A complete lattice admitting no co-monoid structure

J.M. Egger*

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Abstract

In this paper we show that $\mathbf{M}_3$, the five-element modular non-distributive lattice, does not admit a co-monoid structure in the category of complete lattices and join-preserving maps, equipped with the natural tensor product; this is surprising because it does admit several co-monoid structures when the same category is equipped with a seemingly less natural tensor product.

1 Introduction

The author was led to consider co-monoids in the tensor category $(\mathbf{Sup}, 2, \boxtimes)$ as, apparently, the most general means of producing certain models of linear logic—called categories of *adherence spaces*, [6].

In the case of completely distributive lattices, these co-monoids are easy to understand, via duality. Wanting to understand the full generality of the concept, however, the author endeavoured to find a co-monoid structure on $\mathbf{M}_3$—in some sense, the simplest counter-example to (complete) distributivity—and this proved impossible. The present article has been written to explain why.

There does not appear to be any profundity attached to this result; it is just a curious combinatorial fact. In particular, it is not part of a general trend, for it later emerged that there are co-monoid structures on $\mathbf{N}_5$—an example of which is included in the present paper.

Nevertheless, given the importance of the corresponding notion of co-algebra (both on its own, but also as a component of larger definitions such as bi-algebra and Hopf algebra), we feel that the former result stands on its own.

2 Background

Throughout this paper we shall make reference to the basic properties and structure of the category of complete lattices and join-preserving maps (*alias* sup-homomorphisms), here

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denoted **Sup**,[1] many of which were first described in [8].

As in the case of vector spaces, the natural tensor product is motivated by a notion of "bilinearity".

Definition 2.1

Given complete lattices x, y and z, a function $x \times y \xrightarrow{\omega} z$ is said to *preserve joins in each variable*, or to be a *bi-sup-homomorphism*, if it satisfies

$$\begin{aligned} \omega(\bigvee_{j \in J} \alpha_j, \beta) &= \bigvee_{j \in J} \omega(\alpha_j, \beta) \\ \omega(\alpha, \bigvee_{k \in K} \beta_k) &= \bigvee_{k \in K} \omega(\alpha, \beta_k) \end{aligned}$$

for all sets J, K, and all $\alpha, \alpha_j \in x$, $\beta, \beta_k \in y$.

It is a typical piece of "general abstract nonsense" that a universal such map exists.

Theorem 2.2

There exists a complete lattice $x \boxtimes y$ together with a bi-sup-homomorphism $x \times y \xrightarrow{\boxtimes} x \boxtimes y$ such that any other bi-sup-homomorphism $x \times y \xrightarrow{\omega} z$ factors through it, as depicted below.

$$\begin{tikzcd} x \times y \arrow[r, "\omega"] \arrow[d, "\boxtimes"'] & z \\ x \boxtimes y \arrow[ur, dotted, "\hat{\omega}"'] & \end{tikzcd}$$

[Our reason for not using the usual tensor product symbol, $\otimes$, will hopefully become clear soon.]

Similarly, one can show that $\boxtimes$ defines a functor $\mathbf{Sup} \times \mathbf{Sup} \xrightarrow{\boxtimes} \mathbf{Sup}$ which is *pseudo-associative* (in the sense of [9]), and that the two-element lattice, 2, is a *pseudo-unit* for it.

Corollary 2.3

A (unital) quantale—which is a complete lattice q together with an associative, binary operation $q \times q \xrightarrow{\&} q$ that perserves joins in each variable (and a unit $\eta \in q$)—can be regarded as an internal semigroup (monoid) in the tensor category $(\mathbf{Sup}, 2, \boxtimes)$; that is, if we write $\mu = \hat{\&}$ as above—i.e.,

$$\begin{tikzcd} q \times q \arrow[r, "\&"] \arrow[d, "\boxtimes"'] & q \\ q \boxtimes q \arrow[ur, dotted, "\mu"'] & \end{tikzcd}$$

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[1] It is not true that categories are necessarily named after their objects; for instance, the category of sets and relations is usually denoted **Rel**.

—then the associativity of & can be equivalently expressed by the diagram

$$\begin{tikzcd}
(q \boxtimes q) \boxtimes q \arrow[rr, "\sim"] \arrow[d, "\mu \boxtimes q"'] & & q \boxtimes (q \boxtimes q) \arrow[d, "q \boxtimes \mu"] \\
q \boxtimes q \arrow[r, "\mu"] & q & q \boxtimes q \arrow[l, "\mu"']
\end{tikzcd}$$

—which now lies entirely in the category **Sup**.

(And the unitness of η can be expressed by the diagrams

$$\begin{tikzcd}
2 \boxtimes q \arrow[r, "\ulcorner\eta\urcorner \boxtimes q"] \arrow[dr, "\sim"'] & q \boxtimes q \arrow[d, "\mu"] & q \boxtimes 2 \arrow[l, "q \boxtimes \ulcorner\eta\urcorner"'] \arrow[dl, "\sim"] \\
& q &
\end{tikzcd}$$

—where $\ulcorner\eta\urcorner$ is the unique sup-homomorphism $2 \longrightarrow q$ mapping $\top$ to η.)

The advantage of such a description is that it permits dualisation. This may appear silly, at first glance, but the analogous concept for vector spaces, co-algebras, has proven useful—especially in the co-unital case.

Definition 2.4

A *(co-unital) co-quantale* is a co-semigroup (co-monoid) in $(\mathbf{Sup}, 2, \boxtimes)$—i.e., a complete lattice q together with an sup-homomorphism $q \xrightarrow{\delta} q \boxtimes q$ (and an element $\varepsilon \in q$) such that the diagram

$$\begin{tikzcd}
q \boxtimes q \arrow[d, "\delta \boxtimes q"'] & q \arrow[l, "\delta"'] \arrow[r, "\delta"] & q \boxtimes q \arrow[d, "q \boxtimes \delta"] \\
(q \boxtimes q) \boxtimes q \arrow[rr, "\sim"] & & q \boxtimes (q \boxtimes q)
\end{tikzcd}$$

commutes (and also the diagrams

$$\begin{tikzcd}
& q \arrow[dl, "\sim"'] \arrow[d, "\delta"] \arrow[dr, "\sim"] & \\
2 \boxtimes q & q \boxtimes q \arrow[l, "\llcorner\varepsilon\lrcorner \boxtimes q"] \arrow[r, "q \boxtimes \llcorner\varepsilon\lrcorner"'] & q \boxtimes 2
\end{tikzcd}$$

—where $\llcorner\varepsilon\lrcorner$ denotes the largest sup-homomorphism mapping ε to $\bot$).

