STAR-AUTONOMOUS FUNCTOR CATEGORIES

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Abstract. We construct a star-autonomous structure on the functor category $\mathcal{K}^{\mathcal{J}}$, where $\mathcal{J}$ is small, $\mathcal{K}$ is small-complete, and both are star-autonomous. A weaker result, that $\mathcal{K}^{\mathcal{J}}$ admits a linear distributive structure, is also shown under weaker hypotheses. The latter leads to a deeper understanding of the notion of *linear functor*.

1. Introduction

This paper is, to a large extent, a continuation of [6]. In that paper it is shown *inter alia* that the eighteen coherence axioms required of a *linear functor* $\mathcal{J} \longrightarrow \mathcal{K}$ can, in the case $\mathcal{J} = 1$, be neatly summarised using such familiar notions as (co)actions of (co)monoids, and (co)equivariance. Briefly, linear functors $1 \longrightarrow \mathcal{K}$ are in bijective correspondence with *cyclic nuclear monoids* in $\mathcal{K}$. That observation brought fresh intuition to the concept of linear functor: just as a monoidal functor can be thought of as a generalised monoid, so a linear functor should be thought of as a generalised cyclic nuclear monoid.

But there exists a partial converse to the idea that a monoidal functor is a generalised monoid: under certain circumstances it is possible to endow a functor category $\mathcal{K}^{\mathcal{J}}$ with a *convolution* tensor product [5]; in these cases, monoidal functors $\mathcal{J} \longrightarrow \mathcal{K}$ may be regarded as monoids in $\mathcal{K}^{\mathcal{J}}$. Here we show that, under comparable circumstances, it is possible to endow $\mathcal{K}^{\mathcal{J}}$ with a linear distributive structure so that linear functors $\mathcal{J} \longrightarrow \mathcal{K}$ may be regarded as cyclic nuclear monoids in $\mathcal{K}^{\mathcal{J}}$.

Moreover, we show that if $\mathcal{J}$ and $\mathcal{K}$ also admit duals—*i.e.*, if they are star-autonomous categories—then so is $\mathcal{K}^{\mathcal{J}}$. In fact, the formulae describing the (right- and left-) duals of an object X of $\mathcal{K}^{\mathcal{J}}$ are very intuitive: they are its two "de Morgan duals", $X^*(p) = (X({}^*p))^*$ and ${}^*X(p) = {}^*(X(p^*))$.

While this paper follows organically from its predecessor, the real impetus for continuing this vein of research came from collaboration with David Kruml [7]. In order to show that Kruml's notion of *Girard couple of quantales* (which was defined in response to concrete considerations arising in quantale theory) is an instance of a very general concept in [6], the author was led to construct a star-autonomous structure on $\mathbf{Sup}^{\rightarrow}$. This having been achieved, with the aid of the usual star-autonomous structures on $\mathbf{Sup}$ and $\rightarrow$, it was natural to try to understand the full extent of the latter construction.

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1.1. Remarks.

1.1.1. Throughout this paper our focus will be on functors $\mathcal{J} \xrightarrow{X} \mathcal{K}$ where $\mathcal{J}$ is small and $\mathcal{K}$ large. In this context, it is useful to think of $\mathcal{J}$ as a category of indices; we shall therefore write $X_\bullet$ in place of the more usual $X(\bullet)$.

1.1.2. We shall assume the reader familiar with the general theory of linear distributive categories and linear functors [3, 4]. We recall that, just as a boolean algebra can be (equivalently) defined either as a Heyting algebra whose negation operation is involutive or as a distributive lattice in which each element has a complement, so a star-autonomous category can be (essentially equivalently) defined either as a monoidal closed category equipped with a dualising object or as a linear distributive category in which every object has a left dual and a right dual. In the present paper, we shall (for reasons explained in Section 3) take the latter approach; but we shall insist that duals be functorially specified, as occurs tautologically in the former approach.

1.1.3. We do not assume symmetry; in this context, it is convenient to take *(co)closed* to mean *both left- and right-(co)closed*.

2. Preliminaries

2.1. Definition. A category $\mathcal{K}$ is said to have *(co)limits of size* $\mathcal{J}$ if $\mathcal{J}$ is small and $\mathcal{K}$ small-(co)complete with respect to some reasonable notion of *small* (including *finite* and *countable*, as well as *set-sized*).

A monoidal category $(\mathcal{K}; \otimes, i)$ is said to have *distributive* (co)limits of size $\mathcal{J}$ if it has (co)limits of size $\mathcal{J}$ and if these are distributed by $\otimes$.

2.2. Theorem. Let $(\mathcal{J}; \boxtimes, e_\mathcal{J})$ and $(\mathcal{K}; \boxtimes, e_\mathcal{K})$ be monoidal categories, and suppose that the latter has distributive colimits of size $\mathcal{J}$.

Then the functor category $\mathcal{K}^\mathcal{J}$ can be equipped with a monoidal structure $(\boxtimes_C, E)$ such that natural transformations $E \xrightarrow{\zeta} Z$ and $X \boxtimes_C Y \xrightarrow{\vartheta} Z$ are in bijective correspondence with arrows $e_\mathcal{K} \xrightarrow{\hat{\zeta}} Z_{e_\mathcal{J}}$ and natural transformations of the form

$$X_m \boxtimes Y_n \xrightarrow{\hat{\vartheta}_{m,n}} Z_{m \boxtimes n}$$

respectively.

In particular, (M, μ, η) is a monoid in $(\mathcal{K}^\mathcal{J}; \boxtimes_C, E)$ if and only if $(M, \hat{\mu}, \hat{\eta})$ is a monoidal functor $(\mathcal{J}; \boxtimes, e_\mathcal{J}) \longrightarrow (\mathcal{K}; \boxtimes, e_\mathcal{K})$.

2.3. Remark. The hypothesis that $\mathcal{K}$ has colimits of size $\mathcal{J}$ implies that $\mathcal{J}$ can be made into a $\mathcal{K}$-category; one sets the hom-$\mathcal{K}$-object $[\![m, n]\!]$ to be the appropriate co-power of $e_\mathcal{K}$—*i.e.*, that corresponding to the hom-set $[m, n]$. The tensor product constructed in Theorem 2.2 can then be understood as a convolution tensor product in the sense of Day [5]. But the observation that convolution tensor product represents a very natural multicategory structure on $\mathcal{K}^{\mathcal{J}}$, although widely known, appears to have remained folklore. We shall therefore only sketch a proof of Theorem 2.2, primarily to settle notation.

2.4. Lemmata.

2.4.1. $(X \boxtimes_C Y)_r$ is defined as the colimit of

$$\boxtimes \downarrow r \longrightarrow \mathcal{J} \times \mathcal{J} \xrightarrow{X \times Y} \mathcal{K} \times \mathcal{K} \xrightarrow{\boxtimes} \mathcal{K}$$

—more colloquially, we write

$$(X \boxtimes_C Y)_r = \operatorname*{colim}_{p \boxtimes q \to r} X_p \boxtimes Y_q.$$

Given an arrow $p \boxtimes q \xrightarrow{\lambda} r$, we write ϖ_λ for the corresponding coprojection $X_p \boxtimes Y_q \longrightarrow (X \boxtimes_C Y)_r$.

2.4.2. $(X \boxtimes_C Y)_r \xrightarrow{(X \boxtimes_C Y)_\psi} (X \boxtimes_C Y)_{r'}$ is defined by

$$\begin{array}{ccc} X_p \boxtimes Y_q & \xrightarrow{\varpi_\lambda} & (X \boxtimes_C Y)_r \\ & \searrow^{\varpi_{\psi \circ \lambda}} & \downarrow{\scriptstyle (X \boxtimes_C Y)_\psi} \\ & & (X \boxtimes_C Y)_{r'} \end{array}$$

and $A \boxtimes_C B \xrightarrow{\zeta \boxtimes_C \vartheta} X \boxtimes Y$ by

$$\begin{array}{ccc} A_p \boxtimes B_q & \xrightarrow{\varpi_\lambda} & (A \boxtimes_C B)_r \\ {\scriptstyle \zeta_p \boxtimes \vartheta_q}\downarrow & & \downarrow{\scriptstyle (\zeta \boxtimes_C \vartheta)_r} \\ X_p \boxtimes Y_q & \xrightarrow[\varpi_\lambda]{} & (X \boxtimes_C Y)_r \end{array}$$

for all $p \boxtimes q \xrightarrow{\lambda} r$.

