_T(t)(y_{in}+1_b) =$$
$$= Dom(y_{in})\, \varphi_\Sigma^*(e)\, y_{out}\, \varphi_T(t') = y_{in}\, \varphi_\Sigma^*(e')\, \varphi_T(t').$$

In order to show that the restriction to $Dom(y_{in})$ gives an itself system, we prove the second condition

$$Im_Q(Dom(y_{in})\, \varphi_\Sigma^*(e)\, \varphi_T(t)) = Im_Q(Dom(y_{in})\, \varphi_\Sigma^*(e)\, Dom(y_{out})\, \varphi_T(t)) \subseteq$$
$$\subseteq Im_Q(\varphi_T(Dom(v_{out})\, t)) \subseteq Im_T(Dom(y_{out})\, t) \underset{ac)}{\subseteq} Dom(y_{in})+1_b ,$$

Therefore, we may use 5.2 in the following computation

$$\varphi^\#(f') = \varphi_T(i') <(\varphi_\Sigma^*(e')\, \varphi_T(t'))^\dagger,\ 1_b> =$$
$$= \varphi_T(i) <y_{in}(\varphi_\Sigma^*(e')\, \varphi_T(t'))^\dagger,\ 1_b> = \qquad \text{by 5.2}$$
$$= \varphi_T(i) <Dom(y_{in})(\varphi_\Sigma^*(e)\, \varphi_T(t))^\dagger,\ 1_b> =$$
$$= \varphi_T(i)\, (Dom(y_{in})+1_b) <(\varphi_\Sigma^*(e)\, \varphi_T(t))^\dagger,\ 1_b> =$$
$$= \varphi_T(i) <(\varphi_\Sigma^*(e)\, \varphi_T(t))^\dagger,\ 1_b> = \varphi^\#(f) ,$$

where the passing to the last line is based on

$$Im_Q(\varphi_T(i)) \subseteq Im_T(i) \subseteq Dom(y_{in})+1_b .$$

In the general case, if $Dom(y) \subseteq Ac(f)$ and $f' \xmapsto[z]{} f''$ is a reduction by an injective z with $Dom(z) = Im_{PStr}(Ac(f)y)$, then by 9.3 $f \xmapsto[Ac(f)yz]{} f''$. Remark that $Dom(z)$ and $Dom(Ac(f)yz)$ fulfil ac). The above proof give $\varphi^\#(f) = \varphi^\#(f'')$ and $\varphi^\#(f') = \varphi^\#(f'')$, hence the proposition is concluded. □

12.2 MAIN THEOREM. *If $RFl_{\Sigma,T}$ is with strong iterate, then $RFl_{\Sigma,T}$ is the theory with strong iterate, freely generated by $T \cup \Sigma$. In particular, this is true if T is an almost syntactical theory with strong iterate.*

Proof. We have to show that there exist a rank-preserving function $I_\Sigma : \Sigma \to RFl_{\Sigma,T}$ and a morphism of theories with strong iterate $I_T : T \to RFl_{\Sigma,T}$, such that, for every rank-preserving function $\varphi_\Sigma : \Sigma \to RFl_{\Sigma,T}$ and morphism of theories with strong iterate $\varphi_T : T \to Q$, there exists a unique morphism of theories with strong iterate $\varphi^{\#} : RFl_{\Sigma,T} \to Q$, such that $I_\Sigma \varphi^{\#} = \varphi_\Sigma$ and $I_T \varphi^{\#} = \varphi_T$.

Clearly the application

$$I_T(i) = (i, 0_b, \lambda)$$

for $i \in T(a,b)$ gives even a theory with strong iterate morphism. I_Σ, defined by

$$I_\Sigma(\sigma) = (1_{r_{in}(\sigma)} + 0_{r_{out}(\sigma)},\ 0_{r_{in}(\sigma)} + 1_{r_{out}(\sigma)},\ \sigma),$$

is obviously a rank-preserving function. By 12.1 the extension

$$\varphi^{\#}(i.t,e) = \varphi_T(i) \langle (\varphi^*_\Sigma(e)\ \varphi_T(t))^\dagger,\ 1_b \rangle$$

is well defined even in $RFl_{\Sigma,T}$. The remained proof is reproduce here from [6] making use of 6.1 .