The co-unit criterion may seem rather odd until you realise that every sup-homomorphism $q \xrightarrow{\omega} 2$ arises as $\llcorner\varepsilon\lrcorner$ for a unique $\varepsilon \in q$ (namely, $\varepsilon = \bigvee \mathrm{ker}\omega$)—a fact which can be more formally stated as follows.

Lemma 2.5

For any complete lattice x, the set of all sup-homomorphisms $x \longrightarrow 2$ ordered pointwise (henceforth denoted x^*) is isomorphic to x^{op}.

Thus the usual concept of dual (vector) space is entirely analogous to that of dual lattice. Now it is natural to ask whether there is also a naturally lattice-theoretic analogue to the notion of dual map.

But if x and y are complete lattices, then a sup-homomorphism $x \xrightarrow{\omega} y$ is merely the lower (or left) part of a *Galois connection* [3]. The upper (or right) part of this Galois connection is an inf-homomorphism (*i.e.*, arbitrary meet-preserving map) $y \longrightarrow x$. Of course, this function may be equally well regarded as a sup-homomorphism $y^{op} \longrightarrow x^{op}$, and we write $\omega^\sharp$ for this sup-homomorphism.

Lemma 2.6

If $x \xrightarrow{\omega} y$ is a sup-homomorphism, then

$$\llcorner \omega^\sharp(\beta) \lrcorner = \llcorner \beta \lrcorner \circ \omega$$

for all $\beta \in y$—*i.e.*, we have a commutative square

$$\begin{tikzcd}
y^{op} \arrow[r, "\omega^\sharp"] \arrow[d, "\llcorner - \lrcorner"'] & x^{op} \arrow[d, "\llcorner - \lrcorner"] \\
y^* \arrow[r, "(-) \circ \omega"] & x^*
\end{tikzcd}$$

which, in light of the previous lemma, expresses the essential sameness of $\omega^\sharp$ and $(-) \circ \omega$.

But it is in considering duality that we encounter the first important differences between **Sup** and **Vec**. Recall that $x^{**} \cong x$ holds only for finite-dimensional vector spaces, whereas $(x^{op})^{op} = x$ plainly holds for every complete lattice.

Theorem 2.7

The mappings $x \mapsto x^{op}$ and $\omega \mapsto \omega^\sharp$ define a contravariant automorphism of the category **Sup**. In particular, we have $(\psi \circ \omega)^\sharp = \omega^\sharp \circ \psi^\sharp$ for all sup-homomorphisms ω, ψ, and $(\mathrm{id}_x)^\sharp = \mathrm{id}_{x^{op}}$ for all complete lattices x.

On the other hand, we do *not* have $(x \boxtimes y)^{op} \cong x^{op} \boxtimes y^{op}$ for arbitrary complete lattices, whereas $(x \otimes y)^* \cong x^* \otimes y^*$ for every pair of finite-dimensional vector spaces x and y.

The following theorem records much of what is known about the situation, *cf.* [2, 4, 7].

Theorem 2.8

1. The functor $\mathbf{Sup} \times \mathbf{Sup} \xrightarrow{\rotatebox{180}{$\boxtimes$}} \mathbf{Sup}$ defined by

   $$x \rotatebox{180}{$\boxtimes$} y = (x^{op} \boxtimes y^{op})^{op}$$

   on objects, and

$$\omega \mathbin{\cup\mkern-13mu\times} \psi = (\omega^\sharp \mathbin{\cap\mkern-13mu\times} \psi^\sharp)^\sharp$$

on arrows is pseudo-associative, and 2 is a pseudo-unit for it.

2. There exists a coherent natural transformation $\mathbin{\cap\mkern-13mu\times} \xrightarrow{\xi} \mathbin{\cup\mkern-13mu\times}$, induced by the fact that $\mathbin{\cap\mkern-13mu\times}$ and $\mathbin{\cup\mkern-13mu\times}$ have the same pseudo-unit.
3. A complete lattice x is completely distributive if and only if $x \mathbin{\cap\mkern-13mu\times} y \xrightarrow{\xi_{x,y}} x \mathbin{\cup\mkern-13mu\times} y$ is an isomorphism for every complete lattice y.

3 Faux quantales

The need to distinguish between $\mathbin{\cap\mkern-13mu\times}$ and $\mathbin{\cup\mkern-13mu\times}$, for arbitrary complete lattices, is what really drives this article.

Consider, for example, that the dual of a sup-homomorphism of the form $q \longrightarrow q \mathbin{\cap\mkern-13mu\times} q$ has form

$$q^{op} \mathbin{\cup\mkern-13mu\times} q^{op} = (q \mathbin{\cap\mkern-13mu\times} q)^{op} \longrightarrow q^{op}.$$

Thus we cannot hope, for arbitrary q, that co-quantale structures on q correspond bijectively to quantale structures on q^{op}—as might have been expected from our finite-dimensional vector space analogy.

Our discussion will be made more transparent by introducing the following auxiliary concept.

Definition 3.1

A (unital) *faux quantale* is a semigroup (monoid) in the tensor category $(\mathbf{Sup}, 2, \mathbin{\cup\mkern-13mu\times})$; *i.e.*, a complete lattice q together with a map $q \mathbin{\cup\mkern-13mu\times} q \xrightarrow{\mu} q$ (and an element $\eta \in q$) such that the diagram

$$\begin{tikzcd}
(q \mathbin{\cup\mkern-13mu\times} q) \mathbin{\cup\mkern-13mu\times} q \arrow[rr, "\sim"] \arrow[d, "\mu \mathbin{\cup\mkern-13mu\times} q"] & & q \mathbin{\cup\mkern-13mu\times} (q \mathbin{\cup\mkern-13mu\times} q) \arrow[d, "q \mathbin{\cup\mkern-13mu\times} \mu"'] \\
q \mathbin{\cup\mkern-13mu\times} q \arrow[r, "\mu"] & q & q \mathbin{\cup\mkern-13mu\times} q \arrow[l, "\mu"']
\end{tikzcd}$$

commutes (and the diagrams

$$\begin{tikzcd}
2 \mathbin{\cup\mkern-13mu\times} q \arrow[r, "\ulcorner \eta \urcorner \mathbin{\cup\mkern-13mu\times} q"] \arrow[dr, "\sim"'] & q \mathbin{\cup\mkern-13mu\times} q \arrow[d, "\mu"] & q \mathbin{\cup\mkern-13mu\times} 2 \arrow[l, "q \mathbin{\cup\mkern-13mu\times} \ulcorner \eta \urcorner"'] \arrow[dl, "\sim"] \\
& q &
\end{tikzcd}$$

also).