2.4.3. The (inverse of the) associativity isomorphism is defined by

$$\begin{array}{ccc} X_j \boxtimes (Y_k \boxtimes Z_p) & \xrightarrow{\alpha^{-1}} & (X_j \boxtimes Y_k) \boxtimes Z_p \\ {\scriptstyle \mathrm{id} \boxtimes \varpi_{\lambda_0}}\downarrow & & \downarrow{\scriptstyle \varpi_{\mathrm{id}} \boxtimes \mathrm{id}} \\ X_j \boxtimes (Y \boxtimes_C Z)_q & & (X \boxtimes_C Y)_{j \boxtimes k} \boxtimes Z_p \\ {\scriptstyle \varpi_{\lambda_1}}\downarrow & & \downarrow{\scriptstyle \varpi_{\lambda_2}} \\ (X \boxtimes_C (Y \boxtimes_C Z))_r & \xrightarrow[(\alpha^{-1})_r]{} & ((X \boxtimes_C Y) \boxtimes_C Z)_r \end{array}$$

—where λ_0, λ_1 are arbitrary and λ_2 is the composite

$$(j \boxtimes k) \boxtimes p \xrightarrow{\alpha} j \boxtimes (k \boxtimes p) \xrightarrow{\mathrm{id} \boxtimes \lambda_0} j \boxtimes q \xrightarrow{\lambda_1} r.$$

2.4.4. E_r is defined to be the colimit of $e_{\mathcal{J}} \downarrow r \longrightarrow 1 \xrightarrow{e_{\mathcal{K}}} \mathcal{K}$. Note that $e_{\mathcal{J}} \downarrow r$ is just the hom-set $[e_{\mathcal{J}}, r]$ regarded as a discrete category; hence, E_r is just a copower of $e_{\mathcal{K}}$. Given an arrow $e_{\mathcal{J}} \xrightarrow{\zeta} r$ we write ϖ_ζ for the corresponding coprojection $e_{\mathcal{K}} \longrightarrow E_r$.

$E_r \xrightarrow{E_\psi} E_{r'}$ is defined by

$$\begin{array}{ccc} e_{\mathcal{K}} & \xrightarrow{\varpi_\zeta} & E_r \\ & \searrow^{\varpi_{\psi\circ\zeta}} & \Big\downarrow{\scriptstyle E_\psi} \\ & & E_s \end{array}$$

for all $e_{\mathcal{J}} \xrightarrow{\zeta} r$.

2.4.5. The (inverse of the) right unit law is defined by

$$\begin{array}{ccc} X_r & \xrightarrow{\overleftarrow{\upsilon}_r^{-1}} & (X \boxtimes_C E)_r \\ {\scriptstyle \overleftarrow{\upsilon}^{-1}}\Big\downarrow & & \Big\uparrow{\scriptstyle \varpi_{\overleftarrow{\upsilon}}} \\ X_r \boxtimes e_{\mathcal{K}} & \xrightarrow[\mathrm{id} \boxtimes \varpi_{\mathrm{id}}]{} & X_r \boxtimes E_{e_{\mathcal{J}}} \end{array}$$

and the (inverse of the) left unit isomorphism is defined symmetrically.

2.5. Example. As noted in [7, Remark 4], a *unital couple of quantales* is essentially the same thing as a monoidal functor $(\rightarrow; \wedge, \top) \longrightarrow (\mathbf{Sup}; \boxtimes, 2)$. It can therefore also be regarded as a monoid in $(\mathbf{Sup}^{\rightarrow}; \boxtimes_C, E)$, where $E = (1 \xrightarrow{\perp} 2)$ and $(x_1 \xrightarrow{x_!} x_0) \boxtimes_C (y_1 \xrightarrow{y_!} y_0)$ is the dotted arrow in the diagram below.

$$\begin{array}{ccccc} & & \overset{x_! \boxtimes \mathrm{id}}{\frown\!\!\!\longrightarrow} & & \\ x_0 \boxtimes y_1 & \longrightarrow & p & \cdots\cdots\rightarrow & x_1 \boxtimes y_1 \\ {\scriptstyle \mathrm{id} \boxtimes y_!}\Big\uparrow & p.o. & \Big\uparrow & \nearrow_{\mathrm{id} \boxtimes y_!} & \\ x_0 \boxtimes y_0 & \xrightarrow[x_! \boxtimes \mathrm{id}]{} & x_1 \boxtimes y_0 & & \end{array}$$

The reader is cautioned that $(\mathbf{Sup}^{\rightarrow}; \boxtimes_C, E)$ bears little relation to $(\mathbf{Sup}(\mathbf{Set}^{\rightarrow}); \boxtimes, 2_{\mathbf{Set}^{\rightarrow}})$. It follows from [8, Proposition VI.2.1] that $\mathbf{Sup}(\mathbf{Set}^{\rightarrow})$ is (equivalent to) the subcategory of $\mathbf{Sup}^{\rightarrow}$ which has: for objects, surjective sup- and inf-homomorphisms; for arrows, commutative squares satisfying the Beck-Chevalley condition. But this is far from being a monoidal subcategory of $\mathbf{Sup}^{\rightarrow}$, since it does not even contain the object E.

3. Plan

The main objective of this paper, as indicated in the title and abstract, is to construct a meaningful star-autonomous structure on the functor category $\mathcal{K}^{\mathcal{J}}$, where $\mathcal{J}$ and $\mathcal{K}$ are themselves endowed with star-autonomous structures and $\mathcal{K}$ has colimits of size $\mathcal{J}$.

Under these hypotheses, $\otimes_{\mathcal{K}}$ distributes all colimits; hence we can apply Theorem 2.2 to obtain a monoidal structure $(\otimes_C, E)$ on $\mathcal{K}^{\mathcal{J}}$. Since such a convolution tensor product is necessarily closed, it would suffice to find a suitable dualising object. The author, however, frankly balked at the prospect of computing the double dual of an object in $\mathcal{K}^{\mathcal{J}}$ and observed that, under the same hypotheses, $\mathcal{K}$ also has limits of size $\mathcal{J}$ which are distributed by $\oplus_{\mathcal{K}}$. Thus, we can also invoke the dual of Theorem 2.2 to obtain a second monoidal structure $(\oplus_C, D)$ on $\mathcal{K}^{\mathcal{J}}$. Moreover, we already have natural candidates for the (right- and left-) duals of an object of $\mathcal{K}^{\mathcal{J}}$, as described in Section 1.

Thus we opt instead to construct linear distributions for $(\mathcal{K}^{\mathcal{J}}, \otimes_C, E, \oplus_C, D)$, and then to find *linear adjoints* overlying ${}^*X$, X, and X^*. Unsurprisingly, the first half of this project can be carried out under somewhat weaker hypotheses; so, as a bonus, we obtain a more general theorem which turns out to be of some interest in its own right.

3.1. Remark. We shall need to consider the common composite of a naturality square

$$\begin{array}{ccc}
X_p \otimes (Y_q \oplus Z_t) & \xrightarrow{\ \mathrm{id} \otimes (Y_\beta \oplus \mathrm{id})\ } & X_p \otimes (Y_s \oplus Z_t) \\
{\scriptstyle \vec{\kappa}}\downarrow & & \downarrow{\scriptstyle \vec{\kappa}} \\
(X_p \otimes Y_q) \oplus Z_t & \xrightarrow[\ (\mathrm{id} \otimes Y_\beta) \oplus \mathrm{id}\ ]{} & (X_p \otimes Y_q) \oplus Z_t
\end{array}$$

sufficiently often that it will merit abbreviation to $X_p \otimes (Y_q \oplus Z_t) \xrightarrow{[\beta]} (X_p \otimes Y_s) \oplus Z_t$.

Henceforth, we also omit the subscripts on $e_{\mathcal{J}}$, $e_{\mathcal{K}}$, $\otimes_C$ and $\oplus_C$.

4. Linear distributive functor categories

We recall that a linear distributive category $(\mathcal{J}; \otimes, e, \oplus, d)$ is called *bilinear* if $\otimes$ is closed and $\oplus$ is coclosed. Any star-autonomous category is bilinear with, for instance, the right coclosed structure defined by $x \oslash z = x \otimes z^*$, but not *vice versa*.

Although we are dealing with arbitrary (*i.e.*, not necessarily symmetric) bilinear categories, our arguments shall concentrate on their left closed and right coclosed structures ($\multimap$ and $\oslash$, respectively). In order to distinguish between the units and counits of the two adjunctions at hand—$x \otimes (\ ) \dashv x \multimap (\ )$ and $(\ ) \oslash z \dashv (\ ) \oplus z$—we write $\overline{\eta}, \overline{\varepsilon}$ for those of the former and $\underline{\eta}, \underline{\varepsilon}$ for those of the latter.

Finally, we recall that in a bilinear category, there is a canonical isomorphism $(x \multimap y) \oplus z \xrightarrow{\sim} x \multimap (y \oplus z)$. [In the star-autonomous case, this is just the associativity isomorphism $(x^* \oplus y) \oplus z \xrightarrow{\sim} x^* \oplus (y \oplus z)$.]

