Of Operator Algebras and Operator Spaces

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Abstract

One of the recent advances in Functional Analysis has been the introduction of the notion of an (abstract) operator space. This can be seen as a refinement of the notion of a Banach space which (among other things) solves the problem that not every Banach algebra is an operator algebra.

Which theorems about Banach spaces generalise to operator spaces? This question would be easier to answer if one could prove Pestov's Conjecture: that there exists a Grothendieck topos whose internal Banach spaces are equivalent to operator spaces.

I will report on progress towards proving Pestov's conjecture.

Introduction

The category of Banach spaces and linear contractions is a really super category—locally countably presentable, and symmetric monoidal closed—and it is widely used (implicitly) as the foundations of Operator Algebra, an important and trendy field of mathematics.

Unfortunately, the latter turns out to be in error. There is a subtly different category which provides a superior foundations for Operator Algebra: the category of *operator spaces* and linear *complete contractions*.

When you ask an analyst what the difference betwen a Banach space and an operator space is, they usually start talking about quantum this and non-commutative that, and I (for one) barely understand what that is supposed to mean[1].

So the purpose of this talk is to motivate the concept of operator space in terms that a category theorist might understand, and then to show how category theory might be able to help resolve some of the outstanding open problems in operator space theory.

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[1] I'm all in favour of slogans—so long as I know what the slogans are supposed to mean!

Background

Main stuff

More stuff

Conclusion

Appendix A: More Background

Appendix B: Involutive monoidal categories

This bonus section presents a surprisingly long solution to a very small problem—but one which I think is interesting.

Definition

An *involutive monoid* (in sets) is a monoid $(m, 1, \cdot)$ together with a function $m \xrightarrow{\overline{(\,)}} m$ satisfying the axioms:

1. $\overline{\alpha \cdot \beta} = \overline{\beta} \cdot \overline{\alpha}$ for all $\alpha, \beta \in m$; and
2. $\overline{\overline{\alpha}} = \alpha$ for all $\alpha \in m$.

Examples

1. If $(m, 1, \cdot)$ is a commutative monoid, then the identity map $m \xrightarrow{\iota_m} m$ makes it into an involutive monoid.
2. If $(m, 1, \cdot)$ is a group, then $m \xrightarrow{(\,)^{-1}} m$ makes it into an involutive monoid.
3. Transposition makes the multiplicative monoid of $\mathsf{R}^{n\times n}$ into an involutive monoid.
4. Conjugation makes the multiplicative monoids of C and H (the quaternions) into involutive monoids. [This example is related to the previous one, since C and H can be construed as sub-(involutive rings) of $\mathsf{R}^{2\times 2}$ and $\mathsf{R}^{4\times 4}$, respectively.]

It is easy to see that one can define involutive monoids in any symmetric, or indeed braided, monoidal category—e.g., an involutive monoid in abelian groups (real vector spaces) is an involutive ring (real involutive algebra). But is this the greatest generality in which one can do so?

Note that since $(\mathbf{Cat}, 1, \times)$ is a symmetric monoidal category, one can define involutive monoids in it. It then appears self-evident that the latter are simply the strict (and small) case of some concept which we will call *involutive monoidal category.*

Definition (Provisional)

An involutive monoidal category is a monoidal category $(\mathcal{K}, e, \otimes)$ together with a *covariant* functor $\mathcal{K} \xrightarrow{\overline{(\ )}} \mathcal{K}$ and natural isomorphisms

$$\overline{x \otimes y} \xrightarrow{\ \omega_{x,y}\ } \overline{y} \otimes \overline{x}$$

$$\overline{\overline{x}} \xrightarrow{\ \psi_x\ } x$$

satisfying the following coherence conditions:

