Computational Linguistics and Computer Languages Vol. XIV. 1980.

IDENTITIES IN ITERATIVE AND RATIONAL ALGEBRAIC THEORIES

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ABSTRACT

In this paper we present a basis of identities of rational algebraic theories. It is conjectured that this basis forms a basis of identities of iterative algebraic theories, as well. It is shown as a result that free rational theories coincide with the free theories over the equational class corresponding to the basis.

1. ALGEBRAIC THEORIES

An algebraic theory T is a special many-sorted algebra whose sorting set is the set of all ordered pairs (n,p) of non-negative integers. Let us denote by T(n,p) the carrier of sort (n,p) of T for each n,p. The operations in T are: composition, source-tupling and injections. For each n,p,q, composition (denoted by $\cdot$ or juxtaposition) maps T(n,p) x T(p,q) into T(n,q). For each n,p, source-tupling associates with $f_i \in T(1,p)$ (i=1,...,n) a unique element $\langle f_1,\ldots,f_n\rangle \in T(n,p)$. Particularly, if n=0, source-tupling picks out an element $0_p \in T(0,p)$. Finally, injections are nullary operations; there is a corresponding injection $\pi_n^i \in T(1,n)$ to each i and n such that $1 \leq i \leq n$. The operations are required to satisfy the following conditions (cf. [1]):

(i) $(fg)h = f(gh)$ if $f \in T(n,p)$, $g \in T(p,q)$, $h \in T(q,r)$;

(ii) $f<\pi_p^1,\ldots,\pi_p^p> = f$, $f \in T(n,p)$;

(iii) $\pi_n^i<f_1,\ldots,f_n> = f_i$, $1 \leq i \leq n$, $f_j \in T(1,p)$

$(j=1,\ldots,n)$;

(iv) $<\pi_n^1 f,\ldots,\pi_n^n f> = f$, $f \in T(n,p)$.

In particular, if $n=0$, the last condition asserts that $T(0,p)$ is singleton.

Under these assumptions T becomes a category whose objects are the non-negative integers and in which each object n is the n-th copower of object 1. In fact, it was the original definition of algebraic theories (cf. [7]). In this category the identities are the elements $1_n = <\pi_n^1,\ldots,\pi_n^n>$ $(n \geq 0)$. According to the categorical analogy, the elements of T are called morphisms and $f \in T(n,p)$ is written as $f : n \to p$.

It seems convenient to extend source-tupling as follows. Let $f : n \to p$, $g : m \to p$. Then $<f,g> = <\pi_n^1 f,\ldots,\pi_n^n f,\pi_m^1 g,\ldots,\pi_m^m g>$. Evidently, this derived operation is associative. Hence, we may write $<f,g,h>$ to denote either $<f,<g,h>>$ or $<<f,g>,h>$. Another derived operation is the separated sum. First, let us consider 1_n and 0_p. Then, $1_n + 0_p = <\pi_{n+p}^1,\ldots,\pi_{n+p}^n>$, while $0_n + 1_p = <\pi_{n+p}^{n+1},\ldots, \pi_{n+p}^{n+p}>$.

In general, if $f : n \to p$ and $g : m \to q$, then $f + g =$ $= <f(1_p+0_q), g(0_p+1_q)>$. The separated sum is associative, too. Concerning other identities the reader is referred to [3].

The algebraic theory T is called non-degenerate, provided $\pi_2^1 \neq \pi_2^2$. A morphism $f : n \to p$ is said to be ideal if none of the morphisms $\pi_n^1 f,\dots,\pi_n^n f$ is an injection. Finally, T is called ideal if it is non-degenerate and for arbitrary f and ideal g, gf is ideal.

One can introduce homomorphisms - called theory maps - between two theories. These are exactly homomorphisms of algebraic theories considered as many-sorted algebras. Let T and T' be ideal theories and take a theory map $F : T \to T'$. If F preserves ideal morphisms, then it is called ideal as well.

Algebraic theories, as they were introduced, have an equational presentation. Hence, for every ranked alphabet or type $\Sigma = \bigcup_{n\geq 0}^{\infty} \Sigma_n$ there exists a free theory generated by Σ. This is denoted by T_Σ and has the following property. There is a ranked alphabet map $\eta : \Sigma \to T_\Sigma$ such that any ranked alphabet map $F : \Sigma \to T$ into an algebraic theory T has a unique homomorphic extension $\overline{F} : T_\Sigma \to T$; i.e. a theory map $\overline{F}$ which satisfies $F = \eta\overline{F}$. Here, by a ranked alphabet map we mean any mapping $F : \Sigma \to T$ such that $F(\Sigma_n) \subseteq T(1,n)$.

T_Σ can be described as the theory of finite Σ-trees on the variables $\{x_1,x_2,\dots\}$ (cf. [4], [6]). η can be chosen as the mapping $f \mapsto f(x_1,\dots,x_n)(f \in \Sigma_n, n \geq 0)$. Since η is injective, we can consider Σ as a subset (more precisely as a subsystem) of T_Σ. In this way $F = \eta\overline{F}$ corresponds to $\overline{F}|_\Sigma = F$.

In particular, if Σ is the void alphabet, T_Σ becomes the initial theory. This is isomorphic to the theory Θ, in which $\Theta(n,p)$ is the set of all mappings of $[n] = \{1,\dots,n\}$ into $[p]$, composition is composition of mappings, source-tupling is source-tupling of mappings, finally, the injection $\pi_n^i : 1 \to n$ is the mapping which picks out the integer i from [n]. Since Θ is initial in the category of all theories and theory maps, each theory T has exactly one subtheory which is the homomorphic image of Θ. Further on this subtheory will be denoted by Θ_T or, simply, Θ. If T is non-degenerate, Θ_T is isomorphic to Θ, otherwise both Θ and Θ_T is isomorphic to the terminal theory, a theory whose each carrier is either void or singleton. The elements of Θ and Θ_T are called base morphisms and, in the sequal, they are identified. Lower case Greek letters, except Θ, always denote base morphisms. For arbitrary $\rho : n \to p \in \Theta$, $i\rho$ stands for the image of $i \in [n]$ under ρ. A base morphism is called surjective, injective etc. if it is surjective or injective, resp. as a mapping.

We distinguish a subset (or subsystem) from T_Σ. This will be denoted by $\tilde{T}_\Sigma$. $f \in \tilde{T}_\Sigma(n,p)$ if and only if the frontier of f, i.e. the sequence of variables appearing in the leaves of f, is exactly $x_1 \ldots x_p$. $\tilde{T}_\Sigma$ has the following important property: Every element of T_Σ can be uniquely written in the form $\tilde{f}\rho$, where $\tilde{f} \in \tilde{T}_\Sigma$ and $\rho \in \Theta$.