4.1. Definition. Let $(\mathcal{J}; \ltimes, e, \bowtie, d)$ be a bilinear category. Then the *Beck transform* of a map $p \ltimes q \xrightarrow{\alpha} s \bowtie t$ in $\mathcal{J}$ is the map $q \oslash t \xrightarrow{\beta} p \multimap s$ obtained by transposition as follows:

$$\begin{array}{c} p \ltimes q \longrightarrow s \bowtie t \\ \hline q \longrightarrow p \multimap (s \bowtie t) \\ \hline q \longrightarrow (p \multimap s) \bowtie t \\ \hline q \oslash t \longrightarrow p \multimap s \end{array}$$

—equivalently, it is the composite depicted below.

$$\begin{array}{ccccc} q \oslash t & & & & p \multimap s \\ \Big\downarrow{\scriptstyle \overline{\eta} \oslash \mathrm{id}} & & & & \Big\uparrow{\scriptstyle \underline{\varepsilon}} \\ (p \multimap (p \ltimes q)) \oslash t & \xrightarrow{(\mathrm{id} \multimap \alpha) \oslash \mathrm{id}} & (p \multimap (s \bowtie t)) \oslash t & \xrightarrow{\sim} & ((p \multimap s) \bowtie t) \oslash t \end{array}$$

4.2. Theorem. Suppose that $(\mathcal{J}; \ltimes, e, \bowtie, d)$ is bilinear, that $(\mathcal{K}; \ltimes, e, \bowtie, d)$ is linear distributive, that $\mathcal{K}$ has limits and colimits of size $\mathcal{J}$, and that these are distributed by $\bowtie$ and $\ltimes$, respectively.

Then there exists a natural transformation, $\vec{\kappa}$, such that for all arrows

$$p \ltimes q \xrightarrow{\ \lambda\ } r \xrightarrow{\ \rho\ } s \bowtie t$$

in $\mathcal{J}$, the diagram

$$\begin{array}{ccccc} & (X \ltimes (Y \bowtie Z))_r & \xrightarrow{\vec{\kappa}_r} & ((X \ltimes Y) \bowtie Z)_r & \\ {\scriptstyle \varpi_\lambda} \nearrow & & & & \searrow {\scriptstyle \pi_\rho} \\ X_p \ltimes (Y \bowtie Z)_q & & & & (X \ltimes Y)_s \bowtie Z_t \\ {\scriptstyle \mathrm{id} \ltimes \pi_{\underline{\eta}}} \searrow & & & & \nearrow {\scriptstyle \varpi_{\overline{\varepsilon}} \bowtie \mathrm{id}} \\ & X_p \ltimes (Y_{q \oslash t} \bowtie Z_t) & \xrightarrow{[\beta]} & (X_p \ltimes Y_{p \multimap s}) \bowtie Z_t & \end{array}$$

(where β denotes the Beck transform of $\rho \circ \lambda$) commutes in $\mathcal{K}$.

Proof. The diagram contained in the statement of the theorem amounts to a definition of the components of $\vec{\kappa}$ if we can demonstrate the appropriate conicity and coconicity criteria. It suffices to show the latter, as the former follows by a dual argument.

Therefore, let

$$\begin{array}{ccc} p \ltimes q & \xrightarrow{\phi \ltimes \chi} & p' \ltimes q' \\ & {\scriptstyle \lambda} \searrow \quad \swarrow {\scriptstyle \lambda'} & \\ & r & \end{array}$$

be an arrow in $\boxtimes \downarrow r$ and $r \xrightarrow{\rho} s \uplus t$ an object of $r \downarrow \uplus$. Let β be the Beck transform of $\rho \circ \lambda$ and β' be that of $\rho \circ \lambda'$. Then the commutativity of the diagrams

$$
\begin{array}{ccc}
q \oslash t & \xrightarrow{\chi \oslash \mathrm{id}} & q' \oslash t \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta'} \\
p \multimap s & \xleftarrow[\phi \multimap \mathrm{id}]{} & p' \multimap s
\end{array}
\qquad
\begin{array}{ccc}
p \boxtimes (p' \multimap s) & \xrightarrow{\phi \boxtimes \mathrm{id}} & p' \boxtimes (p' \multimap s) \\
{\scriptstyle \mathrm{id} \boxtimes (\phi \multimap \mathrm{id})}\downarrow & & \downarrow{\scriptstyle \overline{\varepsilon}} \\
p \boxtimes (p \multimap s) & \xrightarrow[\overline{\varepsilon}]{} & s
\end{array}
$$

together with the naturality of $\vec{\kappa}$ and $\underline{\eta}$ combine to prove that

$$
\begin{array}{ccccc}
X_p \boxtimes (Y \uplus Z)_q & & \xrightarrow{X_\phi \boxtimes (Y \uplus Z)_\chi} & & X_{p'} \boxtimes (Y \uplus Z)_{q'} \\
{\scriptstyle \mathrm{id} \boxtimes \pi_{\underline{\eta}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id} \boxtimes \pi_{\underline{\eta}}} \\
X_p \boxtimes (Y_{q \oslash t} \uplus Z_t) & & \xrightarrow{X_\phi \boxtimes (Y_{\chi \oslash \mathrm{id}} \uplus \mathrm{id})} & & X_{p'} \boxtimes (Y_{q' \oslash t} \uplus Z_t) \\
{\scriptstyle \vec{\kappa}}\downarrow & & & & \downarrow{\scriptstyle \vec{\kappa}} \\
(X_p \boxtimes Y_{q \oslash t}) \uplus Z_t & \xrightarrow{(\mathrm{id} \boxtimes Y_{\chi \oslash \mathrm{id}}) \uplus \mathrm{id}} & (X_p \boxtimes Y_{q' \oslash t}) \uplus Z_t & \xrightarrow{(X_\phi \boxtimes \mathrm{id}) \uplus \mathrm{id}} & (X_{p'} \boxtimes Y_{q' \oslash t}) \uplus Z_t \\
\downarrow{\scriptstyle (\mathrm{id} \boxtimes Y_\beta) \uplus \mathrm{id}} & & {\scriptstyle (\mathrm{id} \boxtimes Y_{\beta'}) \uplus \mathrm{id}}\downarrow & & {\scriptstyle (\mathrm{id} \boxtimes Y_{\beta'}) \uplus \mathrm{id}}\downarrow \\
(X_p \boxtimes Y_{p \multimap s}) \uplus Z_t & \xleftarrow[(\mathrm{id} \boxtimes Y_{\phi \multimap \mathrm{id}}) \uplus \mathrm{id}]{} & (X_p \boxtimes Y_{p' \multimap s}) \uplus Z_t & \xrightarrow[(X_\phi \boxtimes \mathrm{id}) \uplus \mathrm{id}]{} & (X_{p'} \boxtimes Y_{p' \multimap s}) \uplus Z_t \\
& \searrow{\scriptstyle \varpi_{\overline{\varepsilon}} \uplus \mathrm{id}} & & \swarrow{\scriptstyle \varpi_{\overline{\varepsilon}} \uplus \mathrm{id}} & \\
& & (X \boxtimes Y)_s \uplus Z_t & &
\end{array}
$$

commutes as well.

Now it remains to show that $\vec{\kappa}$ is natural in r, and that $\vec{\kappa}$ is natural in X, Y, Z. For the former, it suffices to show that for all arrows

$$
p \boxtimes q \xrightarrow{\lambda} r \xrightarrow{\psi} r' \xrightarrow{\rho} s \uplus t
$$

in $\mathcal{J}$, the composites

$$
\begin{array}{ccc}
X_p \boxtimes (Y \uplus Z)_q & \overset{\varpi_{\psi \circ \lambda}}{\dashrightarrow} & (X \boxtimes (Y \uplus Z))_{r'} \\
{\scriptstyle \varpi_\lambda}\downarrow & & \\
(X \boxtimes (Y \uplus Z))_r & \xrightarrow{(X \boxtimes (Y \uplus Z))_\psi} & (X \boxtimes (Y \uplus Z))_{r'} \\
{\scriptstyle \vec{\kappa}_r}\downarrow & & \downarrow{\scriptstyle \vec{\kappa}_{r'}} \\
((X \boxtimes Y) \uplus Z)_r & \xrightarrow{((X \boxtimes Y) \uplus Z)_\psi} & ((X \boxtimes Y) \uplus Z)_{r'} \\
& & \downarrow{\scriptstyle \pi_\rho} \\
((X \boxtimes Y) \uplus Z)_r & \overset{\pi_{\rho \circ \psi}}{\dashrightarrow} & (X \boxtimes Y)_s \uplus Z_t
\end{array}
$$

agree. But since $(\rho \circ \psi) \circ \lambda = \rho \circ (\psi \circ \lambda)$, the top-right and bottom-left composites of this diagram are, by definition, equal.

The remaining naturality is left as an exercise to the reader. ■

Symmetrically, we can also define a natural transformation $\overleftarrow{\kappa}$ of the correct type.

4.3. THEOREM. Under the same hypotheses as Theorem 4.2, $\vec{\kappa}$ and $\overleftarrow{\kappa}$ satisfy the necessary coherence conditions to make $(\mathcal{K}^{\mathcal{J}}; \otimes, E, \oplus, D)$ a linear distributive category.