1) When $f : a \to b$ and $f' : b \to c$, are two flowcharts in $Fl_{\Sigma,T}$ one can see that

$$\begin{aligned}
\varphi^{\#}(f\,f') &= \varphi_T(i(1_p+i')) \langle (\varphi^*_\Sigma(ee')\ \varphi_T(\langle t(1_p+i'), t'(0_p+1_{p'c})\rangle))^\dagger,\ 1_c \rangle = \\
&= \varphi_T(i(1_p+i')) \langle \langle \varphi^*_\Sigma(e)\ \varphi_T(t(1_p+i')),\ \varphi^*_\Sigma(e')\ \varphi_T(t')(0_p+1_{p'c}) \rangle^\dagger,\ 1_c \rangle = \\
&= \varphi_T(i)(1_p + \varphi_T(i')) \langle (\varphi^*_\Sigma(e)\ \varphi_T(t(1_p+i')))^\dagger,\ 1_{p'c} \rangle \langle (\varphi^*_\Sigma(e')\ \varphi_T(t'))^\dagger,\ 1_c \rangle \\
&= \varphi_T(i) \langle (\varphi^*_\Sigma(e)\ \varphi_T(t)(1_p + \varphi_T(i')))^\dagger,\ \varphi_T(i') \rangle \langle (\varphi^*_\Sigma(e')\ \varphi_T(t'))^\dagger.\ 1_c \rangle = \\
&= \varphi_T(i) \langle (\varphi^*_\Sigma(e)\ \varphi_T(t))^\dagger \varphi_T(i'),\ \varphi_T(i') \rangle \langle (\varphi^*_\Sigma(e')\ \varphi_T(t'))^\dagger,\ 1_c \rangle = \\
&= \varphi_T(i) \langle (\varphi^*_\Sigma(e)\ \varphi_T(t))^\dagger,\ 1_b \rangle\ \varphi_T(i') \langle (\varphi^*_\Sigma(e')\ \varphi_T(t'))^\dagger,\ 1_c \rangle = \\
&= \varphi^{\#}(f)\ \varphi^{\#}(f').
\end{aligned}$$

2) In the tupling case, for $f : a \to c$ and $f' : b \to c$ making use of the following notations $x = 1_p + 0_{p'} + 1_c$, $x' = 0_p + 1_{p'c}$, the computation is

$$\begin{aligned}
\varphi^{\#}(\langle f,f' \rangle) &= \varphi_T(\langle ix, i'x' \rangle) \langle (\varphi^*_\Sigma(ee')\ \varphi_T(\langle tx, t'x' \rangle))^\dagger,\ 1_c \rangle = \\
&= \varphi_T(\langle ix, i'x' \rangle) \langle \langle \varphi^*_\Sigma(e)\ \varphi_T(t)\,x,\ \varphi^*_\Sigma(e')\ \varphi_T(t')\,x' \rangle^\dagger,\ 1_c \rangle = \\
&= \langle \varphi_T(i)x,\ \varphi_T(i')x' \rangle \langle (\varphi^*_\Sigma(e)\ \varphi_T(t))^\dagger,\ (\varphi^*_\Sigma(e')\ \varphi_T(t'))^\dagger,\ 1_c \rangle =
\end{aligned}$$

$$= < \varphi_T(i) <(\varphi^*_{\Sigma}(e)\ \varphi_T(t))^{\dagger},\ 1_c>,\ \varphi_T(i') <(\varphi^*_{\Sigma}(e')\ \varphi_T(t'))^{\dagger},\ 1_c>> =$$
$$= < \varphi^{\#}(f),\ \varphi^{\#}(f')>.$$

3) If $f : a \rightarrow ab$, then the following computation shows that $\varphi^{\#}$ preserves the iterate.

