A solution in search of a problem: affine linearly distributive categories

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Abstract

This paper describes a new and non-trivial way of constructing linearly distributive categories. It also introduces, and justifies, a more general definition of affine monoidal category than found in the literature.

1 Affine semigroupal categories

Let's start without units, for a change. Strangely, this will lead to more elegant proofs in section 2, and hopefully explain the lack of coherence conditions in section 3. For lack of a better word, I use *semigroupal* to mean *like monoidal but without a unit*.

Definition 1.1

An *affine semigroupal category* is a semigroupal category $(\mathcal{K}, \otimes)$ together with natural transformations $\overleftarrow{\eta}, \overrightarrow{\eta}$ whose components have the form

$$x \xleftarrow{\overleftarrow{\eta}_{x,y}} x \otimes y \xrightarrow{\overrightarrow{\eta}_{x,y}} y$$

and which satisfy the following three coherence axioms:

$$\begin{array}{ccc} (x \otimes y) \otimes z & \xrightarrow{\alpha_{x,y,z}} & x \otimes (y \otimes z) \\ & {\scriptstyle \overleftarrow{\eta}_{x\otimes y,z}} \searrow \qquad \swarrow {\scriptstyle \iota_x \otimes \overleftarrow{\eta}_{y,z}} & \\ & x \otimes y & \end{array}$$

$$\begin{array}{ccc} (x \otimes y) \otimes z & \xrightarrow{\alpha_{x,y,z}} & x \otimes (y \otimes z) \\ & {\scriptstyle \overleftarrow{\eta}_{x,y} \otimes \iota_z} \searrow \qquad \swarrow {\scriptstyle \iota_x \otimes \overrightarrow{\eta}_{y,z}} & \\ & x \otimes z & \end{array}$$

$$\begin{array}{ccc} (x \otimes y) \otimes z & \xrightarrow{\alpha_{x,y,z}} & x \otimes (y \otimes z) \\ & {\scriptstyle \vec{\eta}_{x,y} \otimes \iota_z} \searrow \quad \swarrow {\scriptstyle \vec{\eta}_{x,y \otimes z}} & \\ & y \otimes z & \end{array}$$

We shall call $\overleftarrow{\eta}$ *left ejection*, and $\vec{\eta}$ *right ejection.*

The opposite of an affine semigroupal category will be called a *co-affine semigroupal category.*

Remark 1.2

The definition of an affine semigroupal category is symmetric in the sense that if $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$ is an affine semigroupal category and $\otimes'$ is defined by

$$x \otimes' y = y \otimes x$$

then $(\mathcal{K}, \otimes', \vec{\eta}, \overleftarrow{\eta})$ is also an affine semigroupal category, called the *reverse* of $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$.

`(Because I don't think that either of the words "dual" or "opposite" are appropria in this case.)`

Many of the proofs below will be shortened by appealing to this symmetry.

Example 1.3

`Actually, an interesting counter-example! Real examples will be postponed until section 4.`

Consider the "topologist's Delta"—the category of positive finite ordinals, and order preserving functions—which is semigroupal with respect to addition. Then we can define natural transformations

$$x \xleftarrow{\overleftarrow{\eta}_{x,y}} x + y \xrightarrow{\vec{\eta}_{x,y}} y$$

by

$$\overleftarrow{\eta}(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in x \\ \top_x & \text{if } \alpha \in y \end{cases} \quad \text{and} \quad \vec{\eta}(\alpha) = \begin{cases} \bot_y & \text{if } \alpha \in x \\ \alpha & \text{if } \alpha \in y \end{cases}$$

—but these do not satisfy the second coherence axiom of Definition 1.1.

Suppose that $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$ is an affine semigroupal category such that $\mathcal{K}$ also has binary products (denoted $\&$). Then, of course, $\overleftarrow{\eta}$ and $\vec{\eta}$ can be paired into a single natural transformation η with components of the form

$$x \otimes y \xrightarrow{\eta_{x,y}} x \,\&\, y$$

Theorem 1.4

If $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$ is an affine semigroupal category with binary products, then $(\mathrm{Id}_{\mathcal{K}}, \eta)$ constitute a semigroupal functor $(\mathcal{K}, \&) \longrightarrow (\mathcal{K}, \otimes)$.

Proof

The statement of the theorem means that the diagram

$$\begin{CD}
(x \otimes y) \otimes z @>{\alpha_{x,y,z}}>> x \otimes (y \otimes z) \\
@V{\eta_{x,y} \otimes \iota_{z})}VV @VV{\iota_{x}) \otimes \eta_{y,z}}V \\
(x \,\&\, y) \otimes z @. x \otimes (y \,\&\, z) \\
@V{\eta_{x\&y,z}}VV @VV{\eta_{x,y\&z}}V \\
(x \,\&\, y) \,\&\, z @>>{\alpha_{x,y,z})}> x \,\&\, (y \,\&\, z)
\end{CD}$$

commutes. [Note that we use the symbol α to denote both of the associativity isomorphisms; hopefully this will not confuse.]