PROOF. We shall only prove one of ten axioms, namely

$$\begin{array}{ccc}
W \otimes (X \otimes (Y \oplus Z)) & \xrightarrow{\alpha^{-1}} & (W \otimes X) \otimes (Y \oplus Z) \\
{\scriptstyle \mathrm{id} \otimes \vec{\kappa}} \downarrow & & \\
W \otimes ((X \otimes Y) \oplus Z) & (\text{LD}_1) & \downarrow {\scriptstyle \vec{\kappa}} \\
{\scriptstyle \vec{\kappa}} \downarrow & & \\
(W \otimes (X \otimes Y)) \oplus Z & \xrightarrow[\alpha^{-1} \oplus \mathrm{id}]{} & ((W \otimes X) \otimes Y) \oplus Z.
\end{array}$$

Three more coherence pentagons follow immediately (by symmetry); the remaining two—those which alternate $\otimes$ and $\oplus$, and therefore involve components of both $\vec{\kappa}$ and $\overleftarrow{\kappa}$—can be proven in a similar fashion, as well as the four coherence triangles.

To verify the rth component of (LD_1), it suffices to show that, for all arrows $j \otimes q \xrightarrow{\lambda_1} r$, $k \otimes p \xrightarrow{\lambda_0} q$, and $r \xrightarrow{\rho} s \oplus t$ in $\mathcal{J}$, the composites

$$\begin{array}{ccc}
W_j \otimes (X_k \otimes (Y \oplus Z)_p) & \qquad & W_j \otimes (X_k \otimes (Y \oplus Z)_p) \\
\downarrow {\scriptstyle \mathrm{id} \otimes \varpi_{\lambda_0}} & & \downarrow {\scriptstyle \mathrm{id} \otimes \varpi_{\lambda_0}} \\
W_j \otimes (X \otimes (Y \oplus Z))_q & & W_j \otimes (X \otimes (Y \oplus Z))_q \\
\downarrow {\scriptstyle \varpi_{\lambda_1}} & & \downarrow {\scriptstyle \varpi_{\lambda_1}} \\
(W \otimes (X \otimes (Y \oplus Z)))_r & & (W \otimes (X \otimes (Y \oplus Z)))_r \\
\downarrow {\scriptstyle (\alpha^{-1} \otimes \mathrm{id}) \circ \vec{\kappa} \circ (\mathrm{id} \otimes \vec{\kappa})} & & \downarrow {\scriptstyle \vec{\kappa} \circ \alpha^{-1}} \\
(((W \otimes X) \otimes Y) \oplus Z)_r & & (((W \otimes X) \otimes Y) \oplus Z)_r \\
\downarrow {\scriptstyle \pi_\rho} & & \downarrow {\scriptstyle \pi_\rho} \\
(W \otimes (X \otimes Y))_s \oplus Z_t & & (W \otimes (X \otimes Y))_s \oplus Z_t
\end{array}$$

agree. [Here we use that the functor $W_j \otimes (\ )$ preserves colimits.]

Let β_n denote the Beck transform of $\rho_n \circ \lambda_n$ for $n \in \{0, 1, 2\}$, where $\lambda_0, \lambda_1, \rho_1$ are as above, ρ_0 is the coevaluation map $q \xrightarrow{\eta} (q \oslash t) \oplus t$, λ_2 is as in Lemma 2.4.3, and $\rho_1 = \rho = \rho_2$; also let ξ denote the obvious arrow $(j \otimes k) \otimes (k \multimap (j \multimap s)) \longrightarrow s$ and $\tilde{\xi}$ its transpose—*i.e.*, the canonical isomorphism $(k \multimap (j \multimap s)) \longrightarrow (j \otimes k) \multimap s$.

Then the two composites depicted above can be rewritten as

$$
\begin{array}{c}
W_j \boxtimes (X_k \boxtimes (Y \uplus Z)_p) \\
\big\downarrow\ \mathrm{id} \boxtimes \left(\mathrm{id} \boxtimes \pi_{\underline{\eta}}\right) \\
W_j \boxtimes (X_k \boxtimes (Y_{p \oslash t} \uplus Z_t)) \\
\big\downarrow\ [\beta_0] \\
W_j \boxtimes \big((X_k \boxtimes Y_{k \multimap (q \oslash t)}) \uplus Z_t\big) \\
\big\downarrow\ [\mathrm{id} \multimap \beta_1] \\
\big(W_j \boxtimes (X_k \boxtimes Y_{k \multimap (j \multimap s)})\big) \uplus Z_t \\
\big\downarrow\ \alpha^{-1} \uplus \mathrm{id} \\
\big((W_j \boxtimes X_k) \boxtimes Y_{k \multimap (j \multimap s)}\big) \uplus Z_t \\
\big\downarrow\ (\varpi_{\mathrm{id}} \boxtimes \mathrm{id}) \uplus \mathrm{id} \\
\big((W \boxtimes X)_{j \boxtimes k} \boxtimes Y_{k \multimap (j \multimap s)}\big) \uplus Z_t \\
\big\downarrow\ \varpi_{\xi} \uplus \mathrm{id} \\
((W \boxtimes X) \boxtimes Y)_s \uplus Z_t
\end{array}
\qquad\qquad
\begin{array}{c}
W_j \boxtimes (X_k \boxtimes (Y \uplus Z)_p) \\
\big\downarrow\ \alpha^{-1} \\
(W_j \boxtimes X_k) \boxtimes (Y \uplus Z)_p \\
\big\downarrow\ \varpi_{\mathrm{id}} \boxtimes \mathrm{id} \\
(W \boxtimes X)_{j \boxtimes k} \boxtimes (Y \uplus Z)_p \\
\big\downarrow\ \pi_{\underline{\eta}} \\
(W \boxtimes X)_{j \boxtimes k} \boxtimes (Y_{p \oslash t} \uplus Z_t) \\
\big\downarrow\ [\beta_2] \\
\big((W \boxtimes X)_{j \boxtimes k} \boxtimes Y_{(j \boxtimes k) \multimap s}\big) \uplus Z_t \\
\big\downarrow\ \varpi_{\overline{\varepsilon}} \\
((W \boxtimes X) \boxtimes Y)_s \uplus Z_t
\end{array}
$$

respectively, as demonstrated in Figures A.1 and A.2.

Now it is easy to see that the diagram

$$
\begin{array}{ccc}
p \oslash t & \xrightarrow{\ \beta_2\ } & (j \boxtimes k) \multimap s \\
\beta_0 \big\downarrow & & \big\uparrow \tilde{\xi} \\
k \multimap (q \oslash t) & \xrightarrow[\ \mathrm{id} \multimap \beta_1\ ]{} & k \multimap (j \multimap s)
\end{array}
$$

commutes. This, together with the fact $\mathcal{K}$ itself satisfies (LD_1), entails that the diagram

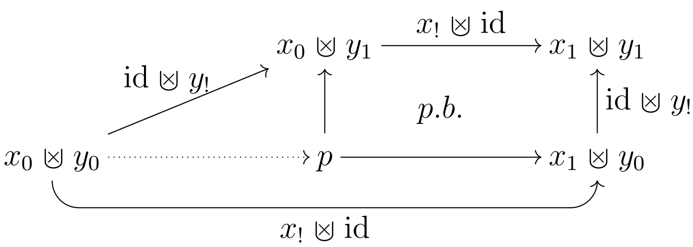
$$
\begin{array}{ccc}
W_j \boxtimes (X_k \boxtimes (Y_{p \oslash t} \uplus Z_t)) & \xrightarrow{\ \alpha^{-1}\ } & (W_j \boxtimes X_k) \boxtimes (Y_{p \oslash t} \uplus Z_t) \\
\big\downarrow\ \mathrm{id} \boxtimes [\beta_0] & & \big\downarrow\ [\beta_2] \\
W_j \boxtimes \big((X_k \boxtimes Y_{k \multimap (q \oslash t)}) \uplus Z_t\big) & (*) & \big((W_j \boxtimes X_k) \boxtimes Y_{(j \boxtimes k) \multimap s}\big) \uplus Z_t \\
\big\downarrow\ [\mathrm{id} \multimap \beta_1] & & \big\uparrow\ \left(\mathrm{id} \boxtimes Y_{\tilde{\xi}}\right) \uplus \mathrm{id} \\
\big(W_j \boxtimes (X_k \boxtimes Y_{k \multimap (j \multimap s)})\big) \uplus Z_t & \xrightarrow[\ \alpha^{-1} \uplus \mathrm{id}\ ]{} & \big((W_j \boxtimes X_k) \boxtimes Y_{k \multimap (j \multimap s)}\big) \uplus Z_t
\end{array}
$$

commutes. Hence the two composites we are interested in do coincide, as demonstrated in Figure A.3. ■

4.4. Remark. If $\mathcal{K}$ is also bilinear, then the distributivity hypotheses of Theorems 4.2 and 4.3 are superfluous. Moreover, in this case $\mathcal{K}^{\mathcal{J}}$ will be bilinear too, since the results of [5] can be applied to both $\mathcal{K}^{\mathcal{J}}$ and $(\mathcal{K}^{\mathcal{J}})^{\mathsf{op}} = (\mathcal{K}^{\mathsf{op}})^{\mathcal{J}^{\mathsf{op}}}$. It therefore seems that bilinear categories provide a more natural setting for these results than the more general one in which we stated them.