Faux quantales arise naturally as the duals of co-quantales.

Theorem 3.2

If $q \xrightarrow{\delta} q \boxtimes q$ is a co-quantale structure for q, then $q^{op} \btimes q^{op} = (q \boxtimes q)^{op} \xrightarrow{\delta^\sharp} q^{op}$ is a faux quantale structure for q^{op}.

Moreover, if ε is a unit for δ then it is also a co-unit for $\delta^\sharp$.

ProofT. he first statement is an easy consequence of Theorem 2.7.

The second statement also follows from Theorem 2.7, together with the observation that

$$\begin{tikzcd}
2 \arrow[r, "\sim"] \arrow[dr, "\ulcorner\alpha\urcorner"'] & 2^{op} \arrow[d, "(\llcorner\alpha\lrcorner)^\sharp"] \\
& q^{op}
\end{tikzcd}$$

commutes for every $\alpha \in q$. Q.E.D.

Faux quantales give rise to quantales, but not necessarily in a one-to-one fashion.

Theorem 3.3

If $q \btimes q \xrightarrow{\mu} q$ is associative, then so is the composite $q \boxtimes q \xrightarrow{\xi_{q,q}} q \btimes q \xrightarrow{\mu} q$ and if $\eta \in q$ is a unit for μ then it is also a unit for $\tilde{\mu} = \mu \circ \xi_{q,q}$.

Proof

The associativity of $\tilde{\mu}$ is established by the following diagram, where

1. the top rectangle commutes by coherence [4];
2. the two squares at the bottom left and bottom right commute by naturality;
3. the bottom middle square commutes by hypothesis; and,
4. all four triangles commute by definition.

$$\begin{tikzcd}
(q \boxtimes q) \boxtimes q \arrow[rrrr, "\sim"] \arrow[d, "\xi_{q,q} \boxtimes q"] \arrow[dd, bend right=60, "\tilde{\mu} \boxtimes q"'] & & & & q \boxtimes (q \boxtimes q) \arrow[d, "q \boxtimes \xi_{q,q}"'] \arrow[dd, bend left=60, "q \boxtimes \tilde{\mu}"] \\
(q \btimes q) \boxtimes q \arrow[r, "\xi_{q \btimes q,q}"] \arrow[d, "\mu \boxtimes q"] & (q \btimes q) \btimes q \arrow[rr, "\sim"] \arrow[d, "\mu \btimes q"] & & q \btimes (q \btimes q) \arrow[d, "q \btimes \mu"'] & q \boxtimes (q \btimes q) \arrow[l, "\xi_{q,q \btimes q}"'] \arrow[d, "q \boxtimes \mu"'] \\
q \boxtimes q \arrow[r, "\xi_{q,q}"] \arrow[rr, bend right, "\tilde{\mu}"'] & q \btimes q \arrow[r, "\mu"] & q & q \btimes q \arrow[l, "\mu"'] & q \boxtimes q \arrow[l, "\xi_{q,q}"'] \arrow[ll, bend left, "\tilde{\mu}"]
\end{tikzcd}$$

Suppose that η is a unit for μ. Then the following diagram establishes that η is also a unit for $\tilde{\mu}$, where

1. the leftmost and rightmost triangles commute by coherence;
2. the two squares commute by naturality; and,
3. the remaining two triangles commute by hypothesis.

$$\begin{tikzcd}
2 \boxtimes q \arrow[rr, "\ulcorner \eta \urcorner \boxtimes q"] \arrow[dr, "\xi_{2,q}"] \arrow[ddrr, bend right, "\sim"'] & & q \boxtimes q \arrow[d, "\xi_{q,q}"] & & q \boxtimes 2 \arrow[ll, "q \boxtimes \ulcorner \eta \urcorner"'] \arrow[dl, "\xi_{q,2}"'] \arrow[ddll, bend left, "\sim"] \\
& 2 \otimes q \arrow[r, "\ulcorner \eta \urcorner \otimes q"] \arrow[dr, "\sim"'] & q \otimes q \arrow[d, "\mu"] & q \otimes 2 \arrow[l, "q \otimes \ulcorner \eta \urcorner"'] \arrow[dl, "\sim"] & \\
& & q & &
\end{tikzcd}$$

Q.E.D.

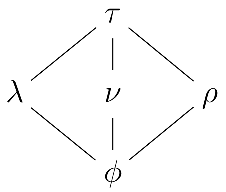
Remark 3.4
The proof given above does not use any features of **Sup** other than that it is a *linearly distributive category* satisfying the *isomix* rule, [4]. Thus the same proof establishes a more general category-theoretic theorem.

Now, given a (unital) quantale q, it is possible to ask whether q arises from a (unital) faux quantale as above. The answer is, in general, no: indeed, this is essentially how the author originally arrived at the result that there are no co-unital co-quantale structures on $\mathbf{M}_3$. It is easy to see that, up to isomorphism, there are precisely eight unital quantale structures on $\mathbf{M}_3$; it is possible, by considering each of these eight structures individually, to show that none of them arise from unital faux quantales in the manner described above.

Luckily, however, there also exists a more direct proof that there are no unital faux quantale structures on $\mathbf{M}_3$, which is described below.

4 Main Result

For the sake of definiteness, let us name the five elements of $\mathbf{M}_3$ as follows:



—a Hasse diagram for $\mathbf{M}_3 \boxtimes \mathbf{M}_3$, which has fifty elements, is included in the appendix.