By the universal property of $\&$, this reduces to the following three diagrams:

$$\begin{CD}
(x \otimes y) \otimes z @>{\alpha_{x,y,z}}>> x \otimes (y \otimes z) \\
@V{\overleftarrow{\eta}_{x\otimes y,z}}VV @VV{\overleftarrow{\eta}_{x,y\otimes z}}V \\
x \otimes y @>>{\overleftarrow{\eta}_{x,y}}> x
\end{CD}$$

$$\begin{CD}
(x \otimes y) \otimes z @>{\alpha_{x,y,z}}>> @>>> x \otimes (y \otimes z) \\
@V{\overleftarrow{\eta}_{x\otimes y,z}}VV @. @VV{\overrightarrow{\eta}_{x,y\otimes z}}V \\
x \otimes y @>>{\overrightarrow{\eta}_{x,y}}> y @<<{\overleftarrow{\eta}_{y,z}}< y \otimes z
\end{CD}$$

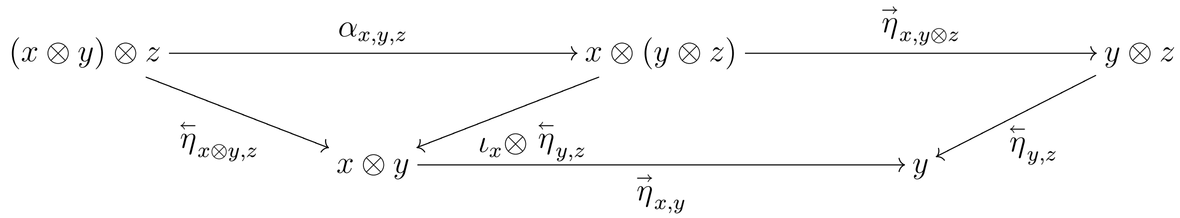
$$\begin{CD}
(x \otimes y) \otimes z @>{\alpha_{x,y,z}}>> x \otimes (y \otimes z) \\
@V{\overrightarrow{\eta}_{x\otimes y,z}}VV @VV{\overrightarrow{\eta}_{x,y\otimes z}}V \\
z @<<{\overrightarrow{\eta}_{y,z}}< y \otimes z
\end{CD}$$

—note that the last of these is the reverse of the first (*cf.* Remark 1.2), so it suffices to prove the first two.

But the first coherence axiom, together with the naturality of $\overleftarrow{\eta}$ and $\overrightarrow{\eta}$, implies

$$\begin{array}{ccccc}
(x \otimes y) \otimes z & \xrightarrow{\alpha_{x,y,z}} & x \otimes (y \otimes z) & \xrightarrow{\overleftarrow{\eta}_{x,y\otimes z}} & x \\
\searrow^{\overleftarrow{\eta}_{x\otimes y,z}} & \swarrow_{\iota_x \otimes \overleftarrow{\eta}_{y,z}} & & & \swarrow^{\iota_x} \\
x \otimes y & & \xrightarrow[\overleftarrow{\eta}_{x,y}]{} & & x
\end{array}$$

and

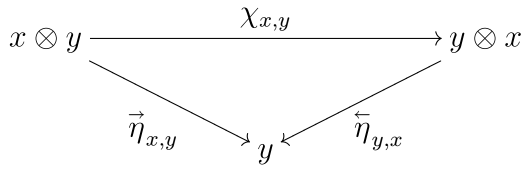


Q.E.D.

So far, we have been scrupulously avoiding symmetry/braiding, although it does make things easier.

Theorem 1.5

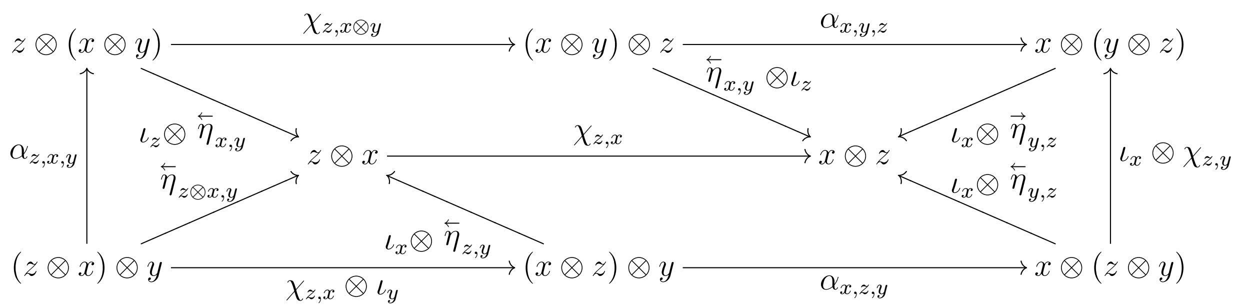
If $(\mathcal{K}, \otimes, \chi)$ is a braided semigroupal category together with a natural transformation $\overleftarrow{\eta}$ satisfying the first coherence axiom of Definition 1.1 and $\vec{\eta}$ is the natural transformation defined by

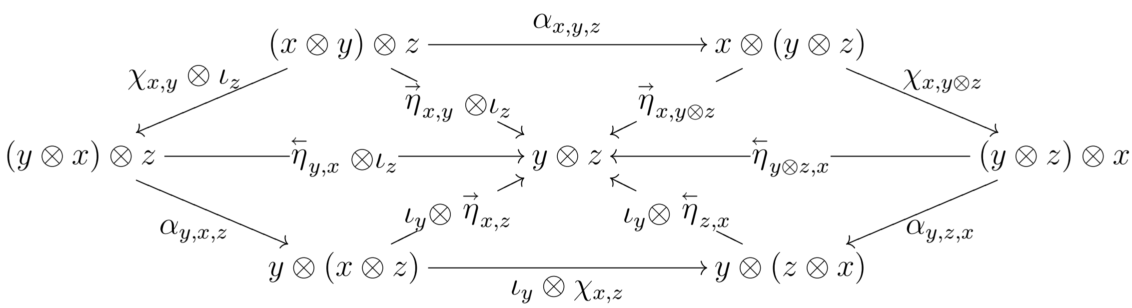


then $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$ is an affine semigroupal category.