$$\varphi^{\#}(f^{\dagger}) = \varphi_T(i^{+a}) <(\varphi^*_{\Sigma}(e)\ \varphi_T(t<x^{pb}_p,\ i^{+a},\ x^{pb}_b>))^{\dagger},\ 1_b> =$$
$$= (\varphi_T(i)(S^a_p+1_b))^{\dagger}\ <(\varphi^*_{\Sigma}(e)\ \varphi_T(t)(S^a_p+1_b)\ <(\varphi_T(i)(S^a_p+1_b))^{\dagger},\ 1_{pb}>)^{\dagger},\ 1_b> =$$
$$= (\varphi_T(i) <(\varphi^*_{\Sigma}(e)\ \varphi_T(t))^{\dagger},\ 1_{ab}>)^{\dagger} = (\varphi^{\#}(f)) \quad .$$

Remark that for $g \in T(a.b)$

$$(I_T \varphi^{\#})(g) = \varphi^{\#}(g, 0_b, \lambda) = \varphi_T(g)<0^{\dagger}_b, 1_b> = \varphi_T(g)<0_b, 1_b> = \varphi_T(g)$$

and if $p = r_n(\sigma)$, $q = r_{out}(\sigma)$, then

$$(I_{\Sigma} \varphi^{\#})(\sigma) = \varphi^{\#}(1_p+0_q,\ 0_p+1_q,\ \sigma) = (1_p+0_q)<(\varphi_{\Sigma}(\sigma)(0_p+1_q))^{\dagger},\ 1_q> =$$
$$= (1_p+0_q) < \varphi_{\Sigma}(\sigma),\ 1_q> = \varphi_{\Sigma}(\sigma).$$

The last step of the proof is the uniqueness of the extension and is a direct consequence of the representation

$$(i,t,e) = I_T(i) <(I^*_{\Sigma}(e)\ I_T(t))^{\dagger},\ 1_b> \ ,$$

making use of the equalities $I_T \varphi^{\#} = \varphi_T$, $I_{\Sigma} \varphi^{\#} = \varphi_{\Sigma}$ and the preservation of composition, tupling and iterate by any morphism of theories with (strong) iterate. □

The interesting particular case is that of PStr.

12.3 Corollary. *$RFl_{\Sigma,PStr}$ is the theory with strong iterate freely generated by Σ.* □

On the other hand, by Esik result [9], when Σ has $|r_{in}(\sigma)| = 1$ for every $\sigma \in \Sigma$, the iteration theory freely generated by Σ is that of Σ-rational trees, denoted RT_{Σ}. Hence

12.4 Corollary. *If $|r_{in}(\sigma)| = 1$ for every $\sigma \in \Sigma$, then $RFl_{\Sigma,PStr}$ and RT_{Σ} are isomorphic iteration theories.* □

Appendix A

Proposition. *In any theory with iterate* † *, I4-Is) holds.*

Proof. Let $f \in T(a,ac)$, $g \in T(b,bc)$ and $y \in Str(a,b)$ be such that $f(y+1_c) = yg$. Suppose that y is an isomorphism. We construct the following system

$$<y(0_a+1_b+0_c),\ y^{-1}f(1_a+0_b+1_c)> : ab \to abc .$$

Using I3) , the b-component of its iterate is

$$(y^{-1}f(1_a+0_b+1_c) <(y(0_a+1_b+0_c))^{\dagger},\ 1_{bc}>)^{\dagger} =$$
$$= (y^{-1}f(1_a+0_b+1_c)<y(1_b+0_c),\ 1_{bc}>)^{\dagger} = (y^{-1}f<y(1_b+0_c),\ 0_b+1_c>)^{\dagger} =$$
$$= (y^{-1}f(y+1_c))^{\dagger} = g^{\dagger} .$$

In order to use I4-W) , let us permute with S^b_a the components

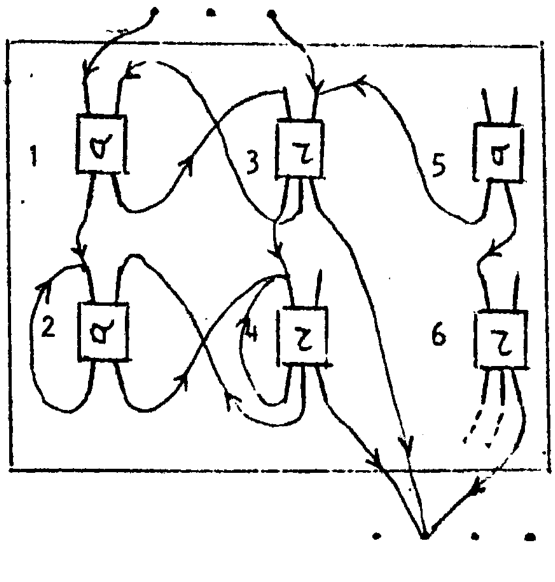
$$<y^{-1}f(0_b+1_{ac})),\ y(1_b+0_{ac})> : ba \to bac .$$