$$\begin{tikzcd}
\overline{(x \otimes y) \otimes z} \arrow[rr, "\overline{\alpha_{x,y,z}}"] \arrow[d, "\omega_{x \otimes y, z}"'] & & \overline{x \otimes (y \otimes z)} \arrow[d, "\omega_{x, y \otimes z}"] \\
\overline{z} \otimes \overline{x \otimes y} \arrow[d, "\iota_{\overline{z}} \otimes \omega_{x,y}"'] & & \overline{y \otimes z} \otimes \overline{x} \arrow[d, "\omega_{y,z} \otimes \iota_{\overline{x}}"] \\
\overline{z} \otimes (\overline{y} \otimes \overline{x}) & & (\overline{z} \otimes \overline{y}) \otimes \overline{x} \arrow[ll, "\alpha_{\overline{z}, \overline{y}, \overline{x}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\overline{x \otimes y}} \arrow[r, "\overline{\omega_{x,y}}"] \arrow[d, "\psi_{x \otimes y}"'] & \overline{\overline{y} \otimes \overline{x}} \arrow[d, "\omega_{\overline{y}, \overline{x}}"] \\
x \otimes y & \overline{\overline{x}} \otimes \overline{\overline{y}} \arrow[l, "\psi_x \otimes \psi_y"]
\end{tikzcd}$$

—and possibly more ...

Examples

1. $(\mathbf{Cat}, 1, \times, (\ )^{\mathsf{op}})$ is an involutive monoidal category.

2. $(\mathbf{Pos}, 1, +_{lex}, (\ )^{\mathsf{op}})$ forms a (not at all symmetric) involutive monoidal category, where $+_{lex}$ denotes the "lexicographic sum" of two posets.

3. The full subcategory of finite linearly ordered sets is closed under $+_{lex}$, and this is a favourite example of a monoidal category which looks as if it ought to be symmetric, but isn't because the symmetry isn't natural.

   Similarly, it is also closed under $(\ )^{\mathsf{op}}$ (and is therefore also an involutive monoidal category), but although $x \cong x^{\mathsf{op}}$ for every object, there is no natural isomorphism $(\ ) \cong (\ )^{\mathsf{op}}$.

4. R-bimodules should form an involutive monoidal category whenever R is an involutive ring. Given an R-bimodule A (with left and right actions denoted by $\triangleright$ and $\triangleleft$ respectively), $\overline{A}$ has the same underlying abelian group as A, but with left and right actions defined by

$$\begin{aligned} r \mathbin{\overline{\triangleright}} a &= a \triangleleft \overline{r} \\ a \mathbin{\overline{\triangleleft}} r &= \overline{r} \triangleright a \end{aligned}$$

[Of course, this is only part of a more general structure on the bicategory of rings, bimodules and bimodule homomorphisms. . .]

5. In particular, complex vector spaces form an involutive monoidal category under conjugation.

6. Banach spaces inherit an involutive structure from complex vector spaces.

Involutive monoidal categories provide (I think) the right level of generality in which to define involutive monoids.

Definition
Let $(\mathcal{K}, e, \otimes, \overline{(\ )})$ be an involutive monoidal category. Then an involutive monoid in $\mathcal{K}$ is a monoid (m, η, ν) together with a map $\overline{m} \xrightarrow{\nu} m$ satisfying

$$\begin{tikzcd} \overline{m \otimes m} \arrow[rr, "\omega_{m,m}"] \arrow[d, "\overline{\mu}"'] & & \overline{m} \otimes \overline{m} \arrow[d, "\nu \otimes \nu"] \\ \overline{m} \arrow[r, "\nu"'] & m & m \otimes m \arrow[l, "\mu"] \end{tikzcd}$$

and

$$\begin{tikzcd} \overline{\overline{m}} \arrow[r, "\overline{\nu}"] \arrow[rr, bend right, "\psi_m"'] & \overline{m} \arrow[r, "\nu"] & m \end{tikzcd}$$

Examples

1. Involutive monoids in $(\mathbf{Cat}, 1, \times, (\ )^{\mathsf{op}})$ are [the strict (and small) version of] what are sometimes called $*$-*monoidal categories*—see, for instance, [1].

2. Involutive monoids in complex vector spaces (with conjugation, not identity, as involution) are precisely what are usually called complex $*$-algebras.