The morphisms $f : n \to p$ $(n > 0)$ which can be obtained as $(\sum_{i=1}^{n} f_i)\rho$, where $f_i \in \Sigma(i=1,..,n)$ and $\rho \in \Theta$, constitute the subset $\Sigma\Theta$.

Now we are ready to prove:

<u>Lemma 1.1.</u>

Let $f : n \to n+p$, $g : m \to m+p$ and $\rho : m \to n$ be morphisms in a free algebraic theory T. Assume that ρ is surjective and $g(\rho+1_p) = \rho f$. Then, there exists a morphism $h : \ell \to \ell+p$ such that for some surjective $\sigma : \ell \to m$ we have

(i) $h(\sigma + 1_p) = \sigma g$,

(ii) if β is a left inverse of σ, i.e. $\beta\sigma = 1_m$ then $\sigma\beta h = h$,

(iii) for every left inverse γ of τ there are base morphisms $\tau_1,\ldots,\tau_\ell : \ell \to \ell$ satisfying both $\tau_i\tau = \tau$ and $\pi^i_\ell \tau\gamma h(\tau_i + 1_p) = \pi^i_\ell h$ $(i=1,\ldots,\ell)$, where τ denotes the composition $\sigma\rho$.

Proof

Since every free algebraic theory is freely generated by a ranked alphabet, it is enough to verify the statement of the lemma for theories $T = T_\Sigma$, the free algebraic theory generated by a ranked alphabet Σ.

Let g_i denote the i-th component of g, i.e. $g_i = \pi^i_m g$ $(i=1,\ldots,m)$. It can be written in the form $g_i = \tilde{g}_i\alpha_i(\beta_i + \beta'_i)$, where $\tilde{g}_i \in \tilde{T}(1,k_i + k'_i)$, $\beta_i : k_i \to m$, $\beta'_i : k'_i \to p$ and finally, $\alpha_i : k_i+k'_i \to k_i+k'_i$ is bijective and satisfies that both $\alpha_i|_{N_i}$ and $\alpha_i|_{N'_i}$, the restrictions of α_i to $N_i = \{j\alpha_i^{-1} | 1 \le j \le k_i\}$ and $N'_i = \{j\alpha_i^{-1} | k_i < j \le k_i+k'_i\}$, are monoton mappings.

Assume that $i\rho = j\rho (i,j \in [m])$. Then, also $g_i(\rho+1_p) = g_j(\rho+1_p)$, i.e. $\tilde{g}_i\alpha_i(\beta_i+\beta'_i)(\rho+1_p) = \tilde{g}_j\alpha_j(\beta_j+\beta'_j)(\rho+1_p)$. But there is a unique way to get a morphism of T_Σ as the composition of an element of $\tilde{T}_\Sigma$ and a base morphism.

Thus, we can conclude that $k_i+k'_i = k_j+k'_j$, $\tilde{g}_i = \tilde{g}_j$, $\alpha_i(\beta_i + \beta'_i)(\rho+1_p) = \alpha_j(\beta_j+\beta'_j)(\rho+1_p)$. Suppose that $t \in N_i$. Then, $t\alpha_i(\beta_i+\beta'_i)(\rho+1_p) \le n$ and hence, $t\alpha_j(\beta_j+\beta'_j)(\rho+1_p) \le n$. Therefore $t\alpha_j \le k_j$, i.e. $t \in N_j$. The converse inclusion is similar. This proves the equalities $N_i = N_j$, $N'_i = N'_j$, $k_i = k_j$ and $k'_i = k'_j$.

Or even, since the mappings $\alpha_i|_{N_i}$, $\alpha_i|_{N_i'}$, $\alpha_j|_{N_j}$ and $\alpha_j|_{N_j'}$ are equally monoton, $\alpha_i = \alpha_j$; and it results from this that $\beta_i\rho = \beta_j\rho$ and $\beta_i' = \beta_j'$.

Define $\ell' = \sum_{i=1}^{m} k_i$, $\ell = m+\ell'$. For every $i \in [m]$ let $\bar{h}_i$ denote the morphism $\bar{h}_i = 0_m+\tilde{g}_i\alpha_i(\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t}+\beta_i')$.

Let $\sigma = \langle 1_m,\beta_1,\ldots,\beta_m\rangle$, $\bar{h} = \langle \bar{h}_1,\ldots,\bar{h}_m\rangle$. A simple computation shows that $\bar{h}_i(\sigma+1_p) = g_i$ for each $i \in [m]$.

Indeed, $\bar{h}_i(\sigma+1_p) =$

$$= (0_m+\tilde{g}_i\alpha_i(\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t}+\beta_i'))(\langle 1_m,\beta_1,\ldots,\beta_m\rangle+1_p) =$$

$$= (0_m+\tilde{g}_i\alpha_i(\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t}+\beta_i'))\langle 1_m+0_p,\langle\beta_1,\ldots,\beta_m\rangle+0_p,0_m+1_p\rangle =$$

$$= \tilde{g}_i\alpha_i(\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t}+\beta_i')(\langle\beta_1,\ldots,\beta_m\rangle+1_p) =$$

$$= \tilde{g}_i\alpha_i((\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t})\langle\beta_1,\ldots,\beta_m\rangle+\beta_i') =$$

$$= \tilde{g}_i\alpha_i(\beta_i+\beta_i') = g_i \quad .$$

This proves $\bar{h}(\sigma+1_p) = g$.

Assume again, that $i\rho = j\rho$ ($i,j \in [m]$). Define $\rho_{i,j} : \ell \to \ell$ by $\rho_{i,j} = 1_m + \rho_{i,j}'$ where $\rho_{i,j}'$ denotes the base morphism

$$\langle \sum_{t=1}^{i-1} 1_{k_t}+\sum_{t=i}^{m} 0_{k_t}, \sum_{t=1}^{j-1} 0_{k_t}+1_{k_j}+\sum_{t=j+1}^{m} 0_{k_t}, \sum_{t=1}^{i} 0_{k_t}+\sum_{t=i+1}^{m} 1_{k_t}\rangle .$$

It is easy to check that

$(\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t})\rho_{i,j}' = \sum_{t=1}^{j-1} 0_{k_t}+1_{k_j}+\sum_{t=j+1}^{m} 0_{k_t}$. Thus,

$\bar{h}_i(\rho_{i,j}+1_p) =$

$= (0_m+\tilde{g}_i\alpha_i(\sum_{t=1}^{i-1} 0_{k_t}+1_{k_i}+\sum_{t=i+1}^{m} 0_{k_t}+\beta_i'))(1_m+\rho_{i,j}' + 1_p) =$