Proof

It suffices to prove the remaining two coherence axioms, which can be done as follows:





Q.E.D.

Of course, if the braiding really is a braiding and not a symmetry, we could apply the reverse of the previous theorem to its result, and may end up with a new affine semigroupal structure $(\mathcal{K}, \otimes, \overleftarrow{\eta}', \overrightarrow{\eta})$, with $\overleftarrow{\eta}'$ defined by

$$\begin{array}{ccc} y \otimes x & \xrightarrow{\chi_{y,x}} & x \otimes y \\ & \overleftarrow{\eta}'_{y,x} \searrow \quad \swarrow \overrightarrow{\eta}_{x,y} & \\ & y & \end{array}$$

—*i.e.*, by

$$\begin{array}{ccccc} y \otimes x & \xrightarrow{\chi_{y,x}} & x \otimes y & \xrightarrow{\chi_{x,y}} & y \otimes x \\ & \overleftarrow{\eta}'_{y,x} \searrow & & \swarrow \overleftarrow{\eta}_{y,x} & \\ & & y & & \end{array}$$

The avoidance of such a situation motivates the definition below.

Definition 1.6

A *braided affine semigroupal category* is an affine semigroupal category $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \overrightarrow{\eta})$ together with a braiding on $\otimes$ (denoted χ) such that both of the diagrams

$$\begin{array}{ccccc} x \otimes y & \xrightarrow{\chi_{x,y}} & y \otimes x & \xrightarrow{\chi_{y,x}} & x \otimes y \\ \overrightarrow{\eta}_{x,y} \searrow & & \swarrow \overleftarrow{\eta}_{y,x} \quad \overleftarrow{\eta}_{y,x} \searrow & & \swarrow \overrightarrow{\eta}_{x,y} \\ & y & & y & \end{array}$$

commute.

2 Canonical co-affine structures

In this section, we shall consider an affine semigroupal category $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \overrightarrow{\eta})$ such that $\mathcal{K}$ has pushouts which distribute over $\otimes$ (*i.e.*, the functors $\mathcal{K} \xrightarrow{x \otimes (-)} \mathcal{K}$ and $\mathcal{K} \xrightarrow{(-) \otimes y} \mathcal{K}$ preserve pushouts).

In this case, we can define a dual tensor product (denoted $\parr$) as the pushout of the two ejections, as depicted below.

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\overrightarrow{\eta}_{x,y}} & y \\ {\scriptstyle \overleftarrow{\eta}_{x,y}} \downarrow & p.o. & \downarrow {\scriptstyle \overleftarrow{\varepsilon}_{x,y}} \\ x & \xrightarrow[\overrightarrow{\varepsilon}_{x,y}]{} & x \parr y \end{array}$$

`Actually, we don't need arbitrary pushouts, but I'm not sure the extra generality is of interest.`

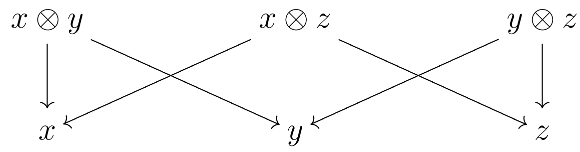
Theorem 2.1

Under the given hypotheses, $(\mathcal{K}, \parr, \vec{\varepsilon}, \overleftarrow{\varepsilon})$ is a co-affine semigroupal category, and $(\mathcal{K}, \otimes, \parr)$ is linearly distributive.

Proof

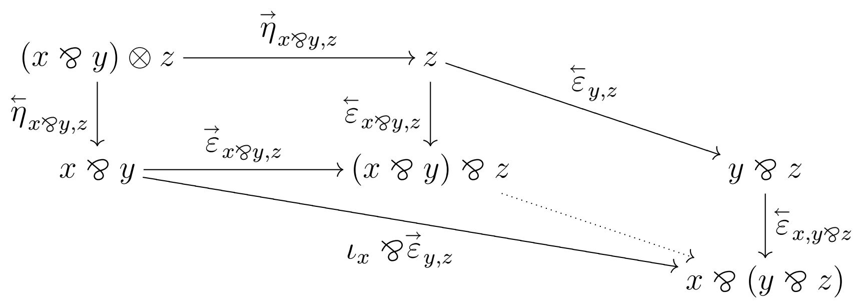
Firstly note that is automatic from their respective definitions that $\parr$ is a functor, and that $\vec{\varepsilon}$ and $\overleftarrow{\varepsilon}$ are natural transformations.

