Infinite mateship

(title suggested by Mark Weber)

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Abstract

In this paper, we prove a Cayley Theorem for compact closed categories, in which infinite strings of adjoint endofunctors replace permutations.

1 Introduction

A compact closed category may be viewed as a generalised group. Therefore, it makes sense to ask whether there exist a representation theory for compact closed categories.

`Unfortunately, I've only got the bare bones of such a theory, without any good combinatorical examples to show it's worthwhile.`

2 Background

Definition 2.1

A *compact closed category* is a monoidal category $(\mathcal{K}, i, \otimes)$ together with an (adjoint) equivalence

$$\mathcal{K}^{\mathsf{op}} \underset{(-)^{\flat}}{\overset{(-)^{\sharp}}{\rightleftarrows}} \mathcal{K}$$

and di-natural transformations $\vec{\varepsilon}, \vec{\eta}$ whose components have the form

$$i \xrightarrow{\vec{\eta}_x} x^{\sharp} \otimes x \qquad\qquad x \otimes x^{\sharp} \xrightarrow{\vec{\varepsilon}_x} i$$

and which satisfy the so-called *triangle identities*—*i.e.*, that the composites

$$x \xrightarrow{\overleftarrow{\upsilon}_x^{-1}} x \otimes i \xrightarrow{\iota_x \otimes \vec{\eta}_x} x \otimes (x^{\sharp} \otimes x) \xrightarrow{\alpha_{x,x^{\sharp},x}} (x \otimes x^{\sharp}) \otimes x \xrightarrow{\vec{\varepsilon}_x \otimes \iota_x} i \otimes x \xrightarrow{\vec{\upsilon}_x} x$$

$$x^{\sharp} \xrightarrow{\vec{\upsilon}_{x^{\sharp}}^{-1}} i \otimes x^{\sharp} \xrightarrow{\vec{\eta}_x \otimes \iota_{x^{\sharp}}} (x^{\sharp} \otimes x) \otimes x^{\sharp} \xrightarrow{\alpha_{x^{\sharp},x,x^{\sharp}}} x^{\sharp} \otimes (x \otimes x^{\sharp}) \xrightarrow{\iota_{x^{\sharp}} \otimes \vec{\varepsilon}_x} x^{\sharp} \otimes i \xrightarrow{\overleftarrow{\upsilon}_{x^{\sharp}}} x^{\sharp}$$

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should equal the identities on x and $x^\sharp$ respectively.

A compact closed category is called *strict* if $(\mathcal{K}, i, \otimes)$ is strict and if the equivalence

$$\mathcal{K}^{\mathsf{op}} \underset{(-)^\flat}{\overset{(-)^\sharp}{\rightleftarrows}} \mathcal{K}$$

is an isomorphism.

Note that we do not require a symmetry, or even braiding, on $\otimes$; neither do we require $(-)^\flat$ to co-incide with $(-)^\sharp$, even up to isomorphism.

Remark 2.2

There are also have dinatural transformations $\overleftarrow{\varepsilon}, \overleftarrow{\eta}$ with components of the form

$$i \xrightarrow{\overleftarrow{\eta}_x} x \otimes x^\flat \qquad\qquad x^\flat \otimes x \xrightarrow{\overleftarrow{\varepsilon}_x} i$$

defined by

$$i \xrightarrow{\overrightarrow{\eta}_{x^\flat}} x^{\flat\sharp} \otimes x \xrightarrow{\sim} x \otimes x^\flat$$

$$x^\flat \otimes x \xrightarrow{\sim} x^\flat \otimes x^{\flat\sharp} \xrightarrow{\overrightarrow{\varepsilon}_{x^\flat}} i$$

which also satisfy the triangle identities.