$= 0_m+\tilde{g}_j\alpha_j(\sum_{t=1}^{j-1} 0_{k_t}+1_{k_j}+\sum_{t=j+1}^{m} 0_{k_t} + \beta_j') = \bar{h}_j$,

i.e. $\bar{h}_i(\rho_{i,j} + 1_p) = \bar{h}_j$. Furthermore, if τ denotes composition $\sigma\rho$, we have $\rho_{i,j}\tau = \tau$. Indeed,

$\rho_{i,j}\tau = (1_m + \rho_{i,j}')<1_m,\beta_1,\ldots,\beta_m>\rho =$

$= <1_m,\rho_{i,j}'<\beta_1,\ldots,\beta_m>>\rho = <1_m,\beta_1,\ldots,\beta_{i-1},\beta_j,\beta_{i+1},\ldots,\beta_m>\rho =$

$= <\rho,\beta_1\rho,\ldots,\beta_{i-1}\rho,\beta_j\rho,\beta_{i+1}\rho,\ldots,\beta_m\rho> = <\rho,\beta_1\rho,\ldots,\beta_m\rho> =$

$= <1_m,\beta_1,\ldots,\beta_m>\rho = \sigma\rho = \tau.$

Let $h = \sigma\bar{h}$ and denote by h_i the i-th component of h. For each (i,j) such that $i\tau = j\tau$ let $\tau_{i,j} = \rho_{i\sigma,j\sigma}$. Then , we have $h_i(\tau_{i,j} + 1_p) = h_j$ and $\tau_{i,j}\tau = \tau$.

$h(\sigma+1_p) = \sigma\bar{h}(\sigma+1_p) = \sigma g$, this proves part (i) of Lemma 1.1. In order to verify (ii), take β an arbitrary left inverse of σ. Obviously, $\sigma\beta h = \sigma\beta\sigma\bar{h} = \sigma\bar{h} = h$. Finally, let $\gamma : n \to \ell$ be a left inverse of τ. For each $i \in [\ell]$, define τ_i by $\tau_i = \tau_{i\tau\gamma,i}$. This can be done by $i\tau\gamma\tau = i\tau$. Evidently, $\tau_i\tau = \tau$ and $\pi_\ell^i\tau\gamma h(\tau_i + 1_p) = \pi_\ell^i\tau\gamma h(\tau_{i\tau\gamma,i} + 1_p) = \pi_\ell^i h$, ending the proof of the lemma.

2. ITERATIVE AND RATIONAL THEORIES, IDENTITIES

By an algebraic theory with iteration we mean a theory T equipped with a new operation $^+$, called iteration, which, with each $f : n \to n+p$, associates a morphism $f^+ : n \to p$. An iterative algebraic theory T (cf. [3]) is an ideal theory with iteration, except that the iteration is partial. For $f \in T(n,n+p)$

f^+ exists if and only if f is ideal in T, considered as an algebraic theory. Furthermore, f^+ is required to be the unique fixed point of f, i.e. f^+ is the unique morphism $g \in T(n,p)$ auch that $g = f\langle g, 1_p\rangle$.

Homomorphisms between two iterative theories are the ideal theory maps. Observe that ideal theory maps preserve iteration.

Rational algebraic theories were introduced in [9]. A rational algebraic theory T is a theory with iteration. Each of the carriers of T is ordered, f^+ is the least fixed point of f, and ordering subject to some other conditions (cf. [9]). Homomorphisms of rational theories, as they were defined in [9], are certain theory maps, but, likewise in case of iterative theories, they preserve the iteration as well.

In a sense, iterative and rational theories have a common generalization which will be introduced here. Consider an arbitrary theory with iteration. It will be called generalized iterative theory, provided it satisfies the following identities, (A) to (E):

(A) $f^+ = f\langle f^+, 1_p\rangle$, where $f : n \to n+p$,

(B) $\langle f,g\rangle^+ = \langle h^+, (g\rho)^+\langle h^+, 1_p\rangle\rangle$, where $f : n \to n+m+p$,
$g : m \to n+m+p$, $h = f\langle 1_n+0_p, (g\rho)^+, 0_n+1_p\rangle$ and
$\rho = \langle 0_m+1_n, 1_m+0_n\rangle + 1_p$,

(C) $(0_n+f)^+ = f$, where $f : n \to p$,

(D) $(f+0_q)^+ = f^+ + 0_q$, where $f : n \to n+p$,

(E) $\langle \pi^1_m \rho g(\rho_1+1_p), \ldots, \pi^m_m \rho g(\rho_m+1_p)\rangle^+ = \rho f^+$ if $f : n \to n+p$,
$g : n \to m+p$, $\rho : m \to n$ is surjective, $\rho_1, \ldots, \rho_m : m \to m$ are base, furthermore, $\rho_1\rho = \ldots = \rho_m\rho = \rho$, as well as

$f = g(\rho+1_p)$ is satisfied.

In the above mentioned identities f and g are treated as variables of the given sort.

Theorem 2.1

Every rational theory is a generalized iterative theory. Every iterative algebraic theory satisfies identities (A) to (E) if ideal morphisms are substituted for f and g.

We do not present a complete proof of this theorem here. The reason is that most of these identities, except possibly the last one, were already discovered in papers [3], [9]. For (B) cf. [2], too.

Let us remark, however, that it would be enough to prove the theorem for free rational and free iterative theories. And what is more, since free iterative theories can be viewed as weak subalgebras - subtheories closed under the iteration of ideal morphisms - of free rational theories, it would be enough to consider free rational theories only.

We have already mentioned that all free iterative theories exist. This fact was first shown in [1]. Another proof can be found in [5].

I_Σ, the free iterative theory generated by the alphabet Σ can be obtained as follows (cf. [4], [5]). First consider T_Σ^∞, the algebraic theory of all, possibly infinite, Σ-trees on the variables $\{x_1,x_2,\ldots\}$. T_Σ^∞ is an ideal theory, even an iterative theory. Then, construct the smallest subtheory of T_Σ^∞ containing T_Σ and closed under iteration of ideal morphisms. This will be the iterative theory I_Σ, called "free" because every ranked alphabet map $F : \Sigma \to T$ into an iterative theory T, such that $F(\Sigma)$ contains ideal morphisms only, has a unique homomorphic extension $\bar{F}$, i.e. an ideal theory map $\bar{F} : I_\Sigma \to T$, satisfying $\bar{F}|_\Sigma = F$. Recall that by $\Sigma \subseteq T_\Sigma$ and $T_\Sigma \subseteq I_\Sigma$, $\Sigma \subseteq I_\Sigma$ holds as well.