Roughly speaking, the associativity of $\parr$ follows from the fact that $(x \parr y) \parr z$ and $x \parr (y \parr z)$ are both colimits of the diagram



and therefore isomorphic.

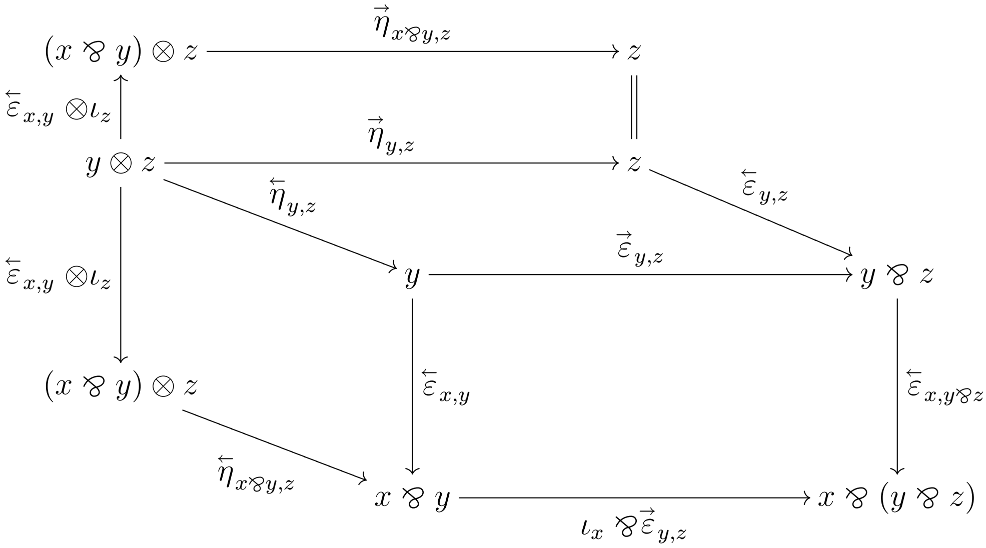
More precisely, $\alpha_{x,y,z}$ is defined by



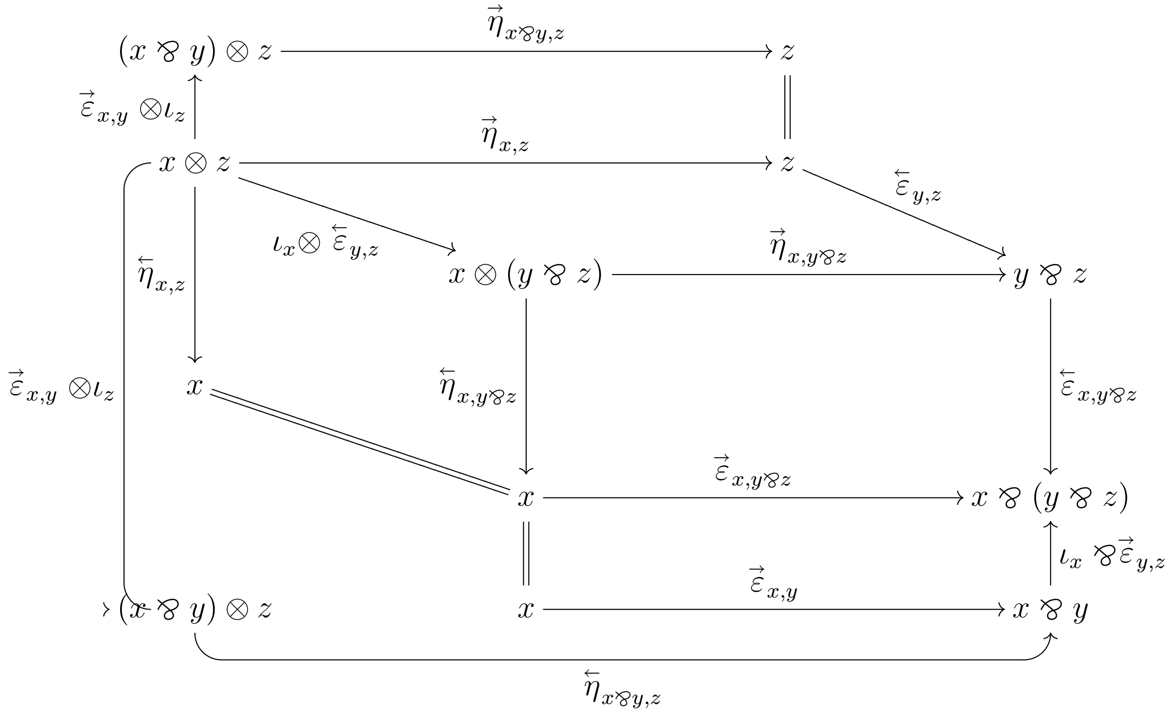
and $(\alpha_{x,y,z})^{-1}$ symmetrically.

Unfortunately, showing that the outer diagram commutes is actually somewhat involved. But note that the triangle in the diagram above is the first coherence axiom for a co-affine semigroupal category. So the effort is, perhaps, worthwhile.