On the other hand, a more general theorem—one which imposed no completeness conditions on $\mathcal{K}$—could undoubtably be proven if we were willing to adopt the more general framework of *polycategories*. [A linear distributive category is essentially the same thing as a *representable* polycategory.] Even in this context, however, the bilinearity of $\mathcal{J}$ appears to remain crucial.

4.5. Example. There is a linear distributive category structure on $\mathbf{Sup}^{\to}$ with: $\otimes_C$ and E as in Example 2.5, $D = (2 \xrightarrow{!} 1)$, and $(x_0 \xrightarrow{x_!} x_1) \parr_C (y_0 \xrightarrow{y_!} y_1)$ the dotted arrow in the diagram below.



5. Star-autonomous functor categories

We now return to the case of greatest interest; throughout this section, $\mathcal{J}$ and $\mathcal{K}$ will denote star-autonomous categories such that $\mathcal{K}$ has colimits of size $\mathcal{J}$. We shall not always pedantically distinguish between, for instance, $e \oslash s$ and s^*.

5.1. Definitions.

5.1.1. Given a functor $\mathcal{J} \xrightarrow{X} \mathcal{K}$, the functors $\mathcal{J} \xrightarrow{X^*} \mathcal{K}$ and $\mathcal{J} \xrightarrow{{}^*X} \mathcal{K}$ are defined by the formulae

$$(X^*)_p = (X_{{}^*p})^*, \qquad ({}^*X)_p = {}^*(X_{p^*}).$$

5.1.2. Given an arrow $e \xrightarrow{\rho} s \parr t$ in $\mathcal{J}$, we define $e \xrightarrow{\tau_\rho} (X^*)_s \parr X_t$ to be the arrow in $\mathcal{K}$ constructed as follows:

$$\frac{\dfrac{\dfrac{e \longrightarrow s \parr t}{{}^*s \longrightarrow t}}{X_{{}^*s} \longrightarrow X_t}}{e \longrightarrow (X_{{}^*s})^* \parr X_t.}$$

5.1.3. Dually, given an arrow $p \otimes q \xrightarrow{\lambda} d$ in $\mathcal{J}$ we define $X_p \otimes (X_{^*q})^* \xrightarrow{\gamma_\lambda} d$ to be the arrow in $\mathcal{K}$ constructed as follows:

$$\cfrac{\cfrac{\cfrac{p \otimes q \longrightarrow d}{p \longrightarrow {}^*q}}{X_p \longrightarrow X_{^*q}}}{X_p \otimes (X_{^*q})^* \longrightarrow d.}$$

5.2. Theorem. There exists a natural transformation $E \xrightarrow{\tau} X^* \oplus X$ whose components are defined by

$$e \xrightarrow{\quad \varpi_\zeta \quad} E_r \xrightarrow{\quad \tau_r \quad} (X^* \oplus X)_r \xrightarrow{\quad \pi_\rho \quad} (X^*)_s \oplus X_t. \qquad \left(\text{composite } e \longrightarrow (X^*)_s \oplus X_t \text{ is } \tau_{\rho\circ\zeta}\right)$$

Dually, there exists a natural transformation $X \otimes X^* \xrightarrow{\gamma} D$ satisfying the dual diagram.

The proof of this theorem is merely a case of verifying the (co)conicity criteria, which we leave as an exercise to the reader.

5.3. Theorem. The natural transformations $E \xrightarrow{\tau} X^* \oplus X$ and $X \otimes X^* \xrightarrow{\gamma} D$ defined above form a linear adjunction in $\mathcal{K}^{\mathcal{J}}$. [Thus X^* *is* the right dual of X.]

Proof. The statement of the theorem means that the composites

$$\begin{array}{ccc}
X & \qquad & X^* \\
\Big\downarrow \overleftarrow{\upsilon}^{-1} & & \Big\downarrow \overrightarrow{\upsilon}^{-1} \\
X \otimes E & & E \otimes X^* \\
\Big\downarrow \mathrm{id} \otimes \tau & & \Big\downarrow \tau \otimes \mathrm{id} \\
X \otimes (X^* \oplus X) & & (X^* \oplus X) \otimes X^* \\
\Big\downarrow \vec{\kappa} & & \Big\downarrow \overleftarrow{\kappa} \\
(X \otimes X^*) \oplus X & & X^* \oplus (X \otimes X^*) \\
\Big\downarrow \gamma \oplus \mathrm{id} & & \Big\downarrow \mathrm{id} \oplus \gamma \\
D \oplus X & & X^* \oplus D \\
\Big\downarrow \vec{\upsilon} & & \Big\downarrow \overleftarrow{\upsilon} \\
X & & X^*
\end{array}$$

equal the identity natural transformations on X and X^* respectively.

As demonstrated in Figure A.4, the rth component of the left-hand composite equals

$$\begin{array}{ccccc} X_r & \xrightarrow{\overleftarrow{\upsilon}^{-1}} & X_r \owedge e & \xrightarrow{\mathrm{id} \owedge \tau_{\underline{\eta}}} & X_r \owedge ((X^*)_{r^*} \ovee X_r) \\ & & & & \downarrow \vec{\kappa} \\ X_r & \xleftarrow[\vec{\upsilon}]{} & d \ovee X_r & \xleftarrow[\gamma_{\overline{\varepsilon}} \ovee \mathrm{id}]{} & (X_r \owedge (X^*)_{r^*}) \ovee X_r. \end{array}$$

[The Beck transform of the composite

$$r \owedge e \xrightarrow{\overleftarrow{\upsilon}} r \xrightarrow{\vec{\upsilon}^{-1}} d \ovee r$$

equals the canonical isomorphism $e \oslash r \longrightarrow r \multimap d (= r^*)$ —but in keeping with our declared intention to minimise pedantry, we have allowed $e \xrightarrow{\underline{\eta}} (e \oslash r) \ovee r$ to subsume the latter.]

An analogous reduction applies to the right-hand composite; thus the statement of the theorem is equivalent to the assertion that

$$e \xrightarrow{\tau_{\underline{\eta}}} (X^*)_{r^*} \ovee X_r \qquad X_r \owedge (X^*)_{r^*} \xrightarrow{\gamma_{\overline{\varepsilon}}} d$$

consitute a linear adjunction in $\mathcal{K}$.

This is essentially tautologous since $(X^*)_{r^*}$ is, by definition, $(X_{*(r^*)})^*$ which is isomorphic to $(X_r)^*$. Moreover, since the transposes of $e \xrightarrow{\eta} r^* \ovee r$ and $r \owedge r^* \xrightarrow{\varepsilon} d$ are the identities on r and r^* respectively, it follows (again by definition) that $\tau_{\underline{\eta}}$ and $\gamma_{\overline{\varepsilon}}$ equal the composites

$$e \xrightarrow{\underline{\eta}} (X_r)^* \ovee X_r \xrightarrow{\sim} (X^*)_{r^*} \ovee X_r$$

$$X_r \owedge (X^*)_{r^*} \xrightarrow{\sim} X_r \owedge (X_r)^* \xrightarrow{\overline{\varepsilon}} d$$

respectively. ∎

A symmetric argument shows that we have a linear adjunction overlying ${}^*X$ and X which demonstrates that ${}^*X$ is the left dual of X.

5.4. Corollary. If $\mathcal{J}$ and $\mathcal{K}$ are star-autonomous categories such that $\mathcal{K}$ has (co)limits of size $\mathcal{J}$, then $\mathcal{K}^{\mathcal{J}}$ is star-autonomous as well.

5.5. Example. There is a star-autonomous structure on $\mathbf{Sup}^{\rightarrow}$, with $\owedge$, E, $\ovee$ and D as in Example 4.5 and the dual of an object $(x_0 \xrightarrow{x_!} x_1)$ given by $({x_1}^* \xrightarrow{{x_!}^*} {x_0}^*)$.

5.6. Remark. One case of interest is when $\mathcal{J}$ and $\mathcal{K}$ happen to be *compact*—*i.e.*, when they have $d = e$ and $\otimes = \ltimes$. In this case, one might expect $\mathcal{K}^{\mathcal{J}}$ to be compact also; but this is not necessarily the case.

Unfortunately, the most natural example of this phenomenon occurs in the more general polycategorical setting referred to in the second half of Remark 4.4. We shall briefly sketch this idea before discussing how it can be modified into an example which does satisfy the hypotheses of this section.

5.7. Example. Let $\mathcal{G}$ denote the ordered group of integers regarded as a compact closed category in the usual way—*i.e.*, as a posetal category with $\otimes = + = \ltimes$, $e = 0 = d$, and $x^* = -x = {}^*x$; let $\mathcal{V}$ denote the usual compact closed category of finite-dimensional k-vector spaces and k-linear transformations. Then, although $\mathcal{V}$ does not have limits of size $\mathcal{G}$, we see that the units of polycategorical structure of $\mathcal{V}^{\mathcal{G}}$ are representable, and that they are different.

$$E_n = \begin{cases} k & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad D_n = \begin{cases} k & \text{if } n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

[In fact, it is possible to carve out an interesting full subpolycategory of $\mathcal{V}^{\mathcal{G}}$ which is entirely representable: let us say that an object X of $\mathcal{V}^{\mathcal{G}}$ is *stable* if all but finitely many of the transition maps $X_n \longrightarrow X_{n+1}$ are isomorphisms; then the full subcategory of stable objects of $\mathcal{V}^{\mathcal{G}}$ does indeed form a star-autonomous category which is not compact.]