R_Σ, the free rational theory generated by Σ, has a similar description. Take $I_{\Sigma_\perp}$, where $\Sigma_\perp$ is Σ except $(\Sigma_\perp)_0 = \Sigma_0 \cup \{\perp\}$, and $\perp$ is a new symbol. There is exactly one way to extend $I_{\Sigma_\perp}$ to a generalized iterative algebraic theory in such a manner that we have $\pi_1^{1^+} = \perp$. When forgetting orderings, R_Σ becomes this theory $I_{\Sigma_\perp}$. R_Σ is free in the following sense. For any ranked alphabet map $F : \Sigma \to T$ into a rational theory T, there is exactly one homomorphism (of rational theories) $\bar{F} : R_\Sigma \to T$ extending F, i.e. such that $\bar{F}|_\Sigma = F$. This was proved in [9]. Actually, this theorem remains valid even if $\bar{F}$ is required to be an iteration preserving theory map, i.e. a homomorphism of theories with iteration.

Further on, let us consider R_Σ as an unordered theory. Do not forget that $I_{\Sigma_\perp}$, and hereby I_Σ as well, is a weak subalgebra of R_Σ and the carriers of $I_{\Sigma_\perp}$ and R_Σ coincide.

We now proceed by stating some consequences of the identities (A) to (E). In these statements, if $(A_1),\dots,(A_{n+1})$ are sentences of first order, expressed in the language of theories with iteration, we write $(A_1),\dots,(A_n) \models (A_{n+1})$ to mean the fact that every theory with iteration which satisfies $(A_1),\dots,(A_n)$, satisfies (A_{n+1}) as well.

(X) $g^+ = \rho f^+$ if $f : n \to n+p$, $g : m \to m+p$, $\rho : m \to n$ is surjective and $g(\rho+1_p) = \rho f$, moreover for any left inverse α of ρ there exists base morphism $\rho_i : m \to m$ ($i \in [m]$) with $\rho_i \rho = \rho$ and $\pi_m^i \rho\alpha g(\rho_i + 1_p) = \pi_m^i g$.

Lemma 2.2.

(E) $\models$ (X).

Proof

Assume that (E) is satisfied by the algebraic theory with iteration T. Let $f: n \to n+p$, $g: m \to m+p$ and $\rho: m \to n$ content the assumptions of (X) and fix an arbitrary left inverse α of ρ . Define g' by $g' = \alpha g$. Then $g'(\rho + 1_p) = f$. But there exist morphisms $\rho_1, \dots, \rho_m : m \to m$ satisfying both $\rho_1 \rho = \dots = \rho_m \rho = \rho$ and

$$< \pi_m^1 \rho g'(\rho_1 + 1_p), \dots, \pi_m^m \rho g'(\rho_m + 1_p) > = g.$$

Thus, by (E) we obtain $g^+ = \rho f^+$.

Further on the following special case of (X) will be used.

(X') $g^+ = \rho f^+$ if $f: n \to n+p$, $g: m \to m+p$, $\rho: m \to n$ is surjective, $g(\rho + 1_p) = \rho f$, moreover there is a base morphism $\alpha: n \to m$ with $\alpha\rho = 1_p$ and $g = \rho\alpha g$.

The next identity can be derived from (X') and from (E), too. Indeed, if in identity (E) we have $n=m$ as well as $g = f(\rho^{-1} + 1_p)$, where ρ^{-1} denotes the inverse of ρ, furthermore $\rho_i = 1_m$ is satisfied for each $i \in [m]$, then we obtain identity

$$\text{(F)} \quad (\rho f(\rho^{-1}+1_p))^+ = \rho f^+$$

and the following lemma:

<u>Lemma 2.3.</u> (E) $\models$ (F).

The next identity is the dual of (B).

$$\text{(B')} \quad \langle f,g\rangle^+ = \langle f^+\langle h^+,1_p\rangle,h^+\rangle \quad \text{if } f : n \to n+m+p,$$
$$g : m \to n+m+p \text{ and } h = g\langle f^+,1_{m+p}\rangle.$$

<u>Lemma 2.4.</u> (B), (E) $\models$ (B').

<u>Proof</u>

We prove that (B), (F) $\models$ (B'). For this purpose let T be an arbitrary theory with iteration which satisfies both (B) and (F). Take f and g as in (B') and define ρ by $\rho = \langle 0_n+1_m, 1_n+0_m\rangle$. The inverse of ρ is $\rho^{-1} = \langle 0_m+1_n, 1_m+0_n\rangle$. Let $f_1 = f(\rho^{-1}+1_p)$, $g_1 = g(\rho^{-1}+1_p)$. Obviously $\rho\langle f,g\rangle(\rho^{-1}+1_p) = \langle g_1,f_1\rangle$.

By (B) we have $\langle g_1,f_1\rangle^+ = \langle h_1^+,(f_1\rho_1)^+\langle h_1^+,1_p\rangle\rangle$, where $\rho_1 = \rho+1_p$, $h_1 = g_1\langle 1_m+0_p,(f_1\rho_1)^+, 0_m+1_p\rangle$.

But, $f_1\rho_1 = f(\rho^{-1}+1_p)(\rho+1_p) = f$ and thus

$$h_1 = g_1\langle 1_m+0_p,f^+,0_m+1_p\rangle = g(\rho^{-1}+1_p)\langle 1_m+0_p,f^+,0_m+1_p\rangle =$$
$$= g\langle 0_m+1_n+0_p,1_m+0_{n+p},0_{n+m}+1_p\rangle\langle 1_m+0_p,f^+,0_m+1_p\rangle =$$
$$= g\langle f^+,1_m+0_p,0_m+1_p\rangle = g\langle f^+,1_{m+p}\rangle = h.$$

Thus, $\langle g_1,f_1\rangle^+ = \langle h^+,f^+\langle h^+,1_p\rangle\rangle$. By (F), $(\rho^{-1}\langle g_1,f_1\rangle(\rho+1_p))^+ = \rho^{-1}\langle g_1,f_1\rangle^+ = \langle f^+\langle h^+,1_p\rangle,h^+\rangle$.

It results from this that $\langle f,g\rangle^+ = \langle f^+\langle h^+,1_p\rangle, h^+\rangle$.

Another identity is:

(G) $(f(1_n+g))^+ = f^+g$, where $f : n \to n+m$, $g : m \to p$.

Lemma 2.5. (B), (C), (D), (E) $\models$ (G).

Proof

We show that (B), (B′), (C), (D) $\models$ (G). Hence, the proof follows by Lemma 2.4.

Let $f_1 = f+0_p : n \to n+m+p$, $g_1 = 0_{n+m}+g : m \to n+m+p$ in an algebraic theory with iteration satisfying (B), (B′), (C) and (D).