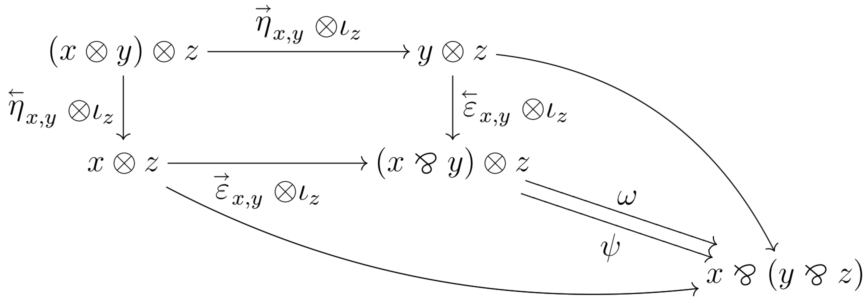
Now


$(x \parr y) \otimes z$
$\vec{\eta}_{x \parr y, z}$
z
$\overleftarrow{\varepsilon}_{x,y} \otimes \iota_z$
$y \otimes z$
$\vec{\eta}_{y,z}$
z
$\overleftarrow{\eta}_{y,z}$
$\overleftarrow{\varepsilon}_{y,z}$
$\overleftarrow{\varepsilon}_{x,y} \otimes \iota_z$
y
$\vec{\varepsilon}_{y,z}$
$y \parr z$
$(x \parr y) \otimes z$
$\overleftarrow{\varepsilon}_{x,y}$
$\overleftarrow{\varepsilon}_{x,y \parr z}$
$\overleftarrow{\eta}_{x \parr y, z}$
$x \parr y$
$x \parr (y \parr z)$
$\iota_x \parr \vec{\varepsilon}_{y,z}$

and


$(x \parr y) \otimes z$
$\vec{\eta}_{x \parr y, z}$
z
$\vec{\varepsilon}_{x,y} \otimes \iota_z$
$x \otimes z$
$\vec{\eta}_{x,z}$
z
$\overleftarrow{\varepsilon}_{y,z}$
$\iota_x \otimes \overleftarrow{\varepsilon}_{y,z}$
$x \otimes (y \parr z)$
$\vec{\eta}_{x, y \parr z}$
$y \parr z$
$\overleftarrow{\eta}_{x,z}$
$\vec{\varepsilon}_{x,y} \otimes \iota_z$
x
$\overleftarrow{\eta}_{x, y \parr z}$
$\overleftarrow{\varepsilon}_{x, y \parr z}$
x
$\vec{\varepsilon}_{x, y \parr z}$
$x \parr (y \parr z)$
$\iota_x \parr \vec{\varepsilon}_{y,z}$
$(x \parr y) \otimes z$
x
$\vec{\varepsilon}_{x,y}$
$x \parr y$
$\overleftarrow{\eta}_{x \parr y, z}$

prove that



—where ω, ψ denote the composites

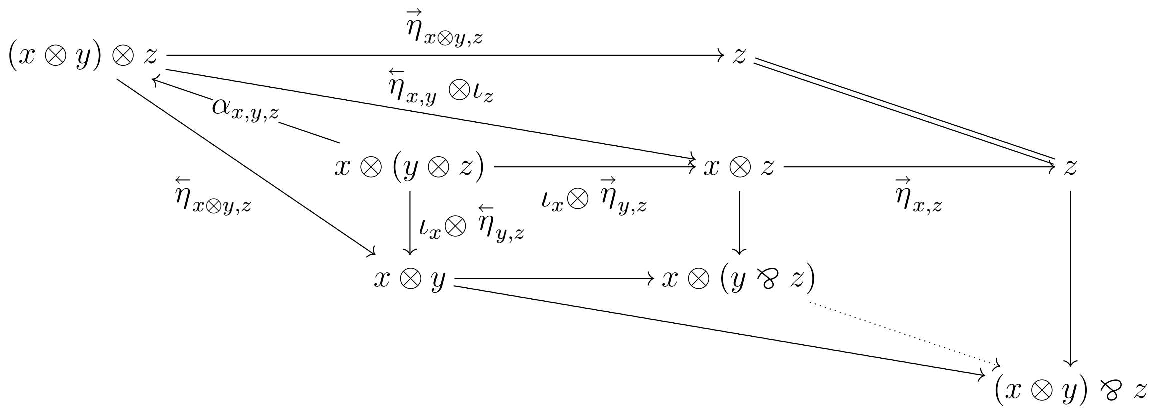
$$(x \parr y) \otimes z \xrightarrow{\overleftarrow{\eta}_{x \parr y,z}} x \parr y \xrightarrow{\iota_x \parr \overrightarrow{\varepsilon}_{y,z}} x \parr (y \parr z)$$
$$(x \parr y) \otimes z \xrightarrow{\overrightarrow{\eta}_{x \parr y,z}} z \xrightarrow{\overleftarrow{\varepsilon}_{y,z}} y \parr z \xrightarrow{\overleftarrow{\varepsilon}_{x,y \parr z}} x \parr (y \parr z)$$

respectively—and hence that $\omega = \psi$; this is precisely what we were trying to show.

To show the second coherence axiom, we again appeal to the uniqueness property of pushouts, and another huge diagram. `To appear.`

The third coherence axiom is the reverse of the first, and hence follows from a previous argument.