Again I3) allows us to compute the b-component of its iterate, as follows

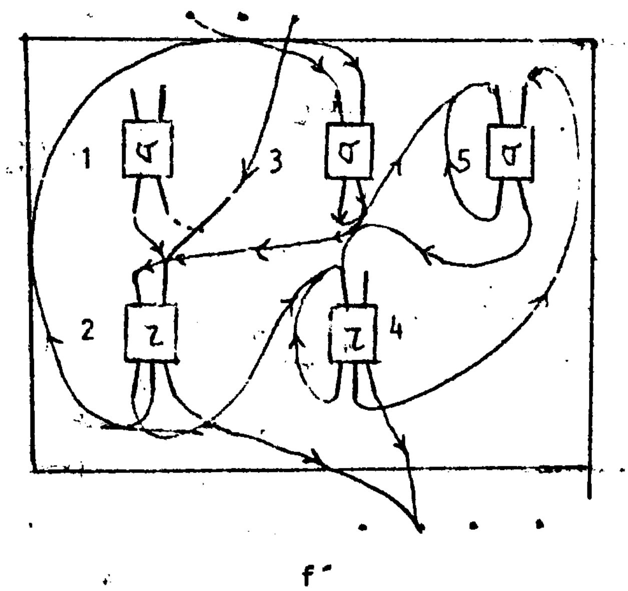
$$(y^{-1}f(0_b+1_{ac}))^{\dagger} <(y(1_b+0_{ac})<(y^{-1}f(0_b+1_{ac}))^{\dagger},\ 1_{ac}>)^{\dagger},\ 1_c> =$$
$$= y^{-1}f <(y(1_b+0_{ac})<y^{-1}f,\ 1_{ac}>)^{\dagger},\ 1_c> = y^{-1}f <y<y^{-1}f,\ 0_{ac}>)^{\dagger},\ 1_c> =$$
$$= y^{-1}f<(yy^{-1}f)^{\dagger},\ 1_c> = y^{-1}f <f^{\dagger},1_c> = y^{-1} f^{\dagger} .$$

Using I4-W) these morphism are equal, i.e. $f^{\dagger} = yg^{\dagger}$. □

Appendix B



f

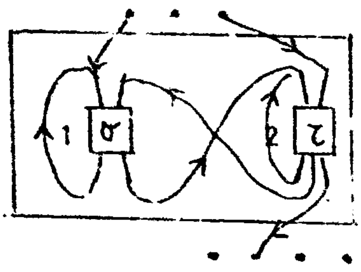


f'

Suppose f is a one-sorted flowchart : $3 \to 4$. Then $[\![Ac(f)]\!] = \{1,2,3,4\}$. .

$f \xrightarrow[y]{} f'$ where $y = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 2 & 4 & 1 & \end{array}$

A total reduction of f is $f \xrightarrow[y]{} f''$ where $y = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 2 & 2 & & \end{array}$



f''

The function $y = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 2 & 3 & & \end{array}$ do not give a reduction.

Appendix C

We shall suppose that T is a theory with iterate.