3. Similarly, involutive monoids in Banach spaces are what are usually called Banach $*$-algebras.

Confession

The *very small problem* referred to at the beginning of this section was, in fact,

> *how do you define the notions of complex *-algebra and Banach *-algebra in a thoroughly categorical way?*

Lemma

If an involutive monoidal category $(\mathcal{K}, e, \otimes, \overline{(\ )})$ is (both left- and right-) closed, then the two internal homs are related by a canonical isomorphism of the form

$$\overline{y \multimap z} \xrightarrow{\sim} \overline{z} \multimapinv \overline{y}.$$

Proof

For arbitrary objects x, y and z we have natural bijections

$$\begin{array}{c} x \longrightarrow \overline{y \multimap z} \\ \hline \overline{x} \longrightarrow y \multimap z \\ \hline y \otimes \overline{x} \longrightarrow z \\ \hline \overline{\overline{x} \otimes \overline{\overline{y}}} \longrightarrow z \\ \hline x \otimes \overline{y} \longrightarrow \overline{z} \\ \hline x \longrightarrow \overline{z} \multimapinv \overline{y} \end{array}$$

which, by a standard categorical argument, gives us what we want. Q.E.D.

In particular, if a closed involutive monoidal category has a dualising object self-conjugate dualising object $d \cong \overline{d}$, then the two duals $x^* := x \multimap d$ and ${}^*x := d \multimapinv x$ are related by

$$\overline{x^*} = \overline{x \multimap d} \cong \overline{d} \multimapinv \overline{x} \cong d \multimapinv \overline{x} = {}^*\overline{x}$$

—or, equivalently ${}^*x \cong \overline{\overline{x}^*}$.

Thus also

$$x \uplus y := {}^*(y^* \otimes x^*) \cong \overline{\overline{(y^* \otimes x^*)}^*} \cong \overline{(\overline{x^*} \otimes \overline{y^*})^*}$$

—which perhaps cements the idea that $\overline{(\ )^*} \cong {}^*\overline{(\ )}$ is the "real" dual in an involutive $*$-autonomous category.

Note that an involutive $*$-autonomous category also has a canonical isomorphism

$$\overline{x \uplus y} \cong (\overline{x^*} \otimes \overline{y^*})^* \cong ({}^*\overline{x} \otimes {}^*\overline{y})^* \cong \overline{y} \uplus \overline{x}$$

which may lead one to also consider involutive linearly distributive categories. But not now.

Appendix C: More involutive monoidal categories

Involutive monoidal categories seem to strike quite close to dagger categories (forgive the pun), but they are also seem to be related to Joyal and Street's notion of a balanced (braided) monoidal category (see [2, Chapter 4], or [3]), which we now recall.

Definition

A *balance* on a braided monoidal category is a natural transformation ϑ with components of the form

$$x \xrightarrow{\vartheta_x} x$$

satisfying the diagram

$$\begin{tikzcd} x \otimes y \arrow[r, "\chi_{x,y}"] \arrow[d, "\vartheta_{x\otimes y}"'] & y \otimes x \arrow[d, "\vartheta_y \otimes \vartheta_x"] \\ x \otimes y & y \otimes x \arrow[l, "\chi_{y,x}"] \end{tikzcd}$$

[Note that if ϑ is the identity transformation, then the diagram reduces to the statement that χ is a symmetry.]

A *balanced monoidal category* is a braided monoidal category together with a chosen balance.

Part of the intuition for balanced monoidal categories comes from considering (braids of) ribbons in place of (braids of) strings. The arrow ϑ_x is to be thought of as a ribbon with a 360 degree twist.

`insert picture of 360 twist`

Now it can hardly have escaped one's notice that if an involutive monoidal category carries a natural isomorphism $x \longrightarrow \overline{x}$ (and most of the examples we have considered so far do)), then there exists at least the possibility of turning

$$\overline{x \otimes y} \xrightarrow{\omega_{x,y}} \overline{y} \otimes \overline{x}$$

into a symmetry, or perhaps braiding.