The linear distribution $x \otimes (y \parr z) \xrightarrow{\overrightarrow{\kappa}_{x,y,z}} (x \otimes y) \parr z$ is defined as follows:



and $(x \parr y) \otimes z \xrightarrow{\overleftarrow{\kappa}_{x,y,z}} x \parr (y \otimes z)$ by the reverse diagram.

All the coherence conditions for $\overrightarrow{\kappa}, \overleftarrow{\kappa}$ follow easily from the uniqueness part of the universal property of pushouts. Q.E.D.

Remark 2.2

Suppose we apply Theorem 2.1 to a category that also has pullbacks; then there exists the possibility that the pullbacks distribute over $\parr$. If this is indeed the case, then we can also apply the dual of Theorem 2.1 to $(\mathcal{K}, \parr, \vec{\varepsilon}, \overleftarrow{\varepsilon})$ and construct a potentially different tensor product. The avoidance of such a situation motivates the second of the definitions below.

Definitions 2.3

1. a *bi-affine linearly distributive category* is a linearly distributive category $(\mathcal{K}, \otimes, \parr, \vec{\kappa}, \overleftarrow{\kappa})$ together with ejections $\overleftarrow{\eta}, \vec{\eta}$ and injections $\vec{\varepsilon}, \overleftarrow{\varepsilon}$ making $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$ affine and $(\mathcal{K}, \parr, \vec{\varepsilon}, \overleftarrow{\varepsilon})$ co-affine.

2. a *canonically bi-affine linearly distributive category* is a bi-affine linearly distributive category such that the diagram

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\vec{\eta}_{x,y}} & y \\ {\scriptstyle \overleftarrow{\eta}_{x,y}} \downarrow & & \downarrow {\scriptstyle \overleftarrow{\varepsilon}_{x,y}} \\ x & \xrightarrow[\vec{\varepsilon}_{x,y}]{} & x \parr y \end{array}$$

   is both a pushout and a pullback. [This does not imply that $\mathcal{K}$ has arbitrary pushouts and pullbacks.]

3 Putting units back in

Having established our main theorem solely on the basis of Definition 1.1, and therefore without units, we now investigate how the presence of units affects said definition. We provisionally define an *affine monoidal category* to be an affine semigroupal category which is also monoidal.

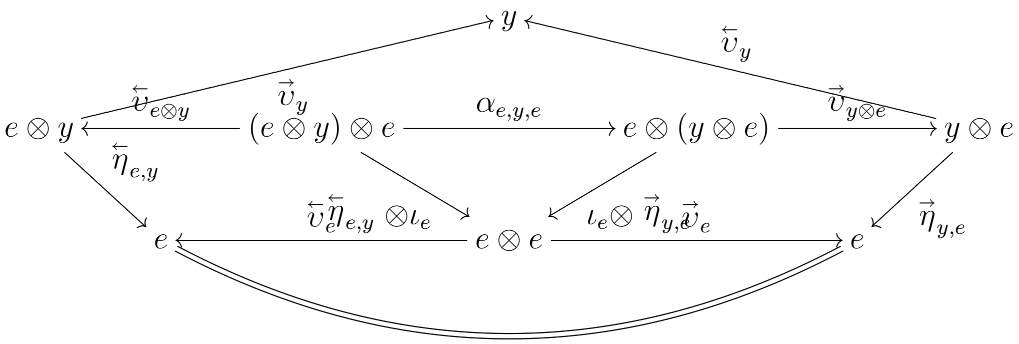
Lemma 3.1
Let $(\mathcal{K}, e, \otimes, \overleftarrow{\eta}, \vec{\eta})$ be an affine monoidal category. Then, for any object y, the arrows

$$y \xrightarrow{(\vec{\upsilon}_y)^{-1}} e \otimes y \xrightarrow{\overleftarrow{\eta}_{e,y}} e$$

$$y \xrightarrow{(\overleftarrow{\upsilon}_y)^{-1}} y \otimes e \xrightarrow{\vec{\eta}_{y,e}} e$$

are equal.

Proof

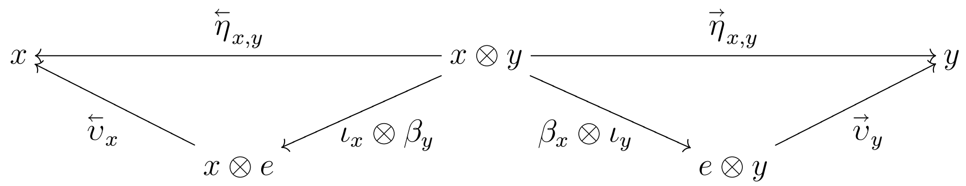


Q.E.D.

Thus, not only is the unit of an affine monoidal category weakly terminal, but its weak terminality is witnessed by a single canonical natural transformation. Moreover, the ejections of an affine monoidal category can be recovered from this natural transformation, henceforth denoted β.