IO) $f^{+b} = (f(1_{a'}+S_{b}^{\bar{c}}+1_{\tilde{c}}))^{+b} = (f(1_{\bar{a}}+S_{\tilde{a}}^{b}+1_{c'}))^{+b}$ if $f : abc \to a'bc'$ $a' = \bar{a}\tilde{a}$ *and* $c' = \bar{c}\tilde{c}$ *(this means that no matter where* b *is in cosource).*

Proof. $f^{+b} = (f < x_{a'}^{abca'\bar{c}\tilde{c}} , x_{b}^{abca'\bar{c}\tilde{c}} , x_{\bar{c}\tilde{c}}^{abca'\bar{c}\tilde{c}} >)^{+} =$

$$= (f(1_{a'}+S_{b}^{\bar{c}}+1_{\tilde{c}}) < x_{a'\bar{c}}^{abca'\bar{c}\tilde{c}} , x_{b}^{abca'\bar{c}\tilde{c}} , x_{\tilde{c}}^{abca'\bar{c}\tilde{c}} >) = (f(1_{a'}+S_{b}^{\bar{c}}+1_{\tilde{c}}))^{+b}$$

and similarly for left shifts. □

II) $f < x_{a'}^{a'c'} , x_{b}^{abc} f^{+b} , x_{c'}^{\cancel{abc}a'c'} > = f^{+b}$ *for* $f : abc \to a'bc'$.

Proof. Let f_a , f_b , f_c be the corresponding components of f. Making use of the isomorphism $y = 1_a+S_b^c$ the second component of f^{+b} is the third of the permuted system

$$(1_a+S_b^c) f < x_{a'}^{abca'c'} , x_{b}^{abca'c'} , x_{c'}^{abca'c'} > (1_a+S_b^c+1_{a'c'}) =$$

$$= < f_{a'}, f_{c'}, f_b > < x_{a'}^{acba'c'} , x_{b}^{acba'c'} , x_{c'}^{acba'c'} > =$$

$$= \langle\langle f_{a'}, f_{c}\rangle, f_{b}\rangle\,(0_{ac} + S^{b}_{a'} + 1_{c'})\,.$$

With 13) the b-component of its iterate is $(f_b(S^{b}_{a'} + 1_{c'}))^{\dagger}$. Hence $x^{abc}_{b} f^{\dagger b}$ fulfils the identity

$$x^{abc}_{b} f^{\dagger b} = f_b(S^{b}_{a'} + 1_{c'})\langle x^{abc}_{b} f^{\dagger b},\ 1_{a'c'}\rangle = x^{abc}_{b} f\ \langle x^{a'c'}_{a'}\ ,\ x^{abc}_{b}\ f^{\dagger b}\ ,\ x^{abca'c'}_{c'}\rangle\,.$$

The ac-component of permuted system now is

$$\langle f_{a'}, f_{c}\rangle(S^{b}_{a'} + 1_{c'})\langle(f_b(S^{b}_{a'} + 1_{c'}))^{\dagger},\ 1_{a'c'}\rangle = \langle f_{a'}, f_{c}\rangle\langle x^{a'c'}_{a'}\ ,\ x^{abc}_{b}\ f^{\dagger b}\ ,\ x^{abca'c'}_{c'}\rangle\,.$$

Return to the starting system and write

$$f^{\dagger b} = (1_a + S^{c}_{b})\langle\langle f_{a'}, f_{c}\rangle, f_{b}\rangle\langle x^{a'c'}_{a'}\ ,\ x^{abc}_{b}\ f^{\dagger b}\ ,\ x_{c}abca'c'\rangle = f\ \langle x^{a'c'}_{a'}\ ,\ x^{abc}_{b}\ f^{\dagger b}\ ,\ x^{abca'c'}_{c'}\rangle.\ \square$$

With similar methods can be obtained the following identities.

I2p) $(f(g+1_b+h))^{\dagger b} = f^{\dagger b}(g+h)$, *for* $f : abc \rightarrow a'bc'$, $g : a' \rightarrow a''$, $h : c' \rightarrow c''$.

V2) $\langle f,g\rangle^{\dagger a} = \langle x^{dac}_{a}, g\rangle\langle x^{dc}_{d},\ f^{\dagger a},\ x^{dc}_{c}\rangle$, *for* $f : a \rightarrow dac$, $g : b \rightarrow dac$.