By (B) we have $\langle f_1,g_1\rangle^+ = \langle h^+,(g_1\rho)^+\langle h^+,1_p\rangle\rangle$, where $\rho = \langle 0_m+1_n, 1_m+0_n\rangle +1_p$, $h = f_1\langle 1_n+0_p, (g_1\rho)^+, 0_n+1_p\rangle$. But, $g_1\rho = (0_{n+m}+g)(\langle 0_m+1_n, 1_m+0_n\rangle +1_p)=0_{n+m}+g$, hence, by (C), $(g_1\rho)^+ = 0_n+g$. Therefore, $h = f_1\langle 1_n+0_p,(g_1\rho)^+, 0_n+1_p\rangle =$ $= (f+0_p)\langle 1_n+0_p, 0_n+g, 0_n+1_p\rangle = f\langle 1_n+0_p, 0_n+g\rangle = f(1_n+g)$. We have already seen that $\langle f_1,g_1\rangle^+ = \langle (f(1_n+g))^+,(0_n+g)\langle h^+,1_p\rangle\rangle =$ $= \langle (f(1_n+g))^+,g\rangle$. On the other hand, by (B′), $\langle f_1,g_1\rangle^+ = \langle f_1^+ \langle h^+,1_p\rangle, h^+\rangle$, where $h = g_1\langle f_1^+,1_{m+p}\rangle$, now.

By $g_1\langle f_1^+, 1_{m+p}\rangle = (0_{n+m}+g)\langle f_1^+,1_{m+p}\rangle = 0_m+g$ and (C), $h^+ = g$. It results from this and by (D) that $\langle f_1,g_1\rangle^+ = \langle f_1^+\langle g,1_p\rangle,g\rangle = \langle (f^++0_p)\langle g,1_p\rangle, g\rangle = \langle f^+g, g\rangle$.

If we put the above mentioned two facts together, we get $(f(1_n+g))^+ = f^+g$.

The next identity contains (G) as a special case.

(H) $\langle f_1, g_1\rangle^+ = \langle f^+\langle g^+, h\rangle, g^+\rangle$, where $f : n \to n+m+p$, $g : m \to m+q$, $h : p \to q$, $f_1 = f(1_{n+m}+h)$ and $g_1 = 0_n+g$.

Lemma 2.6. (B), (C), (D), (E) $\models$ (H).

Proof

Instead of this we prove that (B), (G) $\models$ (H). Suppose that T is an algebraic theory with iteration satisfying both (B) and (G). Take the morphisms f,g and h and let $f_1 = f(1_{n+m}+h)$, $g_1 = 0_n+g$.

By (B) we have $\langle f_1, g_1\rangle^+ = \langle h_1^+, (g_1\rho)^+\langle h_1^+, 1_q\rangle\rangle$, where $h_1 = f_1\langle 1_n+0_q, (g_1\rho)^+, 0_n+1_q\rangle$, $\rho = \langle 0_m+1_n, 1_m+0_n\rangle +1_q$.

$g_1\rho = (0_n+g)\langle 0_m+1_n+0_q, 1_m+0_{n+q}, 0_{n+m}+1_q\rangle =$ $= g\langle 1_m+0_{n+q}, 0_{n+m}+1_q\rangle = g(1_m+0_n+1_q)$. Applying (G) we get $(g_1\rho)^+ = g^+(0_n+1_q) = 0_n+g^+$. Therefore $h_1 =$ $= f(1_{n+m}+h)\langle 1_n+0_q, 0_n+g^+, 0_n+1_q\rangle = f(1_n+\langle g^+,h\rangle)$. Again by (G) we get $h_1^+ = f^+\langle g^+,h\rangle$.

Hence $\langle f_1,g_1\rangle^+ = \langle h_1^+, (0_n+g^+)\langle h_1^+, 1_q\rangle\rangle = \langle f^+\langle g^+,h\rangle, g^+\rangle\rangle$.

Further on, we shall use the following consequence of (H):

(H′) $\langle f_1, g_1\rangle^+ = \langle f^+\langle \pi_m^1 g^+, h\rangle, g^+\rangle$ if $f : n \to n+1+p$, $g : m \to m+q$ $(m \geq 1)$, $h : p \to q$ and $f_1 = f(1_{n+1}+0_{m-1}+h)$, $g_1 = 0_n+g$.

Lemma 2.7. (H) $\models$ (H′).

Proof

Assume that T satisfies (H), $f : n \to n+1+p$, $g : m \to m+q$ $(m \geq 1)$ and $h : p \to q$ are morphisms in T. Let

$f_1 = f(1_{n+1}+0_{m-1}+h)$, $g_1 = 0_n+g$. Furthermore, let $f' = f(1_{n+1}+0_{m-1}+1_p)$, $f'_1 = f'(1_{n+m}+h)$. It is easy to check that $f_1 = f'_1$. It follows from this and by (H) that $\langle f_1,g_1\rangle^+ =$ $= \langle f'^+\langle g^+,h\rangle, g^+\rangle$. By (G), a consequence of (H), $f'^+ =$ $=(f(1_n+1_1+0_{m-1}+1_p))^+ = f^+(1_1+0_{m-1}+1_p)$.

Therefore, $f'^+\langle g^+,h\rangle = f^+\langle \pi^1_m g^+ ,h\rangle$ ending the proof of Lemma 2.7.

Finally, we prove a consequence of (B), as well as that one of (A) and (B).

(I) $(1_n+0_m)\langle f_1,g\rangle^+ = f^+$ if $f : n \to n+p$, $g : m \to n+m+p$, $f_1 = f(1_n+0_m+1_p)$.

<u>Lemma 2.8.</u> (B) $\models$ (I).

<u>Proof</u>

By (B) we have $\langle f_1,g\rangle^+ = \langle h^+,(g\rho)^+ \langle h^+,1_p\rangle\rangle$, where $h = f_1\langle 1_n+0_p, (g\rho)^+, 0_n+1_p\rangle$, $\rho= \langle 0_m+1_n, 1_m+0_n\rangle + 1_p$. We must prove that $h = f$. But this can be immediately seen since
$h = f_1\langle 1_n+0_p, (g\rho)^+, 0_n+1_p\rangle =$
$= f(1_n+0_m+1_p)\langle 1_n+0_p, (g\rho)^+, 0_n+1_p\rangle = f\langle 1_n+0_p,0_n+1_p\rangle = f$.

(J) $\langle f'_1,f'\rangle^+ = \langle f_1^{++},f^+\langle f_1^{++},1_p\rangle\rangle$, where $f = \langle f_1,\ldots,f_n\rangle$: $: n \to n+1+p$, $f^+ = \langle f_1^+,\ldots,f_n^+\rangle$,
$f' =\langle f'_1,\ldots,f'_n\rangle = f(\langle 0_1+1_n, 1_1+0_n\rangle+1_p)$, $n \geq 1$.