Theorem 4.1
There does not exist a co-unital co-quantale structure on $\mathbf{M}_3$.

The theorem will be proven by cases; namely, by showing that, for each potential co-unit ε, there does not exist a sup-homomorphism $q \xrightarrow{\delta} q \boxtimes q$ satisfying the co-unitality axioms with respect to ε. Note that one need only consider three cases, since there are automorphisms of $\mathbf{M}_3$ permuting λ, ν and ρ.

Perhaps surprisingly, we shall not actually need the co-associativity of δ; the theorem would remain true if we dropped that hypothesis. On the other hand, the theorem would be false if we did not require a co-unit.

Example 4.2

The map $q \xrightarrow{\delta} q \boxtimes q$ defined by

$$\begin{array}{rcl} \phi & \mapsto & \bot \\ \lambda & \mapsto & (\lambda \boxtimes \tau) \vee (\tau \boxtimes \lambda) \\ \nu & \mapsto & (\nu \boxtimes \tau) \vee (\tau \boxtimes \nu) \\ \rho & \mapsto & (\rho \boxtimes \tau) \vee (\tau \boxtimes \rho) \\ \tau & \mapsto & \tau \boxtimes \tau \end{array}$$

is a sup-homomorphism, and satisfies co-associativity.

But it is easy to check that neither λ nor ϕ are co-units for δ because, for $\alpha \in \{\lambda, \phi\}$, we have

$$\begin{array}{rcl} (\llcorner \alpha \lrcorner \boxtimes \mathbf{M}_3)(\delta(\nu)) & = & (\llcorner \alpha \lrcorner \boxtimes \mathbf{M}_3)((\nu \boxtimes \tau) \vee (\tau \boxtimes \nu)) \\ & = & (\llcorner \alpha \lrcorner \boxtimes \mathbf{M}_3)(\nu \boxtimes \tau) \vee (\llcorner \alpha \lrcorner \boxtimes \mathbf{M}_3)(\tau \boxtimes \nu) \\ & = & (\llcorner \alpha \lrcorner(\nu) \boxtimes \tau) \vee (\llcorner \alpha \lrcorner(\tau) \boxtimes \nu) \\ & = & (\top \boxtimes \tau) \vee (\top \boxtimes \nu) \\ & = & \top \boxtimes \tau \end{array}$$

whereas the canonical isomorphism $\mathbf{M}_3 \xrightarrow{\sim} 2 \boxtimes \mathbf{M}_3$ maps ν to $\top \boxtimes \nu \neq \top \boxtimes \tau$.

The case τ is dealt with below.

Lemma 4.3

The top element of a non-trivial complete lattice q cannot be a co-unit for any co-quantale structure on q.

Proof

By definition, $\llcorner \top \lrcorner$ is the constant function with value $\bot$; hence the composite

$$\begin{tikzcd} q \arrow[r, "\delta"] & q \boxtimes q \arrow[r, "\llcorner \top \lrcorner \boxtimes q"] & 2 \boxtimes q \end{tikzcd}$$

also equals the constant function with value $\bot$. So, in order for this composite to be an isomorphism (in particular, for it to equal the canonical isomorphism $q \longrightarrow 2 \boxtimes q$), q must be trivial. Q.E.D.

Lemma 4.4

ϕ is not a possible co-unit for any co-quantale structure on $\mathbf{M}_3$.

Proof

In light of the results of section 3, it suffices to show that ϕ is not a possible unit for any quantale structure on $\mathbf{M}_3^{op}$; equivalently, that τ is not a possible unit for any quantale structure on $\mathbf{M}_3$.

Were τ a unit for $q \times q \xrightarrow{\&} q$, then we would have

$$\begin{array}{rcccl} \alpha \mathbin{\&} \beta & \leq & \alpha \mathbin{\&} \tau & = & \alpha \\ \alpha \mathbin{\&} \beta & \leq & \tau \mathbin{\&} \beta & = & \beta \end{array}$$

(and hence $\alpha \mathbin{\&} \beta \leq \alpha \wedge \beta$) for all $\alpha, \beta \in \mathbf{M}_3$. In particular, we would have $\lambda \mathbin{\&} \nu = \phi = \lambda \mathbin{\&} \rho$, and so

$$\begin{array}{rcl} \lambda \mathbin{\&} \tau & = & \lambda \mathbin{\&} (\nu \vee \rho) \\ & = & (\lambda \mathbin{\&} \nu) \vee (\lambda \mathbin{\&} \rho) \\ & = & \phi \vee \phi \end{array}$$

—a contradiction. Q.E.D.

Lemma 4.5

λ is not a possible co-unit for any co-quantale structure on $\mathbf{M}_3$.

Proof

Let χ, ψ denote the composites

$$\begin{tikzcd} \mathbf{M}_3 \boxtimes \mathbf{M}_3 \arrow[r, "\llcorner \lambda \lrcorner \boxtimes \mathbf{M}_3"] & 2 \boxtimes \mathbf{M}_3 \arrow[r, "\sim"] & \mathbf{M}_3 \\ \mathbf{M}_3 \boxtimes \mathbf{M}_3 \arrow[r, "\mathbf{M}_3 \boxtimes \llcorner \lambda \lrcorner"] & \mathbf{M}_3 \boxtimes 2 \arrow[r, "\sim"] & \mathbf{M}_3 \end{tikzcd}$$

respectively.

Then the requirement that λ be a co-unit for δ can be re-stated as

$$\chi(\delta(\alpha)) = \alpha = \psi(\delta(\alpha))$$

or equivalently,

$$\delta(\alpha) \in \chi^{-1}(\{\alpha\}) \cap \psi^{-1}(\{\alpha\}) \tag{1}$$

for all $\alpha \in \mathbf{M}_3$.

Now suppose that $q \xrightarrow{\delta} q \boxtimes q$ is a sup-homomorphism which satifies (1) for $\alpha \in \{\nu, \rho\}$; we will show that δ cannot satisfy (1) for $\alpha = \lambda$.