Appendix D

Remark. If $[f] \in RFl_{\Sigma,T}(a,b)$ denotes the class of $f \in Fl_{\Sigma,T}(a,b)$ then

$$Im_{RFl_{\Sigma,T}}([f]) \subseteq Im_{Fl_{\Sigma,T}}(f) \subseteq Im_T(i)|_b \cup Im_T(t)|_b\,.$$

Lemma 1. *If T is an almost syntactical theory, then for one minimal* $f \in Fl_{\Sigma,T}(a,b)$, *we have*

$$Im_{RFl_{\Sigma,T}}([f]) = Im_{Fl_{\Sigma,T}}(f)$$

and for any $f \in Fl_{\Sigma,T}(a,b)$,

$$Im_{Fl_{\Sigma,T}}(f) = Im_T(i)/_b \cup Im_T(t)/_b\,.$$

Proof. An identity $[f]y = [f]$, means $fy \equiv f$, more precisely $fy \underset{z}{\mapsto} f$. As $z \in PStr_{\Sigma}(e,e)$ is surjective, it follows that z is even an isomorphism. The reduction leads to two identities in T

$$i = i(1_p + y)(z_{in} + 1_b) = i(z_{in} + y)$$

$$z_{out}t = t(1_p + y)(z_{in} + 1_b) = t(z_{in} + y).$$

Using AS) we obtain two equalities

$$Im_T(i) = Im_{PStr}(Im_T(i)(z_{in}+y)) \text{ and } Im_T(t) = Im_{PStr}(Im_T(t)(z_{in}+y))$$

which give

$$Im_T(i)|_b = Im_{PStr}(Im_T(i)|_b y) \text{ and } Im_T(t)|_b = Im_{PStr}(Im_T(t)|_b y)\,.$$

This says that $y \supseteq Im_T(i)|_b \cup Im_T(t)|_b$, hence $Im_{RFl_{\Sigma,T}}(f) = Im_T(i)|_b \cup Im_T(t)|_b$.
The second part is a particular result from this proof, for $z = 1_e$. □

Lemma 2. *If* $f \underset{y}{\longmapsto} f'$ *and* y *is a total function, then*

$$Im_{Fl_{\Sigma,T}}(f) = Im_{Fl_{\Sigma,T}}(f')\,,$$

everytime when T *is an almost syntactical theory.*

Proof. As y_{out} is surjective it follows that

$$Im_{Fl_{\Sigma,T}}(f') = Im_T(i')|_b \cup Im_T(t')|_b = Im_T(i')|_b \cup Im_T(y_{out}t')|_b =$$
$$= Im_T(i(1_p+y))|_b \cup Im_T(t(1_p+y))|_b = Im_T(i)|_b \cup Im_T(t)|_b = Im_{Fl_{\Sigma,T}}(f). \;\square$$

Proposition. *If* T *is an almost syntactical theory, then* $Fl_{\Sigma,T}$ *and* $RFl_{\Sigma,T}$ *are almost syntactical theories.*

Proof. For $Fl_{\Sigma,T}$ the proposition is easy concluded, making use of the lemma 1 given in this appendix,

$$Im_{Fl_{\Sigma,T}}(fy) = Im_T(i(1_p+y))|_b \cup Im_T(t(1_p+y))|_b =$$
$$= Im_{PStr}((Im_T(i)|_b \cup Im_T(t)|_b)\, y) = Im_{PStr}(Im_{Fl_{\Sigma,T}}(f)\, y)\,.$$

In the case of $RFl_{\Sigma,T}$, if f is minimal, then fy is an accessible flowchart (its accessible part is 1_e). Indeed, if $z \subseteq 1_e$

$$Im_T(i) \subseteq z_{in}+1_b \iff Im_T(i(1_p+y)) \subseteq z_{in}+1_b \quad \text{and}$$
$$Im_T(t_j) \subseteq z_{in}+1_b \iff Im_T(t_j(1_p+y)) \subseteq z_{in}+1_b \;, \text{ for every } j \in [z].$$

If $f' \in [fy]$ is minimal, then $fy \underset{z}{\longmapsto} f'$ and z is a total function. Using lemma 2, we finish the proof,

$$Im_{RFl_{\Sigma,T}}([f]y) = Im_{Fl_{\Sigma,T}}(f') = Im_{Fl_{\Sigma,T}}(fy) = Im_{PStr}(Im_{Fl_{\Sigma,T}}(f)\, y) =$$
$$= Im_{PStr}(Im_{RFl_{\Sigma,T}}([f])\, y)\,. \;\square$$

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