Now if we could replace $\mathcal{V}$ by a (non-trivial) compact closed category with all countable limits, then the same argument would apply; embarrassingly, the author does not know whether such a category exists. Alternatively, we could replace $\mathcal{G}$ by any finite compact closed category containing an object r with a different number of arrows $i \longrightarrow r$ and $r \longrightarrow i$. Such categories are known to exist—the author is indebted to Robin Houston for providing us with an example.

6. Linear functors

We now reap the reward for having pursued the greater generality advocated in section 3.

We recall that a *linear functor* [4] between linear distributive categories consists of a monoidal functor and a comonoidal functor together with a set of four natural transformations called *linear strengths* and *linear costrengths* satisfying a number of coherence conditions. Our notation will be as follows: if T is a linear functor, then its monoidal part will be denoted $(\forall_T; \mu, \eta)$, its comonoidal part $(\exists_T; \delta, \varepsilon)$, and its linear strengths and costrengths as below.

$$\forall_T(x) \otimes \exists_T(y) \xrightarrow{\sigma^{(\ell)}} \exists_T(x \otimes y) \xleftarrow{\sigma^{(r)}} \exists_T(x) \otimes \forall_T(y)$$

$$\exists_T(x) \ltimes \forall_T(y) \xleftarrow{\vartheta^{(\ell)}} \forall_T(x \ltimes y) \xrightarrow{\vartheta^{(r)}} \forall_T(x) \ltimes \exists_T(y)$$

This use of quantifier symbols is intended as a mnemonic aid; for instance, $\sigma^{(\ell)}$ should be seen as corresponding to the tautology $(\forall t{:}\,T.x(t)) \wedge (\exists t{:}\,T.y(t)) \Rightarrow (\exists t{:}\,T.x(t) \wedge y(t))$.

6.1. Theorem. Suppose that $\mathcal{I}$ is an arbitrary linear distributive category, and that $\mathcal{J}$ and $\mathcal{K}$ satisfy the hypotheses of Theorem 4.2. Then there is a natural bijection between linear functors $\mathcal{I} \longrightarrow \mathcal{K}^{\mathcal{J}}$ and linear functors $\mathcal{I} \times \mathcal{J} \longrightarrow \mathcal{K}$.

6.2. Lemma. Suppose that $\mathcal{J}$ and $\mathcal{K}$ satisfy the hypotheses of Theorem 4.2. Then the linear distributions of $\mathcal{J}$, $\mathcal{K}$ and $\mathcal{K}^{\mathcal{J}}$ are related by the fact that the diagram

$$
\begin{array}{ccccc}
 & & ((X \otimes Y) \oplus Z)_{p\otimes(q\oplus r)} & & \\
 & \nearrow^{\vec{\kappa}_{(p\otimes q)\oplus r}} & & \searrow^{((X \otimes Y) \oplus Z)_{\vec{\kappa}}} & \\
(X \otimes (Y \oplus Z))_{p\otimes(q\oplus r)} & & & & ((X \otimes Y) \oplus Z)_{(p\otimes q)\oplus r} \\
\varpi_{\mathrm{id}}\uparrow & \searrow_{(X \otimes (Y \oplus Z))_{\vec{\kappa}}} & & \nearrow_{\vec{\kappa}_{(p\otimes q)\oplus r}} & \downarrow \pi_{\mathrm{id}} \\
X_p \otimes (Y \oplus Z)_{q\oplus r} & & (X \otimes (Y \oplus Z))_{(p\otimes q)\oplus r} & & (X \otimes Y)_{p\otimes q} \oplus Z_r \\
\mathrm{id} \otimes \pi_{\mathrm{id}}\downarrow & & & & \uparrow \varpi_{\mathrm{id}} \oplus \mathrm{id} \\
X_p \otimes (Y_q \oplus Z_r) & & \xrightarrow{\quad\vec{\kappa}\quad} & & (X_p \otimes Y_q) \oplus Z_r
\end{array}
$$

commutes.

Proof. The top diamond of the diagram above is a naturality square.

Note that by the definition contained in Lemma 2.4.2, the diagram

$$
\begin{array}{ccc}
(X \otimes (Y \oplus Z))_{p\otimes(q\oplus r)} & & \\
\varpi_{\mathrm{id}}\uparrow & \searrow^{(X \otimes (Y \oplus Z))_{\vec{\kappa}}} & \\
X_p \otimes (Y \oplus Z)_{q\oplus r} & \xrightarrow[\varpi_{\vec{\kappa}}]{} & (X \otimes (Y \oplus Z))_{(p\otimes q)\oplus r}
\end{array}
$$

commutes.

Now, by the definition contained in Theorem 4.2, we also have that

$$
\begin{array}{ccc}
(X \otimes (Y \oplus Z))_{(p\otimes q)\oplus r} & \xrightarrow{\vec{\kappa}_{(p\otimes q)\oplus r}} & ((X \otimes Y) \oplus Z)_{(p\otimes q)\oplus r} \\
\varpi_{\vec{\kappa}}\uparrow & & \downarrow \pi_{\mathrm{id}} \\
X_p \otimes (Y \oplus Z)_{q\oplus r} & & (X \otimes Y)_{p\otimes q} \oplus Z_r \\
\mathrm{id} \otimes \pi_{\underline{\eta}}\downarrow & & \uparrow \varpi_{\overline{\varepsilon}} \oplus \mathrm{id} \\
X_p \otimes (Y_{(q\oplus r)\oslash r} \oplus Z_r) & \overset{[\beta]}{\dashrightarrow} & (X_p \otimes Y_{p\multimap(p\otimes q)}) \oplus Z_r
\end{array}
$$

commutes, where β is the Beck transform of $p \otimes (q \oplus r) \xrightarrow{\vec{\kappa}} (p \otimes q) \oplus r$.

But in any bilinear category, the latter equals

$$(q \odot r) \obslash r \xrightarrow{\underline{\varepsilon}} q \xrightarrow{\overline{\eta}} p \multimap (p \otimes q)$$

and so $[\beta]$ may be factorised as follows:

$$\begin{array}{ccc}
X_p \otimes (Y_{(q\odot r)\obslash r} \odot Z_r) & \dashrightarrow & (X_p \otimes Y_{(q\odot r)\obslash r}) \odot Z_r \\
{\scriptstyle \mathrm{id}\otimes(Y_{\underline{\varepsilon}}\odot \mathrm{id})}\downarrow & & \vdots \\
X_p \otimes (Y_q \odot Z_r) & \xrightarrow{\vec{\kappa}} & (X_p \otimes Y_q) \odot Z_r \\
\vdots & & \downarrow{\scriptstyle (\mathrm{id}\otimes Y_{\overline{\eta}})\odot \mathrm{id}} \\
X_p \otimes (Y_{p\multimap(p\otimes q)} \odot Z_r) & \dashrightarrow & (X_p \otimes Y_{p\multimap(p\otimes q)}) \odot X_r.
\end{array}$$

Now regarding the triangle identities

$$\begin{array}{ccc}
p \otimes q & \xrightarrow{\mathrm{id}\otimes\overline{\eta}} & p \otimes (p \multimap (p \otimes q)) \\
& {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \overline{\varepsilon}} \\
& & p \otimes q
\end{array}
\qquad
\begin{array}{ccc}
q \odot r & \xrightarrow{\underline{\eta}} & ((q \odot r) \obslash r) \odot r \\
& {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \underline{\varepsilon}\odot \mathrm{id}} \\
& & q \odot r
\end{array}$$

as arrows in, respectively, $\otimes \downarrow (p \otimes q)$ and $(q \odot r) \downarrow \odot$, we conclude that

$$\begin{array}{ccc}
X_p \otimes Y_q & \xrightarrow{\mathrm{id}\otimes Y_{\overline{\eta}}} & X_p \otimes Y_{p\multimap(p\otimes q)} \\
& {\scriptstyle \varpi_{\mathrm{id}}}\searrow & \downarrow{\scriptstyle \varpi_{\overline{\varepsilon}}} \\
& & (X \otimes Y)_{p\otimes q}
\end{array}
\qquad
\begin{array}{ccc}
(Y \odot Z)_{q\odot r} & \xrightarrow{\pi_{\underline{\eta}}} & Y_{(q\odot r)\obslash r} \odot Z_r \\
& {\scriptstyle \pi_{\mathrm{id}}}\searrow & \downarrow{\scriptstyle Y_{\underline{\varepsilon}}\odot \mathrm{id}} \\
& & Y_q \odot Z_r
\end{array}$$

commute. ∎

PROOF OF THEOREM 6.1.