<u>Lemma 2.9.</u> (A),(B) $\models$ (J).

<u>Proof</u>

Assume that T is an algebraic theory with iteration satisfying (A) and (B), and assume that the variables appearing in (J) are interpreted in T. By application of (B) we get

$<f_1', f'>^+ = <h^+, (f'\rho)^+ <h^+, 1_p>>$, where

$h = f_1' <1_1+0_p, (f'\rho)^+ , 0_1+1_p>$, $\rho = <0_n+1_1, 1_n+0_1> + 1_p$.

$f'\rho = f(<0_1+1_n, 1_n+0_1> + 1_p)(<0_n+1_1, 1_n+0_1> + 1_p) =$

$= f(<1_n+0_1, 0_n+1_1> + 1_p) = f$. Therefore, $h = f(<0_1+1_n, 1_1+0_n>+1_p) \cdot$

$\cdot <1_1+0_p, f^+, 0_1+1_p> = f_1 <0_1+1_n+0_p, 1_1+0_{n+p}, 0_{1+n}+1_p><1_1+0_p, f^+, 0_1+1_p> =$

$= f_1 <f^+, 1_1+0_p, 0_1+1_p> = f_1 < f^+, 1_{1+p}> = f_1^+$. The last equality is obtained by application of (A).

Thus we get $(f_1', f')^+ = <f_1^{++}, f^+ <f_1^{++}, 1_p>>$.

Summarizing the results of this section, we have proved that any generalized iterative theory satisfies the identities (B'), (F), (G), (H), (H'), (I) and (J), as well as the implication (X). In fact, the same proofs can be used to show that all these sentences are valid in iterative theories, too.

3. THE MAIN RESULTS

We now turn to prove that the identities (A) to (E) form a basis of identities of rational theories. This is accomplished by verifying that free rational theories are exactly the free generalized iterative theories. As an intermediate step, we also show that every ranked alphabet map $F : \Sigma \to T$ into a generalized iterative theory T has a unique homomorphic extension (a theory map, preserving iteration of ideal morphism) $\bar{F} : I_\Sigma \to T$. In fact, the proof of the last mentioned theorem is based upon the observation that all considerations in [5] can be carried out under weaker assumptions, i.e. by using the identities (A) to (E) and their consequences only.

For the rest of this section, Σ is taken as an arbitrary fixed alphabet. With the exception of the last two theorems all statements relate to theory I_Σ.

Lemma 3.1.

Let $f : n \to n+p$, $g : m \to m+p \in \Sigma\Theta$. Assume that $\{\pi_n^i f^+ \mid i \in [n]\} = \{\pi_m^i g^+ \mid i \in [m]\}$. Then, there exist surjective base morphisms $\rho : n \to \ell$ and $\sigma : m \to \ell$, as well as a morphism $h : \ell \to \ell+p$ such that both $f(\rho+1_p) = \rho h$ and $g(\sigma+1_p) = \sigma h$ hold.

Proof

Let ℓ denote the number of distinct components of f^+. We can choose the base morphisms $\alpha_0 : \ell \to n$, $\beta_0 : \ell \to m$, $\rho : n \to \ell$ and $\sigma : m \to \ell$ in such a way that each of the following conditions is satisfied, i.e.
$\alpha_0 \rho = 1_\ell$, $\beta_0 \sigma = 1_\ell$, $\rho\alpha_0 f^+ = f^+$, $\sigma\beta_0 g^+ = g^+$ and $\alpha_0 f^+ = \beta_0 g^+$.

For an arbitrary $\alpha : \ell \to n$, if $\alpha\rho = 1_\ell$, let f_α denote the composition $f_\alpha = \alpha f(\rho+1_p)$. Similarly, $\beta g(\sigma+1_p)$ is denoted by g_β, provided $\beta\sigma = 1_\ell$. It is easy to check that both $\rho\alpha f^+ = f^+$ and $\sigma\beta g^+ = g^+$ hold. Thus, $f_\alpha\langle\alpha f^+, 1_p\rangle = \alpha f^+$ and $g_\beta\langle\beta g^+, 1_p\rangle = \beta g^+$ showing that $f_\alpha^+ = \alpha f^+$ and $g_\beta^+ = \beta g^+$. But we have $\alpha f^+ = \alpha_0 f^+ = \beta_0 g^+ = \beta g^+$ for every choice of α and β, therefore the morphisms f_α and g_β have the same iteration. Hence, by Lemma 3.5. in [5], it follows that there exists an ideal element $h \in T_\Sigma$ such that f_α and g_β are the partial unwindings of h, for any α and β. But both, f_α and g_β are in $\Sigma\Theta$ resulting that $f_\alpha = h = g_\beta$.

We have shown that for every $\alpha : \ell \to n$ and $\beta : \ell \to m$ if $\alpha\rho = 1_\ell$ and $\beta\sigma = 1_\ell$ are satisfied then so is $f_\alpha = f_{\alpha_0} = g_{\beta_0} = g_\beta$ and by definition, this morphism was chosen as h. Now, we have to verify the equality $f(\rho+1_p) = \rho h$. Let $i \in [n]$ be arbitrary. Choose α in such a way that both $\alpha\rho = 1_\ell$ and $i\rho\alpha = i$

are valid. For this α we have

$$\pi_n^i f(\rho+1_p) = \pi_n^{i\rho\alpha} f(\rho+1_p) = \pi_n^i \rho\alpha f(\rho+1_p) = \pi_n^i \rho f_\alpha = \pi_n^i \rho h.$$

Since $i \in [n]$ was arbitrary, this proves that $f(\rho+1_p) = \rho h$. The proof of $g(\sigma+1_p) = \sigma h$ is similar.

At this point recall a definition from [5]. Let $f : n \to n+p$ be in T_Σ, $i,j \in [n]$. The j-th component of f is said to be **reachable** from the i-th one if there exists a non-negative integer m such that $\pi_n^i f^m$ contains an occurrence of variable x_j. Here, f^m is defined by induction on m : $f^0 = 1_n + 0_p$, $f^{m+1} = f<f^m, 0_n+1_p>$. Furthermore, the j-th component of f is called "superfluous" if it is unreachable from the first component of f and $j \neq 1$.

Lemma 3.2.

Let $f : n \to n+p$, $g : m \to m+p \in \Sigma\Theta$. Assume that neither f nor g contains superfluous components. Furthermore, let $F : T_\Sigma \to T$ be an arbitrary theory map into the generalized iterative theory T. Then $\pi_n^1(F(f))^+ = \pi_m^1(F(g))^+$, provided that $\pi_n^1 f^+ = \pi_m^1 g^+$.