For, from

$$\begin{array}{rcl} \chi(\alpha \boxtimes \beta) & = & \upsilon((\llcorner\lambda\lrcorner \boxtimes \mathbf{M}_3)(\alpha \boxtimes \beta)) \\ & = & \upsilon(\llcorner\lambda\lrcorner(\alpha) \boxtimes \beta) \\ & = & \left\{ \begin{array}{ll} \upsilon(\bot \boxtimes \beta) = \upsilon(\bot) = \bot & \text{if } \alpha \leq \lambda \\ \upsilon(\top \boxtimes \beta) = \beta & \text{otherwise} \end{array} \right. \end{array}$$

(where υ denotes the hitherto anonymous isomorphism $\mathbf{2} \boxtimes q \xrightarrow{\sim} q$), it follows that

$$\chi^{-1}(\{\nu\}) = \left\{ \begin{array}{l} \nu \boxtimes \nu, (\nu \boxtimes \nu) \vee (\lambda \boxtimes \lambda), (\nu \boxtimes \nu) \vee (\lambda \boxtimes \rho), \\ \rho \boxtimes \nu, (\rho \boxtimes \nu) \vee (\lambda \boxtimes \lambda), (\rho \boxtimes \nu) \vee (\lambda \boxtimes \rho), \\ \tau \boxtimes \nu, (\tau \boxtimes \nu) \vee (\lambda \boxtimes \tau) \end{array} \right\}.$$

Similarly, we obtain

$$\psi^{-1}(\{\nu\}) = \left\{ \begin{array}{l} \nu \boxtimes \nu, (\nu \boxtimes \nu) \vee (\lambda \boxtimes \lambda), (\nu \boxtimes \nu) \vee (\rho \boxtimes \lambda), \\ \nu \boxtimes \rho, (\nu \boxtimes \rho) \vee (\lambda \boxtimes \lambda), (\nu \boxtimes \rho) \vee (\rho \boxtimes \lambda), \\ \nu \boxtimes \tau, (\nu \boxtimes \tau) \vee (\tau \boxtimes \lambda) \end{array} \right\}$$

and hence,

$$\delta(\nu) \in \chi^{-1}(\{\nu\}) \cap \psi^{-1}(\{\nu\}) = \{\nu \boxtimes \nu, (\nu \boxtimes \nu) \vee (\lambda \boxtimes \lambda)\}.$$

Interchanging ν and ρ in the computation above, we arrive at

$$\delta(\rho) \in \{\rho \boxtimes \rho, (\rho \boxtimes \rho) \vee (\lambda \boxtimes \lambda)\}.$$

Thus

$$\begin{array}{rcl} \delta(\lambda) & \leq & \delta(\tau) \\ & = & \delta(\nu \vee \rho) \\ & = & \delta(\nu) \vee \delta(\rho) \\ & \leq & ((\lambda \boxtimes \lambda) \vee (\nu \boxtimes \nu)) \vee ((\lambda \boxtimes \lambda) \vee (\rho \boxtimes \rho)) \\ & = & (\lambda \boxtimes \lambda) \vee (\nu \boxtimes \nu) \vee (\rho \boxtimes \rho) \end{array}$$

and hence there are eight possible choices for $\delta(\lambda)$, corresponding to the eight subsets of $\{\lambda \boxtimes \lambda, \nu \boxtimes \nu, \rho \boxtimes \rho\}$.

But χ (and ψ) map these three elements of $\mathbf{M}_3$ to ϕ, ν and ρ, respectively, and λ cannot be written as a join of these three; thus we cannot have $\chi(\delta(\lambda)) = \lambda = \psi(\delta(\lambda))$ as required.[2]

Q.E.D.

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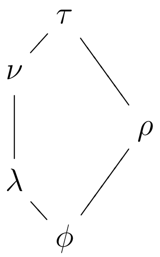
[2] In fact, $\chi^{-1}(\{\lambda\}) \cap \psi^{-1}(\{\lambda\}) = \{(\lambda \boxtimes \nu) \vee (\nu \boxtimes \lambda), (\lambda \boxtimes \nu) \vee (\rho \boxtimes \lambda), (\lambda \boxtimes \rho) \vee (\nu \boxtimes \lambda), (\lambda \boxtimes \rho) \vee (\rho \boxtimes \lambda)\}$.

5 A non-distributive bi-unital bi-quantale

We conclude this paper by showing that it is possible to have a co-unital co-quantale structure on a complete lattice which is not completely distributive—specifically, $\mathbf{N}_5$.

Again, for the sake of definiteness, let us name the elements of $\mathbf{N}_5$ as follows:



Theorem 5.1

The map $\mathbf{N}_5 \xrightarrow{\delta} \mathbf{N}_5 \boxtimes \mathbf{N}_5$ defined by

$$\begin{array}{rcl} \phi & \mapsto & \bot \\ \lambda & \mapsto & \lambda \boxtimes \lambda \\ \nu & \mapsto & \nu \boxtimes \nu \\ \rho & \mapsto & (\rho \boxtimes \tau) \vee (\tau \boxtimes \rho) \\ \tau & \mapsto & \tau \boxtimes \tau \end{array}$$

is a sup-homomorphism that is co-associative and for which ρ is a co-unit.

Proof

To show that δ is a sup-homomorphism, it suffices to show that $\delta(\lambda) \vee \delta(\rho) = \delta(\tau) = \delta(\nu) \vee \delta(\rho)$. This is true because, for $\alpha \in \{\lambda, \nu\}$,

$$\begin{array}{rcl} \delta(\alpha) \vee \delta(\rho) & = & (\alpha \boxtimes \alpha) \vee (\rho \boxtimes \tau) \vee (\tau \boxtimes \rho) \\ & \geq & (\alpha \boxtimes \alpha) \vee (\rho \boxtimes \alpha) \vee (\tau \boxtimes \rho) \\ & = & ((\alpha \vee \rho) \boxtimes \alpha) \vee (\tau \boxtimes \rho) \\ & = & (\tau \boxtimes \alpha) \vee (\tau \boxtimes \rho) \\ & = & \tau \boxtimes (\alpha \vee \rho) \\ & = & \tau \boxtimes \tau. \end{array}$$