$(\Rightarrow)$. Given a linear functor $\mathcal{I} \xrightarrow{T} \mathcal{K}^{\mathcal{J}}$, we define functors $\mathcal{I} \times \mathcal{J} \xrightarrow{\forall_{\hat{T}},\, \exists_{\hat{T}}} \mathcal{K}$ in the obvious way:

$$\forall_{\hat{T}}(x,p) = [\forall_T(x)]_p \qquad \exists_{\hat{T}}(x,p) = [\exists_T(x)]_p\,.$$

[Note that we apply the notational convention of Remark 1.1.1 only to $\mathcal{J}$, since $\mathcal{I}$ is not assumed to be small.]

Now we know, from Theorem 2.2, that $E \xrightarrow{\eta} \forall_T(e)$ corresponds to a unique arrow $e \xrightarrow{\hat{\eta}} [\forall_T(e)]_e = \forall_{\hat{T}}(e,e)$ and that each component of μ, $\forall_T(x) \otimes \forall_T(y) \xrightarrow{\mu} \forall_T(x \otimes y)$ corresponds to a transformation (natural in p and q) of the form

$$\forall_{\hat{T}}(x,p) \otimes \forall_{\hat{T}}(y,q) = [\forall_T(x)]_p \otimes [\forall_T(y)]_q \xrightarrow{\hat{\mu}} [\forall_T(x \otimes y)]_{p\otimes q} = \forall_{\hat{T}}(x \otimes y, p \otimes q)$$

—the naturality of μ (in x and y) naturally entails the same for $\hat{\mu}$. We take it as read that $(\forall_{\hat{T}}; \hat{\mu}, \hat{\eta})$ form a monoidal functor $\mathcal{I} \times \mathcal{J} \longrightarrow \mathcal{K}$.

Similarly, $\sigma^{(\ell)}$ and $\sigma^{(r)}$ give rise to natural transformations

$$\forall_{\hat{T}}(x,p) \otimes \exists_{\hat{T}}(y,q) \xrightarrow{\hat{\sigma}^{(\ell)}} \exists_{\hat{T}}(x \otimes y, p \otimes q) \xleftarrow{\hat{\sigma}^{(r)}} \exists_{\hat{T}}(x,p) \otimes \forall_{\hat{T}}(y,q)$$

and, dually δ, ε, $\vartheta^{(\ell)}$, $\vartheta^{(r)}$ give rise to natural transformations $\hat{\delta}$, $\hat{\varepsilon}$, $\hat{\vartheta}^{(\ell)}$, $\hat{\vartheta}^{(r)}$ of the correct form. Thus, we have all the data necessary for a linear functor $\mathcal{I} \times \mathcal{J} \xrightarrow{\hat{T}} \mathcal{K}$; all that remains to show are the coherence conditions.

Again, we shall prove only one of these, as an illustration:

$$\begin{array}{ccc}
\forall_{\hat{T}}(x,p) \otimes \forall_{\hat{T}}(y \oplus z, q \oplus r) & \xrightarrow{\mathrm{id} \otimes \hat{\vartheta}^{(\ell)}} & \forall_{\hat{T}}(x,p) \otimes (\exists_{\hat{T}}(y,q) \oplus \forall_{\hat{T}}(z,r)) \\
\downarrow{\scriptstyle \hat{\mu}} & & \downarrow{\scriptstyle \vec{\kappa}} \\
\forall_{\hat{T}}(x \otimes (y \oplus z), p \otimes (q \oplus r)) & (\mathrm{LF}_{\hat{1}1}) & (\forall_{\hat{T}}(x,p) \otimes \exists_{\hat{T}}(y,q)) \oplus \forall_{\hat{T}}(z,r) \\
\downarrow{\scriptstyle \forall_{\hat{T}}(\vec{\kappa},\vec{\kappa})} & & \downarrow{\scriptstyle \sigma^{(\ell)} \oplus \mathrm{id}} \\
\forall_{\hat{T}}((x \otimes y) \oplus z, (p \otimes q) \oplus r) & \xrightarrow[\vartheta^{(\ell)}]{} & \exists_{\hat{T}}(x \otimes y, p \otimes q) \oplus \forall_{\hat{T}}(z,r)
\end{array}$$

—to show that this holds, we select the $p \otimes (q \oplus r)$ and $(p \otimes q) \oplus r$ components of the corresponding axiom for T:

$$\begin{array}{ccc}
\forall_T(x) \otimes \forall_T(y \oplus z) & \xrightarrow{\mathrm{id} \otimes \vartheta^{(\ell)}} & \forall_T(x) \otimes (\exists_T(y) \oplus \forall_T(z)) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \vec{\kappa}} \\
\forall_T(x \otimes (y \oplus z)) & (\mathrm{LF}_{11}) & (\forall_T(x) \otimes \exists_T(y)) \oplus \forall_T(z) \\
\downarrow{\scriptstyle \forall_T(\vec{\kappa})} & & \downarrow{\scriptstyle \sigma^{(\ell)} \oplus \mathrm{id}} \\
\forall_T((x \otimes y) \oplus z) & \xrightarrow[\vartheta^{(\ell)}]{} & \exists_T(x \otimes y) \oplus \forall_T(z)
\end{array}$$

which, together with the naturality squares for corresponding to the arrow $(p \otimes q) \oplus r \xrightarrow{\vec{\kappa}} p \otimes (q \oplus r)$, prove that the bottom rectangle, and therefore the whole of, Figure A.5 commutes.

$(\Leftarrow)$. We hold it self-evident that the construction of the data of $\hat{T}$ from those of T is reversible—*i.e.*, that the only non-trivial part of the converse has to do with the coherence conditions.

Suppose that $(\mathrm{LF}_{\hat{1}1})$ holds; to show that the rth component of (LF_{11}) holds, we must (yet again) consider arbitrary arrows $p \otimes q \xrightarrow{\lambda} r$ and $r \xrightarrow{\rho} s \oplus t$, and show that the composites

$$
\begin{array}{ccc}
\begin{array}{c}
[\forall_T(x)]_p \owedge [\forall_T(y \ovee z)]_q \\
\big\downarrow \varpi_\lambda \\
[\forall_T(x) \owedge \forall_T(y \ovee z)]_r \\
\big\downarrow \left[\vartheta^{(\ell)} \circ \forall_T(\vec{\kappa}) \circ \mu\right]_r \\
[\exists_T(x \owedge y) \ovee \forall_T(z)]_r \\
\big\downarrow \pi_\rho \\
[\exists_T(x \owedge y)]_s \ovee [\forall_T(z)]_t
\end{array}
& \qquad &
\begin{array}{c}
[\forall_T(x)]_p \owedge [\forall_T(y \ovee z)]_q \\
\big\downarrow \varpi_\lambda \\
[\forall_T(x) \owedge \forall_T(y \ovee z)]_r \\
\big\downarrow \left[(\sigma^{(\ell)} \ovee \mathrm{id}) \circ \vec{\kappa} \circ (\mathrm{id} \owedge \vartheta^{(\ell)})\right]_r \\
[\exists_T(x \owedge y) \ovee \forall_T(z)]_r \\
\big\downarrow \pi_\rho \\
[\exists_T(x \owedge y)]_s \ovee [\forall_T(z)]_t
\end{array}
\end{array}
$$

coincide. The procedure seen in the proofs of Theorems 4.3 and 5.3 must be followed once more: each of these composites can be rewritten (expanded) according to the definitions of the terms involved; to show that their expanded versions of these composites coincide then quickly follows from the hypothesis, and the commutativity of a certain diagram in $\mathcal{J}$. We refer to Figures A.6 and A.7 for some of details—the reader who has been alert up until this point should be able to fill in what remains. ∎

Considering the case $\mathcal{I} = 1$, and applying [6, Theorem 2.4], we obtain the following.

6.3. Corollary. Under the hypotheses of Theorem 4.2, there exists a natural bijection between linear functors $\mathcal{J} \longrightarrow \mathcal{K}$ and cyclic nuclear monoids in $\mathcal{K}^{\mathcal{J}}$.

We recall that a Frobenius monoid is defined in [6] to be a certain kind of cyclic nuclear monoid; it therefore seems to make sense to distinguish those linear functors which correspond to Frobenius monoids.

6.4. Definition. Let $\mathcal{J}, \mathcal{K}$ be arbitrary linear distributive categories; then a *Frobenius monoidal functor* $\mathcal{J} \longrightarrow \mathcal{K}$ is a linear functor such that $\forall_T = \exists_T$, $\sigma^{(\ell)} = \mu = \sigma^{(r)}$ and $\vartheta^{(\ell)} = \delta = \vartheta^{(r)}$.

