Math 2135: Linear Algebra, Mid-March Stuff

1. For any $u \in \mathsf{R}^n$, let $|u|$ denote the (usual notion of) length of u; and for any $A \in \mathsf{R}^{n\times n}$, let $\|A\|$ denote the maximum value of $f(x_1, \ldots, x_n) = \left| A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right|$ subject to the constraint $g(x_1, \ldots, x_n) = \left| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right| = 1$. Using only logic only (no calculations!), show that each of the following properties hold:
   (a) $\|A + B\| \le \|A\| + \|B\|$, for all $A, B \in \mathsf{R}^{n\times n}$.
   (b) $\|kA\| = |k|\,\|A\|$, for all $A \in \mathsf{R}^{n\times n}$ and all $k \in \mathsf{R}$.
   (c) $\|A\| = 0$ implies A is the zero matrix.
   (d) $\|AB\| \le \|A\|\,\|B\|$, for all $A, B \in \mathsf{R}^{n\times n}$.
   (e) $\|I\| = 1$.

2. Let $A \in \mathsf{R}^{2\times 2}$. We will use the method of Lagrange multipliers to maximise $\phi(x, y) = \left| A \begin{pmatrix} x \\ y \end{pmatrix} \right|^2$ subject to the constraint $\gamma(x, y) = \left| \begin{pmatrix} x \\ y \end{pmatrix} \right|^2 = 1$. [The squaring is just to get rid of the square roots, which makes calculating derivatives easier!]
   (a) Show that $\nabla\phi(x, y) = 2A^T A \begin{pmatrix} x \\ y \end{pmatrix}$.
   (b) Show that λ is a Lagrange multiplier if and only if it is an eigenvalue of $A^T A$.
   (c) Show that if (λ, x, y) is a solution of the system

$$\begin{cases} \nabla\phi(x, y) = \lambda\nabla\gamma(x, y) \\ \gamma(x, y) = 1 \end{cases}$$

   then $\phi(x, y) = \lambda$.
   (d) Conclude that $\|A\|$ is the square root of the largest eigenvalue of $A^T A$. [Remember that a positive matrix—i.e., one of the form $A^T A$—cannot have negative eigenvalues!]

   [All these results generalise to $\mathsf{R}^{n\times n}$ for $n > 2$.]

3. (a) Show that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.
   (b) Show that if $A \in \mathsf{R}^{n\times n}$ is diagonalisable, and λ is an eigenvalue of A^2, then $\pm\sqrt{\lambda}$ is an eigenvalue of A.
   (c) Conclude that if A is a symmetric matrix, then

$$\|A\| = \max\{\,|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}\,.$$

(d) Calculate the norm of each of the following matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

(e) Show that the parallelogram law fails for this notion of norm of a matrix—i.e., there does not exist an inner product $\mathsf{R}^{2\times 2} \times \mathsf{R}^{2\times 2} \xrightarrow{\langle\ ,\ \rangle} \mathsf{R}$ such that $\|A\| = \sqrt{\langle A, A\rangle}$.