Now the two composites

$$\begin{tikzcd} \mathbf{N}_5 \arrow[r, "\delta"] & \mathbf{N}_5 \boxtimes \mathbf{N}_5 \arrow[r, "\delta \boxtimes \mathbf{N}_5"] & (\mathbf{N}_5 \boxtimes \mathbf{N}_5) \boxtimes \mathbf{N}_5 \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{N}_5 \arrow[r, "\delta"] & \mathbf{N}_5 \boxtimes \mathbf{N}_5 \arrow[r, "\mathbf{N}_5 \boxtimes \delta"] & \mathbf{N}_5 \boxtimes (\mathbf{N}_5 \boxtimes \mathbf{N}_5) \end{tikzcd}$$

both amount to suitable parenthesisations of

$$\begin{array}{lcl}
\phi & \mapsto & \bot \\
\lambda & \mapsto & \lambda \boxtimes \lambda \boxtimes \lambda \\
\nu & \mapsto & \nu \boxtimes \nu \boxtimes \nu \\
\rho & \mapsto & (\rho \boxtimes \tau \boxtimes \tau) \vee (\tau \boxtimes \rho \boxtimes \tau) \vee (\tau \boxtimes \tau \boxtimes \rho) \\
\tau & \mapsto & \tau \boxtimes \tau \boxtimes \tau
\end{array}$$

and since the canonical isomorphism $(\mathbf{N}_5 \boxtimes \mathbf{N}_5) \boxtimes \mathbf{N}_5 \xrightarrow{\sim} \mathbf{N}_5 \boxtimes (\mathbf{N}_5 \boxtimes \mathbf{N}_5)$ is defined by permuting said parenthesisations, δ is co-associative.

Finally, because of the symmetry (or more accurately, *co-commutativity*) of δ, it suffices to check only one of the two co-unitality conditions. For $\alpha \in \{\lambda, \nu, \tau\}$, we have

$$\begin{array}{rcl}
\upsilon((\llcorner\rho\lrcorner \boxtimes \mathbf{M}_3)(\delta(\alpha))) & = & \upsilon((\llcorner\rho\lrcorner \boxtimes \mathbf{M}_3)(\alpha \boxtimes \alpha)) \\
 & = & \upsilon(\llcorner\rho\lrcorner(\alpha) \boxtimes \alpha) \\
 & = & \upsilon(\top \boxtimes \alpha) \\
 & = & \alpha
\end{array}$$

and

$$\begin{array}{rcl}
\upsilon((\llcorner\rho\lrcorner \boxtimes \mathbf{M}_3)(\delta(\rho))) & = & \upsilon((\llcorner\rho\lrcorner \boxtimes \mathbf{M}_3)((\rho \boxtimes \tau) \vee (\tau \boxtimes \rho))) \\
 & = & \upsilon((\llcorner\rho\lrcorner \boxtimes \mathbf{M}_3)(\rho \boxtimes \tau)) \vee \upsilon((\llcorner\rho\lrcorner \boxtimes \mathbf{M}_3)(\tau \boxtimes \rho)) \\
 & = & \upsilon(\llcorner\rho\lrcorner(\rho) \boxtimes \tau) \vee \upsilon(\llcorner\rho\lrcorner(\tau) \boxtimes \rho) \\
 & = & \upsilon(\bot \boxtimes \tau) \vee \upsilon(\top \boxtimes \rho) \\
 & = & \phi \vee \rho = \rho.
\end{array}$$

Q.E.D.

As stated in the introduction, the definition of a co-algebra, although important in its own right, is often seen as a stepping stone to more complex definitions. The most basic of these is that of *bialgebra*.

Definition 5.2

A *(bi-unital) bi-quantale* is a semigroup (monoid) in the category of (co-unital) co-quantales—*i.e.*, a complete lattice q carrying a quantale structure $q \boxtimes q \xrightarrow{\mu} q$ (with unit η) and a co-quantale structure $q \xrightarrow{\delta} q \boxtimes q$ (with co-unit ε), such that the diagram

$$\begin{tikzcd}
q \boxtimes q \arrow[r, "\mu"] \arrow[d, "\delta \boxtimes \delta"'] & q \arrow[r, "\delta"] & q \boxtimes q \\
(q \boxtimes q) \boxtimes (q \boxtimes q) \arrow[rr, "\chi"'] & & (q \boxtimes q) \boxtimes (q \boxtimes q) \arrow[u, "\mu \boxtimes \mu"']
\end{tikzcd}$$

—where χ denotes the automorphism $(\alpha \boxtimes \beta) \boxtimes (\zeta \boxtimes \theta) \mapsto (\alpha \boxtimes \zeta) \boxtimes (\beta \boxtimes \theta)$—commutes (and the diagrams

$$\begin{tikzcd}
2 \arrow[r, "\sim"] \arrow[d, "\ulcorner \eta \urcorner"'] & 2 \boxtimes 2 \arrow[d, "\ulcorner \eta \urcorner \boxtimes \ulcorner \eta \urcorner"] \\
q \arrow[r, "\delta"'] & q \boxtimes q
\end{tikzcd}
\qquad
\begin{tikzcd}
q \boxtimes q \arrow[r, "\mu"] \arrow[d, "\llcorner \varepsilon \lrcorner \boxtimes \llcorner \varepsilon \lrcorner"'] & q \arrow[d, "\llcorner \varepsilon \lrcorner"] \\
2 \boxtimes 2 \arrow[r, "\sim"'] & 2
\end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[r, "\ulcorner \eta \urcorner"'] \arrow[rr, bend left, "\mathrm{id}_2"] & q \arrow[r, "\llcorner \varepsilon \lrcorner"'] & 2
\end{tikzcd}$$

also).

Theorem 5.3

There is a bi-unital bi-quantale structure on $\mathbf{N}_5$ with δ as above and μ defined by the multiplication table below.

$\&$	ϕ	λ	ν	ρ	τ
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
λ	ϕ	λ	ν	ρ	τ
ν	ϕ	ν	ν	ρ	τ
ρ	ϕ	ρ	ρ	ρ	ρ
τ	ϕ	τ	τ	ρ	τ

Before proving this theorem, let us rephrase the three axioms in Definition 5.2 which refer to the unit and co-unit.

Lemma 5.4

For a bi-quantale to be bi-unital, it is necessary and sufficient that μ have a unit η and δ a co-unit ε satisfying the following three conditions:

1. $\delta(\eta) = \eta \boxtimes \eta$,
2. $\eta \not\leq \varepsilon$, and
3. $\alpha \,\&\, \beta \leq \varepsilon$ if and only if $\alpha \leq \varepsilon$ or $\beta \leq \varepsilon$.

