Tour 22 - Binomial Coefficients

1. Prove that $\binom{n}{k} = \binom{n}{n-k}$, for all positive integers n and k with $k \leq n$.

2. Prove that for all positive integers r, s, and n, we have:

$$\binom{r}{0}\binom{s}{n} + \binom{r}{1}\binom{s}{n-1} + \ldots + \binom{r}{n-1}\binom{s}{1} + \binom{r}{n}\binom{s}{0} = \binom{r+s}{n}.$$

3. The following is known as *Pascal's Identity*:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

   (a) Find a *combinatorial* proof of Pascal's Identity.

   (b) Explain how this formula relates to Pascal's Triangle.

4. The *Binomial Theorem* states that for any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

   (a) Prove the Binomial Theorem using mathematical induction.

   (b) Prove that the sum of the elements in the n^{th} row of Pascal's Triangle is 2^n.

   (c) Prove that the alternating sum of the elements in the n^{th} row of Pascal's Triangle is 0. For example, in the fourth row, we have $1 - 4 + 6 - 4 + 1 = 0$.

5. Elizabeth's Original Problem

6. Alison's Original Problem

7. (a) Determine the number of solutions (a_1, a_2, a_3) in positive integers to the equation $a_1 + a_2 + a_3 = 10$.

   (b) Determine the number of solutions (a_1, a_2, a_3) in non-negative integers to the equation $a_1 + a_2 + a_3 = 7$.

   What do you notice about these two answers? Is this a coincidence?

8. Victoria's Original Problem

9. Prove this incredible fact: if the binary representation of n contains p ones, then there are 2^p odd numbers in the n^{th} row of Pascal's Triangle.

   (For example, $13 = 1101_2$, so there are $2^3 = 8$ odd numbers in the thirteenth row of Pascal's Triangle).