Definition 2.3

Of *mateship.*

Definition 2.4

Of *embedding of compact closed categories.*

3 Main Result

Definitions 3.1

Let $\mathcal{K}$ be an arbitrary category. We shall write $\mathbf{End}^{\pm\infty}(\mathcal{K})$ for the category of endofunctors with infinitely many left and right adjoints; *i.e.*, the objects of $\mathbf{End}^{\infty}(\mathcal{K})$ are infinite strings of adjoint endofunctors

$$f = (\cdots \dashv f_{-2} \dashv f_{-1} \dashv f_0 \dashv f_1 \dashv f_2 \dashv \cdots)$$

(with specified units and counits, $\eta^{(n,n+1)}, \varepsilon^{(n,n+1)}$) and an arrow $f \xrightarrow{\lambda} g$ is defined to be a natural transformation $f_0 \xrightarrow{\lambda_0} g_0$.

Theorem 3.2
Let $\mathcal{K}$ be an arbitrary category. Then $\mathbf{End}^{\pm\infty}(\mathcal{K})$ is a strict compact closed monoidal category.

Proof. The tensor product $\bullet$ is defined by:

$$(f \bullet g)_n = \begin{cases} f_n g_n & \text{if } n \text{ is even} \\ g_n f_n & \text{if } n \text{ is odd} \end{cases}$$

(with the obvious unit and counit maps), and the tensor unit i by

$$i_n = \mathrm{Id}_{\mathcal{K}}$$

for all n. It should be clear that these operations make $\mathbf{End}^{\pm\infty}(\mathcal{K})$ a strict monoidal category

To show that $\mathbf{End}^{\pm\infty}(\mathcal{K})$ is a (strict) compact closed category, we define $(-)^\sharp$ and $(-)^\flat$ by shifting on objects and mateship on arrows.

For an object

$$f = (\cdots \dashv f_{-1} \dashv f_0 \dashv f_1 \dashv \cdots)$$

of $\mathbf{End}^{\pm\infty}(\mathcal{K})$, $f^\sharp$ and $f^\flat$ are defined by

$$f^\sharp_n = f_{n+1} \text{ and } f^\flat_n = f_{n-1}$$

respectively.

For an arrow $f \xrightarrow{\lambda} g$ $\lambda^\sharp$ and $\lambda^\flat$ are defined to be the mates of $f_0 \xrightarrow{\lambda_0} g_0$,

$$g_1 \xrightarrow{\lambda_1} f_1 \text{ and } g_{-1} \xrightarrow{\lambda_{-1}} f_{-1}$$

respectively.

Now the arrows $i \xrightarrow{\vec{\eta}_f} f^\sharp \bullet f$ and $f \bullet f^\sharp \xrightarrow{\vec{\varepsilon}_f} i$ defined by the chosen unit and counit maps $\mathrm{Id}_{\mathcal{K}} \xrightarrow{\eta^{(01)}} f_1 f_0$ and $f_0 f_1 \xrightarrow{\varepsilon^{(01)}} \iota_{\mathcal{K}}$ satisfy the triangle identities tautologically, and so $\mathbf{End}^{\pm\infty}(\mathcal{K})$ is a compact closed category. Q.E.D.

Remark 3.3
Let $\mathbf{End}^{\infty}(\mathcal{K})$ denote the category of endofunctors with infinitely many right adjoints—*i.e.*, infinite strings of adjoint endofunctors

$$f = (f_0 \dashv f_1 \dashv f_2 \dashv \cdots).$$

Then both $\bullet$ and the left internal hom-functor

$$\mathbf{End}^{\pm\infty}(\mathcal{K})^{\mathsf{op}} \times \mathbf{End}^{\pm\infty}(\mathcal{K}) \xrightarrow{\multimap} \mathbf{End}^{\pm\infty}(\mathcal{K})$$

(which, as in any compact closed category, is defined by the formula $f \multimap h := f^\sharp \bullet h$), restrict along the forgetful functor $\mathbf{End}^{\pm\infty}(\mathcal{K}) \longrightarrow \mathbf{End}^{\infty}(\mathcal{K})$ so that $\mathbf{End}^{\infty}(\mathcal{K})$; is a left-closed monoidal category—and which is not, in general, right-closed.