Proof

That the first statement is equivalent to the top-left bi-unital axiom, and that the second statement is equivalent to the bottom bi-unital axiom, are easy consequences of the definitions of $\ulcorner - \urcorner$ and $\llcorner - \lrcorner$.

To show that the top-right bi-unital axiom is equivalent to the third statement, first note that the canonical isomorphism $2 \boxtimes 2 \longrightarrow 2$ represents the bi-sup-homomorphism $2 \times 2 \xrightarrow{\wedge} 2$. Thus the composite

$$q \boxtimes q \xrightarrow{\llcorner \varepsilon \lrcorner \boxtimes \llcorner \varepsilon \lrcorner} 2 \boxtimes 2 \xrightarrow{\sim} 2$$

maps a pure tensor $\alpha \boxtimes \beta$ to $\llcorner\varepsilon\lrcorner(\alpha) \wedge \llcorner\varepsilon\lrcorner(\beta)$. Now

$$\begin{aligned} \llcorner\varepsilon\lrcorner(\alpha) \wedge \llcorner\varepsilon\lrcorner(\beta) = \bot \quad &\Longleftrightarrow \quad \llcorner\varepsilon\lrcorner(\alpha) = \bot \text{ or } \llcorner\varepsilon\lrcorner(\beta) = \bot \\ &\Longleftrightarrow \quad \alpha \leq \varepsilon \text{ or } \beta \leq \varepsilon \end{aligned}$$

and

$$\llcorner\varepsilon\lrcorner(\mu(\alpha \boxtimes \beta)) = \bot \iff \alpha \mathbin{\&} \beta = \mu(\alpha \boxtimes \beta) \leq \varepsilon$$

—but for two maps of the form $q \boxtimes q \longrightarrow q$ to be equal it suffices to show that they map the same set of pure tensors to $\bot$. Q.E.D.

Proof of Theorem 5.3

The three conditions of Lemma 5.4 are easily seen to hold, with $\eta = \lambda$ and $\varepsilon = \rho$. So it remains to show the first diagram of Definition 5.2. Once again, it suffices to show that the two maps agree on pure tensors, and we proceed by cases.

Now if $\alpha, \beta \in \{\lambda, \nu, \tau\}$, then $\mu(\alpha \boxtimes \beta) = \alpha \mathbin{\&} \beta \in \{\lambda, \nu, \tau\}$ also. Hence,

$$\begin{aligned} (\mu \boxtimes \mu)(\chi((\delta \boxtimes \delta)(\alpha \boxtimes \beta))) &= (\mu \boxtimes \mu)(\chi(\delta(\alpha) \boxtimes \delta(\beta))) \\ &= (\mu \boxtimes \mu)(\chi((\alpha \boxtimes \alpha) \boxtimes (\beta \boxtimes \beta))) \\ &= (\mu \boxtimes \mu)((\alpha \boxtimes \beta) \boxtimes (\alpha \boxtimes \beta)) \\ &= \mu(\alpha \boxtimes \beta) \boxtimes \mu(\alpha \boxtimes \beta) \\ &= \delta(\mu(\alpha \boxtimes \beta)) \end{aligned}$$

If $\alpha \in \{\lambda, \nu, \tau\}$, then

$$\begin{aligned} &(\mu \boxtimes \mu)(\chi((\delta \boxtimes \delta)(\alpha \boxtimes \rho))) \\ &\quad = (\mu \boxtimes \mu)(\chi(\delta(\alpha) \boxtimes \delta(\rho))) \\ &\quad = (\mu \boxtimes \mu)(\chi((\alpha \boxtimes \alpha) \boxtimes ((\rho \boxtimes \tau) \vee (\tau \boxtimes \rho)))) \\ &\quad = (\mu \boxtimes \mu)(\chi((\alpha \boxtimes \alpha) \boxtimes (\rho \boxtimes \tau))) \vee (\mu \boxtimes \mu)(\chi((\alpha \boxtimes \alpha) \boxtimes (\tau \boxtimes \rho))) \\ &\quad = (\mu \boxtimes \mu)((\alpha \boxtimes \rho) \boxtimes (\alpha \boxtimes \tau)) \vee (\mu \boxtimes \mu)((\alpha \boxtimes \tau) \boxtimes (\alpha \boxtimes \rho)) \\ &\quad = (\mu(\alpha \boxtimes \rho) \boxtimes \mu(\alpha \boxtimes \tau)) \vee (\mu(\alpha \boxtimes \tau) \boxtimes \mu(\alpha \boxtimes \rho)) \\ &\quad = (\rho \boxtimes \tau) \vee (\tau \boxtimes \rho) = \delta(\rho) = \delta(\mu(\alpha \boxtimes \rho)) \end{aligned}$$

Finally,

$$\begin{aligned} &(\mu \boxtimes \mu)(\chi((\delta \boxtimes \delta)(\rho \boxtimes \rho))) \\ &\quad = (\mu \boxtimes \mu)(\chi(\delta(\rho) \boxtimes \delta(\rho))) \\ &\quad = (\mu \boxtimes \mu)(\chi((\rho \boxtimes \tau) \vee (\tau \boxtimes \rho) \boxtimes (\rho \boxtimes \tau) \vee (\tau \boxtimes \rho))) \\ &\quad = (\mu \boxtimes \mu)(\chi((\rho \boxtimes \tau) \boxtimes (\rho \boxtimes \tau))) \vee (\mu \boxtimes \mu)(\chi((\rho \boxtimes \tau) \boxtimes (\tau \boxtimes \rho))) \\ &\quad\quad \vee (\mu \boxtimes \mu)(\chi((\tau \boxtimes \rho) \boxtimes (\rho \boxtimes \tau))) \vee (\mu \boxtimes \mu)(\chi((\tau \boxtimes \rho) \boxtimes (\tau \boxtimes \rho))) \\ &\quad = (\mu \boxtimes \mu)((\rho \boxtimes \rho) \boxtimes (\tau \boxtimes \tau)) \vee (\mu \boxtimes \mu)((\rho \boxtimes \tau) \boxtimes (\tau \boxtimes \rho)) \end{aligned}$$