Proof

It follows under the assumption of the lemma that $\{\pi_n^i f^+ \mid i \in [n]\} = \{\pi_m^i g^+ \mid i \in [m]\}$. By virtue of Lemma 3.1. we have $f(\rho+1_p) = \rho h$ and $g(\sigma+1_p) = \sigma h$ for some surjective base morphisms $\rho : n \to \ell$ and $\sigma : m \to \ell$ and a morphism $h : \ell \to \ell+p \in \Sigma\Theta$.

Without loss of generality, we may assume ρ and σ to be such that $1\rho = 1\sigma = 1$.

By virtue of Lemma 1.1. we obtain that there exist $f' : n' \to n'+p$ in $\Sigma\Theta$ and surjective $\rho' : n' \to n$ satisfying

(1) $\rho' f = f'(\rho'+1_p)$; (2) $f' = \rho'\alpha' f'$ if $\alpha'\rho' = 1_n$; (3) for arbitrary

$\alpha : \ell \to n'$, if $\alpha\ \rho'\rho = 1$, then there are base morphism $\rho_1, \dots, \rho_n$ $n' \to n'$ with $\rho_i\rho'\rho = \rho'\rho$ and $\pi^i_{n'}\rho'\rho\alpha f'(\rho_i + 1_p) = \pi^i_n f'$ $(i \in [n'])$.
From this, and using the fact that F is a theory map, by (X) we get $F(f')^+ = \rho'\rho F(h)^+$ and $F(f')^+ = \rho F(f)^+$. Hence, $F(f)^+ = \rho F(h)^+$.
The proof of $F(g)^+ = \sigma F(h)^+$ is similar.
Hence, $\pi^1_n(F(f))^+ = \pi^1_n\rho(F(h))^+ = \pi^1_m\sigma(F(h))^+ = \pi^1_m(F(g))^+$ is obtained.

The next statement is analogous to Lemma 3.10. in [5].

Lemma 3.3.

Let $f : n \to n+p \in \Sigma\Theta$. There exists one $g : m \to m+p \in \Sigma\Theta$ which has no superfluous component and satisfies the condition $\pi^1_n(F(f))^+ = \pi^1_m(F(g))^+$ for any generalized iterative theory T and theory map $F : T_\Sigma \to T$.

Proof

First, assume that those components of f which are not superfluous, are exactly the first components m. In this case, $(1_m+0_{n-m})f$ can be written as $g(1_m+0_{n-m}+1_p)$, where $g : m \to m+p \in \Sigma\Theta$. Since F is a theory map it follows that $(1_m+0_{n-m})F(f) = F(g)(1_m+0_{n-m}+1_p)$, furthermore, $F(f) = \langle F(g)(1_m+0_{n-m}+1_p),\ (0_m+1_{n-m})F(f)\rangle$.

It results from this by (I) that $(1_m+0_{n-m})(F(f))^+ = F(g)^+$.
This implies $\pi^1_n(F(f))^+ = \pi^1_m(F(g))^+$.

In the general case, let $i_1, \dots, i_m$ be all different indices such that $\{\pi^{i_t}_m f \mid t \in [m]\}$ is exactly the set of not superfluous components of f. We may assume that $i_1 = 1$. Let the bijection $\rho : n \to n$ satisfy $i_t\rho = t$ for each $t \in [m]$. Applying the first case for $\rho^{-1}f(\rho+1_p)$, we get a morphism $g : m \to m+p$

in $\Sigma\Theta$ which does not contain superfluous components and satisfies $\pi_n^1(F(\rho^{-1}f(\rho+1_p)))^+ = \pi_m^1(F(g))^+$ for any theory map $F : T_\Sigma \to T$. Since F is a theory map, by the identity (F), this implies $\pi_n^1 \rho^{-1}(F(f))^+ = \pi_m^1(F(g))^+$, i.e. by $1\rho = 1$, $\pi_n^1(F(f))^+ = \pi_m^1(F(g))^+$.

We are now ready to state

Theorem 3.4.

Let $F : \Sigma \to T$ be an arbitrary ranked alphabet map into a generalized iterative theory T. There exists exactly one homomorphism $\bar{F} : I_\Sigma \to T$ extending F, i.e. such that $\bar{F}|_\Sigma = F$.

Proof

Since Σ generates T_Σ and T_Σ generates I_Σ, there can be at most one $\bar{F}$ extending F. Thus, we have to show the existence of $\bar{F}$ only.

We know that there is a theory map from T_Σ into T (considered as an algebraic theory) which extends F. Let us denote this theory map by F, too.

Define $\bar{F}$ as follows:

(i) $\bar{F}(\pi_n^i) = \pi_n^i$ if $n \geq 1$, $i \in [n]$,

(ii) $\bar{F}(f) = \pi_n^1(F(a))^+$ if $a : n \to n+p \in \Sigma\Theta$ and $f = \pi_n^1 a^+$,

(iii) $\bar{F}(<f_1,\ldots,f_n>) = <\bar{F}(f_1),\ldots,\bar{F}(f_n)>$ if $n \neq 1$, $f_i : 1 \to p$.

By Theorem 4.1.1 of [4] and lemmas 3.2. and 3.3, $\bar{F}$ is a mapping of I_Σ into T. By (i) and (iii) $\bar{F}|_\Theta = F|_\Theta$.

Take an arbitrary element $f : 1 \to p \in \Sigma$. Since $f = (0_1+f)^+$ and $0_1+f \in \Sigma\Theta$, we have $\bar{F}(f) = \pi_1^1(F(0_1+f))^+ = \pi_1^1(0_1+F(f))^+ = \pi_1^1 F(f) = F(f)$. Observe that we have used identity (C). By (iii) this results $\bar{F}|_\Sigma = F$.

By virtue of (iii), $\bar{F}$ preserves source-tupling. We now prove that $\bar{F}$ preserves composition. Since it preserves source--tupling, it is enough to show that for any morphism $f : 1 \to p$ and $g : p \to q$ $\bar{F}(fg) = \bar{F}(f)\bar{F}(g)$. This is obvious if f is base, hence we may assume that f is ideal. Or even, by a note in [5], we may confine ourselves to the case that g is base, or its first component is ideal and all other are base.

First, assume that g is base, $g = \rho$. We know that $f = \pi_n^1 a^+$, where $a : n \to n+p \in \Sigma\Theta$. By (G) $f\rho = \pi_n^1(a(1_n+\rho))^+$. Therefore, $\bar{F}(f\rho) = \pi_n^1(F(\,a(1_n+\rho)))^+ = \pi_n^1(F(a)(1_n+\rho))^+$. On the other hand $\bar{F}(f)\rho = \pi_n^1(F(a))^+\rho$, and this, by an application of (G), results that $\bar{F}(f)\rho = \pi_n^1(F(a)(1_n+\rho))^+$. Hence, $\bar{F}(f\rho) = \bar{F}(f)\rho$.