Dually, the category of endofunctors with infinitely many left adjoints—denoted $\mathbf{End}^{-\infty}(\mathcal{K})$—is a right-closed monoidal category which is not, in general, left-closed.

`This is where the whole thing actually started, and perhaps explains the somewhat awkward notation.`

Theorem 3.4

Let $\mathcal{K}$ be a compact closed category. Then there is an embedding of compact closed categories $\mathcal{K} \xrightarrow{C} \mathbf{End}^{\pm\infty}(\mathcal{K})$ defined by

$$C(x)_n = \begin{cases} x^{\sharp n} \otimes (-) & \text{if } n > 0 \\ x \otimes (-) & \text{if } n = 0 \\ x^{\flat n} \otimes (-) & \text{if } n < 0 \end{cases}$$

Proof. In any monoidal closed category we have $x \otimes (-) \dashv x \multimap (-)$, but in a compact closed category we also have $x \multimap (-) \cong x^\sharp \otimes (-)$.

So we do get an infinite string of adjoint endofunctors

$$(\cdots \dashv x^\flat \otimes (-) \dashv x \otimes (-) \dashv x^\sharp \otimes (-) \dashv \cdots)$$

—which is what $C(x)$ is defined to be.

`And it's obviously an "embedding"...` Q.E.D.

4 The same again, but with additives

Theorem 4.1

If a category $\mathcal{K}$ has biproducts, then so does $\mathbf{End}^{\pm\infty}(\mathcal{K})$. Moreover, if $\mathcal{K}$ is a compact closed category with biproducts, then the Cayley embedding $\mathcal{K} \xrightarrow{C} \mathbf{End}^{\pm\infty}(\mathcal{K})$ preserves them.

Proof. If f and g are objects of $\mathbf{End}^{\pm\infty}(\mathcal{K})$ then, for each n, we have composable adjuntions

$$\mathcal{K} \underset{\oplus}{\overset{\Delta}{\rightleftarrows}} \mathcal{K} \times \mathcal{K} \underset{(f_n, g_n)}{\overset{(f_{n+1}, g_{n+1})}{\rightleftarrows}} \mathcal{K} \times \mathcal{K} \underset{\Delta}{\overset{\oplus}{\rightleftarrows}} \mathcal{K}$$

and therefore an infinite string of adjunctions

$$f \oplus g = (\cdots \dashv f_{-1} \oplus g_{-1} \dashv f_0 \oplus g_0 \dashv f_1 \oplus g_1 \dashv \cdots)$$

(where $f_n \oplus g_n$ denotes the composite $\Delta; (f_n, g_n); \oplus$).

It is then easy to verify the universal property that $f \oplus g$ is the biproduct of f and g. Q.E.D.

Remark 4.2

If $\mathcal{K}$ has products and coproducts which do not co-incide, then we get nothing. Given Robin Houston's theorem, this should not be surprising.

Extension to abelian categories?

5 Future work

To find an interesting compact closed category $\mathcal{V}$ (with or without biproducts) which admits an embedding into some $\mathbf{End}^{\pm\infty}(\mathcal{K})$, where $\mathcal{K}$ is demonstrably simpler than $\mathcal{V}$. For example, where $\mathcal{K}$ has smaller cardinality, or does not carry a monoidal structure related to that of $\mathcal{V}$.

But note that if $\mathcal{K}$ is finite/posetal/single-iso-class, then $\mathbf{End}^{\pm\infty}(\mathcal{K})$ is so too.

So, for example $\mathbf{Vec}_{fd}$ can only be represented on infinite categories—and ones which admit uncountably many endo-natural transformations of the identity functor! I'm not sure we can really do better than $\mathbf{Vec}_{fd}$ itself.