$$\begin{aligned}
& \vee(\mu \boxtimes \mu)((\tau \boxtimes \rho) \boxtimes (\rho \boxtimes \tau)) \vee (\mu \boxtimes \mu)((\tau \boxtimes \tau) \boxtimes (\rho \boxtimes \rho)) \\
= \;& (\mu(\rho \boxtimes \rho) \boxtimes \mu(\tau \boxtimes \tau)) \vee (\mu(\rho \boxtimes \tau) \boxtimes \mu(\tau \boxtimes \rho)) \\
& \vee(\mu(\tau \boxtimes \rho) \boxtimes \mu(\rho \boxtimes \tau)) \vee (\mu(\tau \boxtimes \tau) \boxtimes \mu(\rho \boxtimes \rho)) \\
= \;& (\rho \boxtimes \tau) \vee (\rho \boxtimes \rho) \vee (\rho \boxtimes \rho) \vee (\tau \boxtimes \rho) \\
= \;& (\rho \boxtimes \tau) \vee (\tau \boxtimes \rho) = \delta(\rho) = \delta(\mu(\rho \boxtimes \rho)).
\end{aligned}$$

[The antepenultimate step is justified by the fact that $\rho \boxtimes \rho \leq \rho \boxtimes \tau$.] Q.E.D.

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6 Appendix

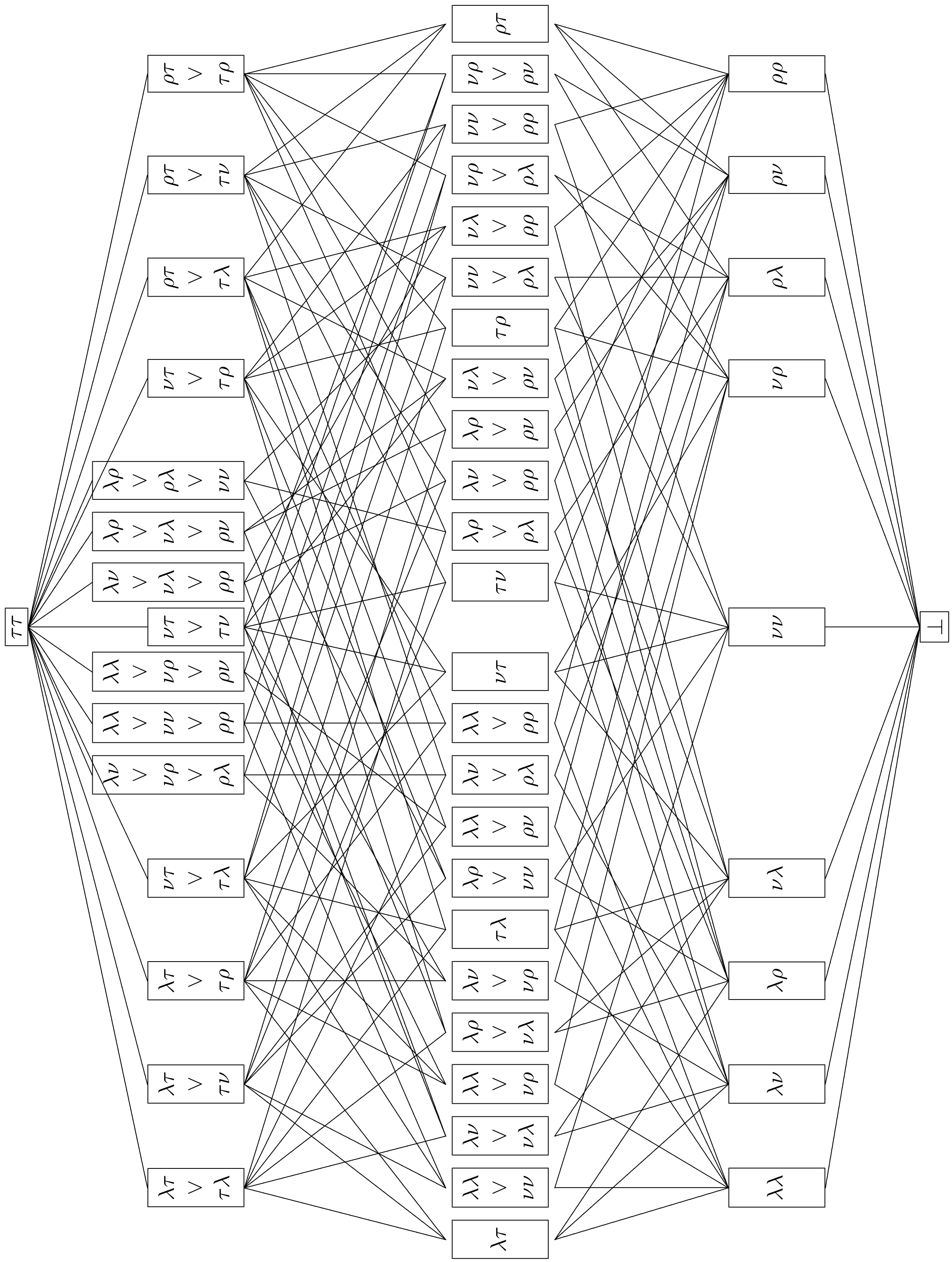
For completeness' sake, we include a Hasse diagram of the lattice $\mathbf{M}_3 \boxtimes \mathbf{M}_3$.

Please note that, for reasons of space, we shall suppress the symbol $\boxtimes$ in the diagram below; thus, for example, $\lambda\rho$ is an abbreviation for the element $\lambda \boxtimes \rho$.

The most interesting feature of the diagram is the presence of six "extraneous" and symmetry-ruining co-atoms. Note that since $\mathbf{M}_3 \cong \mathbf{M}_3^{op}$, we have

$$\mathbf{M}_3 \uplus \mathbf{M}_3 = (\mathbf{M}_3^{op} \boxtimes \mathbf{M}_3^{op})^{op} \cong (\mathbf{M}_3 \boxtimes \mathbf{M}_3)^{op}$$

—so, in light of Theorem 2.8, it is hardly surprising that $\mathbf{M}_3 \boxtimes \mathbf{M}_3$ fails to be self-dual.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