The proof of $\bar{F}(fg) = \bar{F}(f)\bar{F}(g)$ in the second case, i.e. the first component of g is ideal and the others are base, is similar, only apply identity (H') instead of (G).

Finally, we prove that for ideal $f : n \to n+p$ we have $(\bar{F}(f))^+ = \bar{F}(f^+)$. Since $\bar{F}$ is a theory map and by identity (B) it is enough to deal with the case: $n = 1$.

Since f is ideal there exists an $a : m \to m+1+p \in \Sigma\Theta$ such that $f = \pi_m^1 a^+$. Let $b = a(\langle 0_1+1_m, 1_1+0_m\rangle +1_p)$, $c = \langle \pi_m^1 b, b\rangle$. By (J) we get $f^+ = \pi_{m+1}^1 c^+$. Since $c \in \Sigma\Theta$, it follows that $\bar{F}(f^+) = \pi_{m+1}^1(F(c))^+$. Similarly, a repeated application of (J) yields $(\bar{F}(f))^+ = (\pi_m^1(F(a))^+)^+ = \pi_{m+1}^1\langle \pi_m^1 F(b), F(b)\rangle^+ = \pi_{m+1}^1(F(c))^+$. This ends the proof of Theorem 3.4.

Corollary

Theorem 3.4 holds under certain weaker assumptions, too. In fact the iteration need not be defined for arbitrary morphisms in the theory T. But we require F to be such that it being considered as a theory map $F : T_\Sigma \to T$ should satisfy $(F(f))^+$ to exist in T that whenever $f \in \Sigma\Theta$, or $f = 0_p$ for some p. Furthermore, likewise in Theorem 3.4, we have to require the identities (A) to (E) to be satisfied in T in the strong sense: for every evaluation the left hand side of an identity exists if and only if the right hand side exists, and if both of them exist, they are equal. This is always the case if T is an iterative theory and F(f) is ideal for every $f \in \Sigma$.

Theorem 3.5.

R_Σ is the generalized iterative theory freely generated by Σ.

Proof

By virtue of Theorem 3.4 and since $I_{\Sigma_\perp}$ is a weak subalgebra of R_Σ, moreover, the carriers of R_Σ and $I_{\Sigma_\perp}$ coincide, it is enough to prove the following statement: for every ranked alphabet map $F : \Sigma_\perp \to T$ such that $F(\perp) = (\pi_1^1)^+$, remember that $\perp = \pi_1^{1+}$ holds in R_Σ, the free extension $\bar{F} : I_{\Sigma_\perp} \to T$ constructed in the proof of Theorem 3.4 is a homomorphism (of generalized iterative theories) from R_Σ into T.

We know that $\bar{F}$ preserves theory operations, i.e. composition, source-tupling and injections. Hence, we have to show that $\bar{F}$ preserves (arbitrary) iteration. By identity (B) and since $\bar{F}$ is a theory map, it is enough to deal with scalar morphisms.

Take an arbitrary morphism $f : 1 \to 1+p$. If f is ideal then, by Theorem 3.4, $\bar{F}(f^+) = (\bar{F}(f))^+$. Otherwise f is an injection π^i_{1+p}. If $i = 1$ then $\bar{F}(\pi^{1^+}_{1+p}) = \bar{F}((\pi^1_1+0_p)^+) = \bar{F}(\underline{\perp}+0_p) = \bar{F}(\underline{\perp})+ 0_p =$ $= \pi^{1^+}_1 + 0_p$. On the other hand $(\bar{F}(\pi^1_{1+p}))^+ = \pi^{1^+}_{1+p} = (\pi^1_1 + 0_p)^+ =$ $= \pi^{1^+}_1 + 0_p$. Observe that identity (D) was used. Assume now that $i > 1$. Then $\pi^i_{1+p} = 0_1 + \pi^{i-1}_p$. Therefore, by (C), $\bar{F}(\pi^{i^+}_{1+p}) = \bar{F}(\pi^{i-1}_p) = \pi^{i-1}_p = \pi^{i^+}_{1+p} = (\bar{F}(\pi^i_{1+p}))^+$.

We are now able to prove the main result:

Theorem 3.6.

Identities (A) to (E) together with those defining algebraic theories, form a basis of identities of rational theories.

Proof

We have to prove that the equational class of all generalized iterative theories coincides with the equational class generated by the class of rational theories (considered as unordered theories). But this can be done immediately by Theorem 2.1 and Theorem 3.5.

Corollary

ω-continuous algebraic theories were also examined in [8] and [9]. These are special rational theories. It was proved by [8] that the free ω-continuous algebraic theory generated by Σ exactly is the theory $T^\infty_{\Sigma_\perp}$ with a certain ordering.

What is important for us from this fact is that R_Σ is a subalgebra of $T^\infty_{\Sigma_\perp}$. It results from this that the equational class generated by the rational theories exactly is that one generated by the class of all ω-continuous theories. Therefore, Theorem 3.6 remains valid even if rational theories are replaced by ω-continuous theories. The same holds for some other types of continuity (cf. Δ -continuity), too.

At the beginning of this paper we have mentioned that by the author's conjecture, identities (A) to (E) together with the defining identities of algebraic theories form a basis of identities of iterative theories, too. This conjecture is based on Theorem 3.4 and its corollary. Unfortunately, we do not know any definition of validity of an identity in a class of partial algebras by which we could prove Theorem 3.6 for iterative theories, and which is accepted by mathematicians working in partial algebras.

4. FURTHER REMARKS

We know that identities listed in (A) to (E) are not completely independent; e.g. it would be sufficient to require (A) in case n = 1, etc.

On the other hand we conjecture that all of the identities grouped in (A) or in (B) etc. cannot be omitted. A simplification of the basis will probably be introduced in a forthcoming paper.

Another note concerns with the connection of iterative and and generalized iterative theories. We have actually verified in the proof of Theorem 3.5 that R_Σ is the free generalized iterative theory generated by I_Σ. Roughly speaking, R_Σ can be obtained by adjoining a new element $\perp$ to I_Σ. It can be seen that this remains valid in the general case, too: for every iterative theory T there exists a free generalized iterative theory generated by T and this free theory can be obtained by adjoining a new element to T. This helps us to prove another interesting statement. Let T be an iterative theory and assume that T(1,0) is nonvoid, say $\perp \in T(1,0)$. Then, there is exactly one way to extend T to a generalized iterative theory such that we have $\underline{1} = \pi_1^{1^+}$.

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