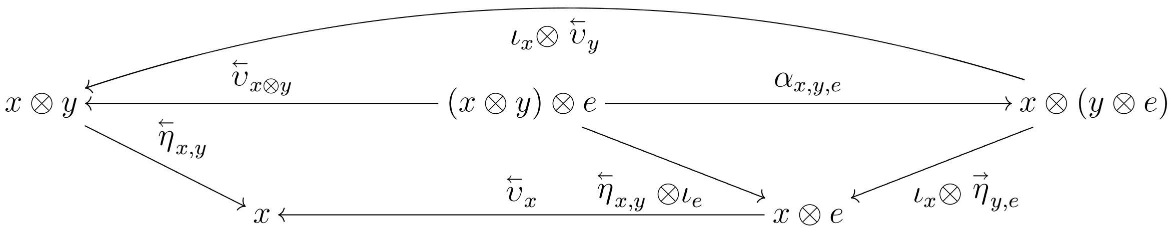
Theorem 3.2

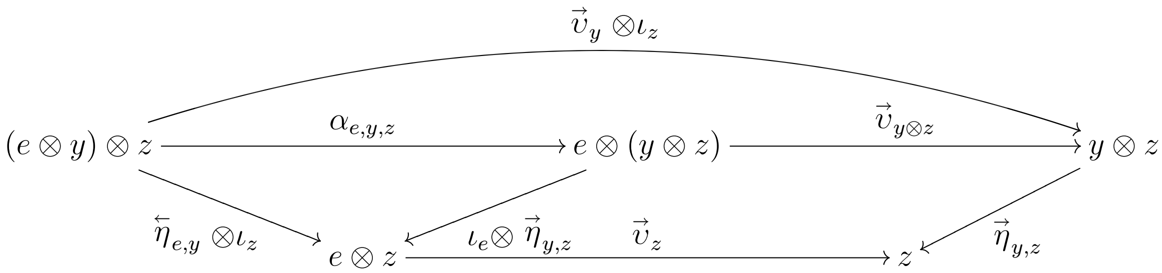
If $(\mathcal{K}, e, \otimes, \overleftarrow{\eta}, \overrightarrow{\eta})$ is an affine monoidal category, and β is defined as above, then the diagrams



commute.

Proof



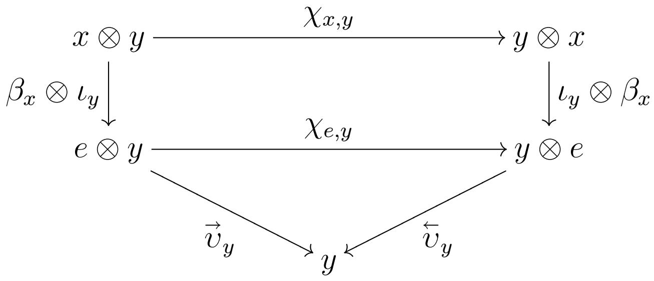


Q.E.D.

Corollary 3.3

Any braiding on an affine monoidal category automatically satisfies the coherence axiom in Definition 1.6.

Proof



Q.E.D.

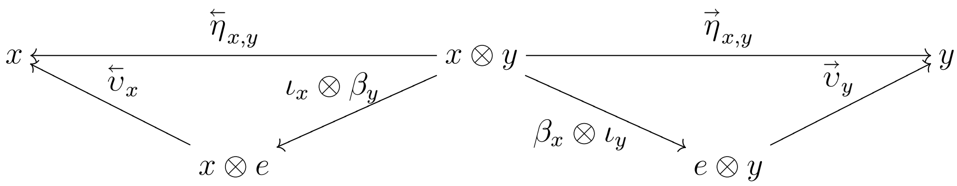
Now it is reasonable to ask whether β satisfies any further axioms than naturality. The answer is no.

Theorem 3.4

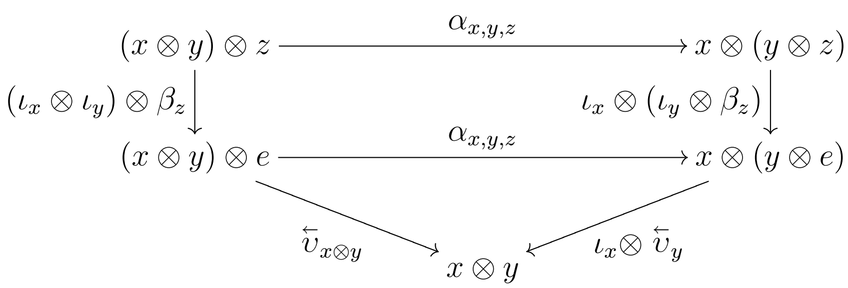
Let $(\mathcal{K}, e, \otimes)$ be a monoidal category, and β a natural transformation with components of the form

$$x \xrightarrow{\beta_x} e.$$

Then $(\mathcal{K}, \otimes, \overleftarrow{\eta}, \vec{\eta})$ is an affine semigroupal category where $\overleftarrow{\eta}$ and $\vec{\eta}$ are defined by



Proof



etc. Q.E.D.

As a result of the theorems above, we can replace our provisional definition of affine monoidal category with the following equivalent definition.

Definition 3.5

An *affine monoidal category* is a monoidal category $(\mathcal{K}, e, \otimes)$ together with a natural transformation β witnessing the weak terminality of e.

Also, as a result of Corollary 3.3, we can define a *braided affine monoidal category* to be an affine monoidal category which is also braided.