But we can sharpen this intuition if we adopt a ribbon-theoretic point of view. Let us think of $\overline{x}$ as "x with the opposite orientation"—so that an isomorphism of the form

$$x \xrightarrow{\phi_x} \overline{x}$$

should be thought of as a ribbon with a 180 degree twist.

`insert picture of 180 twist`

(also, think of the action of $\overline{(\ )}$ on arrows as revealing the other side of the ribbon—*i.e.*, flipping them, but horizontally instead of vertically).

Then it becomes clear that the composite

$$\begin{tikzcd} x \otimes y \arrow[r, "\phi_{x\otimes y}"] \arrow[rrr, bend right, "\chi_{x,y}"'] & \overline{x \otimes y} \arrow[r, "\omega_{x,y}"] & \overline{y} \otimes \overline{x} \arrow[r, "\phi_y^{-1} \otimes \phi_x^{-1}"] & y \otimes x \end{tikzcd}$$

—which represents

`insert picture of 180 twist and two -180 twists combining to form braid`

—should indeed be a braid and not a symmetry; and, moreover, that the composite

$$\begin{tikzcd} x \arrow[r, "\phi_x"] \arrow[rrr, bend right, "\vartheta_x"'] & \overline{x} \arrow[r, "\overline{\phi_x}"] & \overline{\overline{x}} \arrow[r, "\psi_x"] & x \end{tikzcd}$$

—which represents

`insert picture of two 180 twists combining to form a 360 twist`

—should be a balance for it.

[Note that in this attempt toward a graphical calculus, ω and ψ are not represented.]

Being lazy, I have only proven half of what I should.

Theorem

Let $(\mathcal{K}, i, \otimes, \overline{(\ )})$ be an involutive monoidal category, and suppose that

$$x \xrightarrow{\phi_x} \overline{x}$$

is a natural isomorphism such that $\overline{\phi_x} = \phi_{\overline{x}}$ and such that the composite

$$\begin{tikzcd} x \otimes y \arrow[r, "\phi_{x\otimes y}"] \arrow[rrr, bend right, "\chi_{x,y}"'] & \overline{x \otimes y} \arrow[r, "\omega_{x,y}"] & \overline{y} \otimes \overline{x} \arrow[r, "\phi_y^{-1} \otimes \phi_x^{-1}"] & y \otimes x \end{tikzcd}$$

defines a braiding on $(\mathcal{K}, i, \otimes)$.

Then

$$\begin{tikzcd} x \arrow[r, "\phi_x"] \arrow[rrr, bend right, "\vartheta_x"'] & \overline{x} \arrow[r, "\overline{\phi_x}"] & \overline{\overline{x}} \arrow[r, "\psi_x"] & x \end{tikzcd}$$

defines a balance for χ.

Proof

Consider the diagram

$$
\begin{tikzcd}
x \otimes y \arrow[r, "\phi_{x\otimes y}"] \arrow[d, "\phi_{x\otimes y}"] \arrow[rrr, bend left, "\chi_{x,y}"] \arrow[ddd, bend right, "\vartheta_{x\otimes y}"'] & \overline{x \otimes y} \arrow[r, "\omega_{x,y}"] \arrow[dl, equal] \arrow[dr, "\overline{\phi_y \otimes \phi_x}"] & \overline{y} \otimes \overline{x} \arrow[r, "\phi_y^{-1} \otimes \phi_x^{-1}"] \arrow[dr, equal] & y \otimes x \arrow[d, "\phi_y \otimes \phi_x"] \arrow[ddd, bend left, "\vartheta_y \otimes \vartheta_x"] \\
\overline{x \otimes y} \arrow[d, "\overline{\phi_{x\otimes y}}"] & & \overline{\overline{x} \otimes \overline{y}} \arrow[d, "\overline{\omega_{y,x}^{-1}}"] \arrow[dr, "\omega_{\overline{x},\overline{y}}"] & \overline{y} \otimes \overline{x} \arrow[d, "\overline{\phi_y} \otimes \overline{\phi_x}"] \\
\overline{\overline{x \otimes y}} \arrow[d, "\psi_{x\otimes y}"] & \overline{\overline{\overline{y \otimes x}}} \arrow[dr, "\psi_{\overline{y\otimes x}}"] & \overline{\overline{y \otimes x}} \arrow[l, "\overline{\overline{\phi_{y\otimes x}}}"'] \arrow[dr, "\psi_{y\otimes x}"] & \overline{\overline{y}} \otimes \overline{\overline{x}} \arrow[d, "\psi_y \otimes \psi_x"] \\
x \otimes y & \overline{x} \otimes \overline{y} \arrow[l, "\phi_x^{-1} \otimes \phi_y^{-1}"'] & \overline{y \otimes x} \arrow[l, "\omega_{y,x}"'] & y \otimes x \arrow[l, "\phi_{y\otimes x}"'] \arrow[lll, bend left, "\chi_{y,x}"]
\end{tikzcd}
$$