Remarks 3.6

1. If $\mathcal{K}$ has a genuine terminal object t, then any natural transformation of the form

$$x \xrightarrow{\;\beta_x\;} e$$

is uniquely determined by its t-component, because for any x we have

$$\begin{array}{ccc} x & \xrightarrow{\;\beta_x\;} & e \\ {\scriptstyle !_x}\big\downarrow & & \big\| \\ t & \xrightarrow{\;\beta_t\;} & e \end{array}$$

2. If $\mathcal{K}$ has a terminal object and binary products, then $(\mathrm{Id}_{\mathcal{K}}, !_e, \eta)$ do not necessarily constitute a monoidal functor $(\mathcal{K}, t, \&) \longrightarrow (\mathcal{K}, e, \otimes)$. In order for this to be the case, we must have

$$\begin{array}{ccc} e \otimes x & \xrightarrow{\;!_e \otimes \iota_x\;} & t \otimes x \\ {\scriptstyle \vec{\upsilon}_x}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{t,x}} \\ x & \xleftarrow[\;\vec{\upsilon}_x\;]{} & t \,\&\, x \end{array}$$

[Again, we use $\vec{\upsilon}$ to mean two different things.]

But since $t \,\&\, x \xrightarrow{\vec{\upsilon}_x} x$ is (by definition) $t \,\&\, x \xrightarrow{\vec{\pi}_{t,x}} x$, we have

$$\begin{array}{ccc} e \otimes x & \xleftarrow{\;\beta_t \otimes \iota_x\;} & t \otimes x \\ {\scriptstyle \vec{\upsilon}_x}\big\downarrow & \swarrow{\scriptstyle \vec{\eta}_{t,x}} & \big\downarrow{\scriptstyle \eta_{t,x}} \\ x & \xleftarrow[\;\vec{\upsilon}_x\;]{} & t \,\&\, x \end{array}$$

and therefore

$$\begin{array}{ccc} e \otimes x & \xrightarrow{!_e \otimes \iota_x} & t \otimes x \\ {\scriptstyle \vec{\upsilon}_x}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \beta_t \otimes \iota_x} \\ x & \xleftarrow[\vec{\upsilon}_x]{} & e \otimes x \end{array}$$

which can only happen if and only if $!_e$ and β_t are inverse to each other—*i.e.*, if and only if e is a terminal object.

3. The preceding is not to be considered a flaw in the theory. By way of analogy, note that in a general mix category we have only a semigroupal functor $(\mathcal{K}, \parr) \xrightarrow{(\mathrm{Id}_{\mathcal{K}}, \xi)} (\mathcal{K}, \otimes)$; in order to have a monoidal functor $(\mathcal{K}, d, \parr) \longrightarrow (\mathcal{K}, e, \otimes)$, we must have $d \cong e$.

4 Examples

1. Any cartesian closed category satisfies the hypotheses of Theorem 2.1, trivially.

2. For instance, in $(\mathbf{Set}, 1, \times)$, the diagram

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\vec{\eta}_{x,y}} & y \\ {\scriptstyle \overleftarrow{\eta}_{x,y}}\downarrow & & \downarrow{\scriptstyle \overleftarrow{\varepsilon}_{x,y}} \\ x & \xrightarrow[\vec{\varepsilon}_{x,y}]{} & x \parr y \end{array}$$

defines the operation

$$x \parr y = \begin{cases} x & \text{if } y = \emptyset \\ y & \text{if } x = \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Note that, while $\parr$ does not preserve pullbacks in general (consider

$$\begin{array}{ccc} & & 1 \\ & & \downarrow{\scriptstyle \alpha} \\ 1 & \xrightarrow[\beta]{} & y \end{array}$$

with $\alpha \neq \beta$), the diagram

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\vec{\eta}_{x,y}} & y \\ {\scriptstyle \overleftarrow{\eta}_{x,y}}\downarrow & & \downarrow{\scriptstyle \overleftarrow{\varepsilon}_{x,y}} \\ x & \xrightarrow[\vec{\varepsilon}_{x,y}]{} & x \parr y \end{array}$$

is always a pullback, as well as a pushout. Thus $(\mathbf{Set}, 1, \otimes, 0, \otimes)$ is a strongly bi-affine linearly distributive category.

3. The same result holds for any presheaf topos, since limits and colimits are calculated pointwise there.

4. More interestingly, one can construct affine monoidal closed structures on a presheaf topos by starting with affine monoidal structures on the exponent...

5. Let $(\mathcal{K}, e, \otimes)$ be a monoidal category and (m, η, μ) a monoid in it. Then there is an obvious monoidal structure on $\mathcal{K}/m$ with $e \xrightarrow{\eta} m$ as unit. Since $\mathcal{K}/m$ has terminal object $m \xrightarrow{\iota_m} m$, requiring $e \xrightarrow{\eta} m$ to be naturally weakly terminal is the same as requiring η to be a split epi in $\mathcal{K}$.

6. The preceding can occur. For example, let R be a ring which can be decomposed as the direct sum of two of its own ideals; i.e., we have $I, J \subseteq R$, $I + J = R$ and $I \cap J = \{0\}$.

   Then $A = R/I$ is an R-algebra, whose underlying R-module is isomorphic to J; i.e., A is a monoid in R-mod whose unit map $R \longrightarrow A$ is split by the inclusion $A \cong J \longrightarrow R$.