where

- the top-left and top-right triangles are tautologies;
- the trapezoid (near the top-right) is a naturality square for ω;
- the upper of the two diamonds (near the bottom-right) is a variant of the two coherence axioms; [Note that $\overline{\omega_{y,x}^{-1}} = \overline{\omega_{y,x}}^{-1}$.]
- the lower of the two diamonds (also near the bottom-right) is a naturality square for ψ; and
- the remaining figure commutes for reasons explained below.

Let α denote the composite

$$x \otimes y \xleftarrow{\phi_x^{-1} \otimes \phi_y^{-1}} \overline{x} \otimes \overline{y} \xleftarrow{\omega_{y,x}} \overline{y \otimes x}$$

which occurs at the bottom left of the diagram above; then $\overline{\alpha^{-1}}$ equals the composite

$$\overline{x \otimes y} \xrightarrow{\overline{\phi_y \otimes \phi_x}} \overline{\overline{x} \otimes \overline{y}} \xrightarrow{\overline{\omega_{y,x}^{-1}}} \overline{\overline{y \otimes x}}$$

which occurs along the top-right of the remaining figure.

Thus

$$
\begin{tikzcd}
 & \overline{x\otimes y} \arrow[dl, equal] \arrow[ddrr, description, "\overline{\alpha^{-1}}"] \arrow[drr, description, "\overline{\phi_y\otimes\phi_x}"] & & \\
\overline{x\otimes y} \arrow[d, description, "\overline{\phi_{x\otimes y}}"] & & & \overline{\overline{x}\otimes\overline{y}} \arrow[d, description, "\overline{\omega^{-1}_{y,x}}"] \\
\overline{\overline{x\otimes y}} \arrow[d, description, "\psi_{x\otimes y}"] & \overline{\overline{\overline{y\otimes x}}} \arrow[l, description, "\overline{\overline{\alpha}}"] \arrow[drr, description, "\psi_{\overline{y\otimes x}}"] & & \overline{\overline{y\otimes x}} \arrow[ll, description, "\overline{\phi_{\overline{y\otimes x}}}"] \arrow[ulll, description, "\overline{\alpha}"] \\
x\otimes y & & & \overline{y\otimes x} \arrow[lll, description, "\alpha"]
\end{tikzcd}
$$

commutes, since

- the trapezoid is a naturality square for ψ;
- the lower triangle is the result of applying $\overline{(\ )}$ to a naturality square for ϕ;
- the middle triangle is a tautology; and
- the upper triangle has already been explained.

[Note that I also sneakily changed a $\overline{\overline{\phi_{y\otimes x}}}$ into a $\overline{\phi_{\overline{y\otimes x}}}$, thus using the seemingly extraneous axiom in the statement of the theorem.] Q.E.D.

References

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[2] André Joyal and Ross Street. The geometry of tensor calculus. I. *Adv. Math.*, 88(1):55–112, 1991.

[3] André Joyal, Ross Street, and Dominic Verity. Traced monoidal categories. *Math. Proc. Cambridge Philos. Soc.*, 119(3):447–468, 1996.