Equivalently, it is a functor $\mathcal{J} \xrightarrow{F} \mathcal{K}$ equipped with both a monoidal structure (μ, η with respect to $\owedge, e$), and a comonoidal structure (δ, ε with respect to $\ovee, d$) satisfying the diagrams below.

$$
\begin{array}{ccccc}
F(x \owedge (y \ovee z)) & \xleftarrow{\mu_{x,y \ovee z}} & F(x) \owedge F(y \ovee z) & \xrightarrow{\mathrm{id} \owedge \delta_{y,z}} & F(x) \owedge (F(y) \ovee F(z)) \\
{\scriptstyle F(\vec{\kappa})}\big\downarrow & & & & \big\downarrow{\scriptstyle \vec{\kappa}} \\
F((x \owedge y) \ovee z) & \xrightarrow[\delta_{x \owedge y,z}]{} & F(x \owedge y) \ovee F(z) & \xleftarrow[\mu_{x,y} \ovee \mathrm{id}]{} & (F(x) \owedge F(y)) \ovee F(z)
\end{array}
$$

$$
\begin{array}{ccccc}
F((x \ovee y) \owedge z) & \xleftarrow{\mu_{x \ovee y,z}} & F(x \ovee y) \owedge F(z) & \xrightarrow{\delta_{x,y} \owedge \mathrm{id}} & (F(x) \ovee F(y)) \owedge F(z) \\
{\scriptstyle F(\overleftarrow{\kappa})}\big\downarrow & & & & \big\downarrow{\scriptstyle \overleftarrow{\kappa}} \\
F(x \ovee (y \owedge z)) & \xrightarrow[\delta_{x,y \owedge z}]{} & F(x) \ovee F(y \owedge z) & \xleftarrow[\mathrm{id} \ovee \mu_{y,z}]{} & F(x) \ovee (F(y) \owedge F(z))
\end{array}
$$

6.5. EXAMPLE. We have previously shown that a Frobenius monoid in $(\mathbf{Sup}; \otimes, 2, \oplus, 2^{\mathsf{op}})$ amounts to a quantale equipped with a dualising element [6, Theorem 4.1]. A Frobenius monoid in $(\mathbf{Sup}^{\to}; \otimes, E, \oplus, D)$ —equivalently, a Frobenius functor $(\to; \wedge, \top, \vee, \bot) \longrightarrow (\mathbf{Sup}; \otimes, 2, \oplus, 2^{\mathsf{op}})$—similarly amounts to a couple of quantales equipped with a dualising element in the sense of [7, Definition 1]. [Note that that definition requires a dualising element of a couple of quantales $C \xrightarrow{\phi} Q$ to be an element of C; therefore, it determines a unique square

$$\begin{array}{ccc} C & \longrightarrow & 2^{\mathsf{op}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle !} \\ Q & \underset{!}{\longrightarrow} & 1 \end{array}$$

in $\mathbf{Sup}$, which is to say, a unique arrow $(C \xrightarrow{\phi} Q) \longrightarrow D$ in $\mathbf{Sup}^{\to}$.]

Finally we note that the construction, given in [7, Theorem 15], of a single Frobenius quantale G from a Frobenius couple of quantales $C \xrightarrow{\phi} Q$ can be derived from the fact that the functor

$$\begin{array}{ccc} \mathbf{Sup}^{\to} & \longrightarrow & \mathbf{Sup} \\ (C \xrightarrow{\phi} Q) & \mapsto & \{(\alpha, \beta) \in C \times Q \mid \phi(\alpha) \leq \beta\} \end{array}$$

admits a Frobenius monoidal structure.

7. Future Work

A definition of *Girard monoid* (or, *symmetric Frobenius monoid*) which could be interpreted in any symmetric linear distributive category was given in [6]; it was further shown that a Girard monoid in $(\mathbf{Sup}; \otimes, 2, \oplus, 2^{\mathsf{op}})$ amounts to a quantale equipped with a *cyclic* dualising element. It is not hard to show that the star-autonomous structure we have constructed on $\mathbf{Sup}^{\to}$ is also symmetric, nor that a dualising element of a couple of quantales $C \xrightarrow{\phi} Q$ is cyclic if and only if the corresponding Frobenius monoid in $\mathbf{Sup}^{\to}$ is a Girard monoid; thus Kruml's notion of a Girard couple of quantales does fit into the more general framework.

But it has become clear to the author, as a result of joint research with M.B. McCurdy into *cyclic* star-autonomous categories [9, 2], that the notion of Girard monoid can, and should, be studied in even greater generality than that of symmetric linear distributive categories, and so we have decided to postpone a fuller discussion of Girard monoids (and the associated *Girard functors*) until all the pre-requisites for the more general notion are in place.

It is anticipated that the sequel to this paper will also address the observation, made in [6, Remark 3.3], that there exists a more general notion than that of cyclic nuclear monoid, in which we do not have $M^* \cong {}^*M$. A study of the latter is made more urgent by the linear distributive structure of $\mathcal{K}^{\mathcal{I}}$, which implies that there is also a more general concept of "morphism of linear distributive category" than that of linear functor.

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References

[1] Michael Barr. *∗-autonomous categories*, volume 752 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979. With an appendix by Po Hsiang Chu.

[2] Richard F. Blute, François Lamarche, and Paul Ruet. Entropic Hopf algebras and models of non-commutative logic. *Theory Appl. Categ.*, 10:No. 17, 424–460 (electronic), 2002.

[3] J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. *J. Pure Appl. Algebra*, 114(2):133–173, 1997.

[4] J. R. B. Cockett and R. A. G. Seely. Linearly distributive functors. *J. Pure Appl. Algebra*, 143(1-3):155–203, 1999. Special volume on the occasion of the 60th birthday of Professor Michael Barr (Montreal, QC, 1997).

[5] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, Lecture Notes in Mathematics, Vol. 137, pages 1–38. Springer, Berlin, 1970.

[6] J.M. Egger. The Frobenius relations meet linear distributivity. Submitted to Proceedings of CT'06 (White Point), volume 19 of Theory and Applications of Categories, 2007.

[7] J.M. Egger and David Kruml. Girard couples of quantales. *Appl. Categ. Structures*, 16, 2008.

[8] André Joyal and Myles Tierney. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.*, 51(309):vii+71, 1984.

[9] Kimmo I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Research Notes in Mathematics Series*. Longman, Essex, 1990.

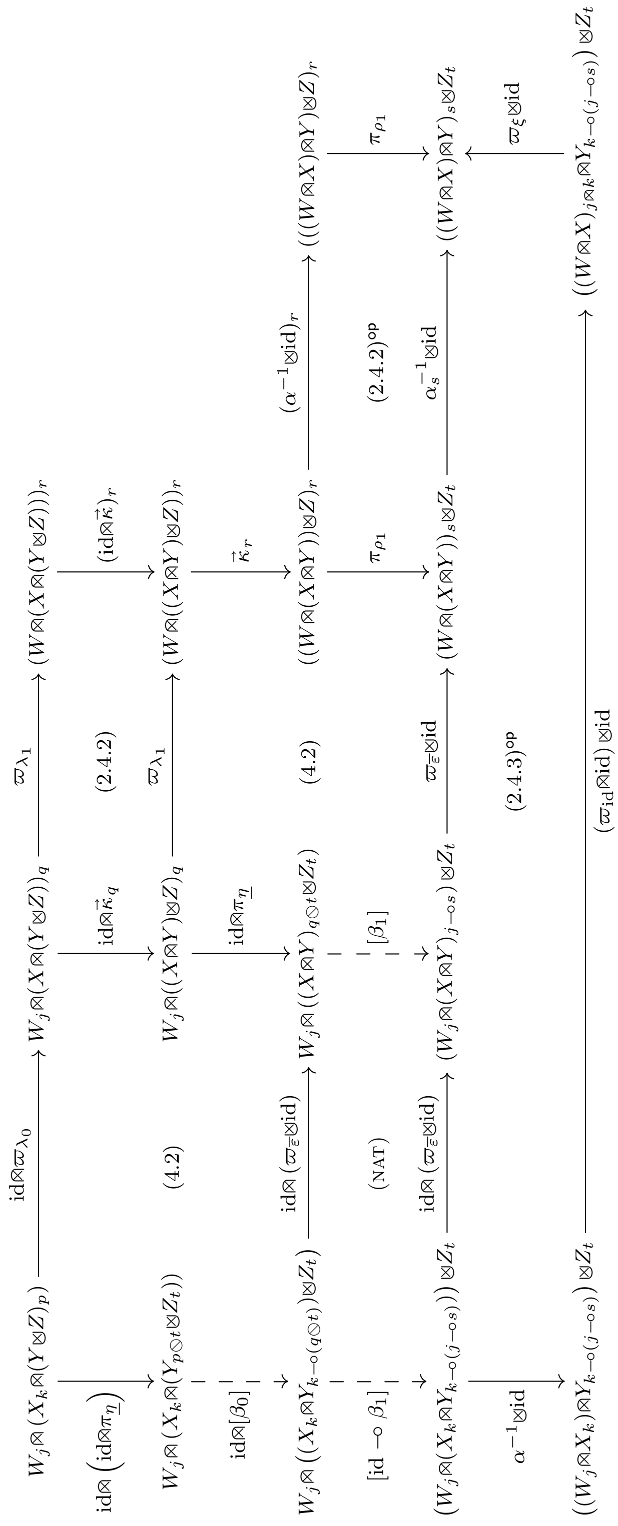
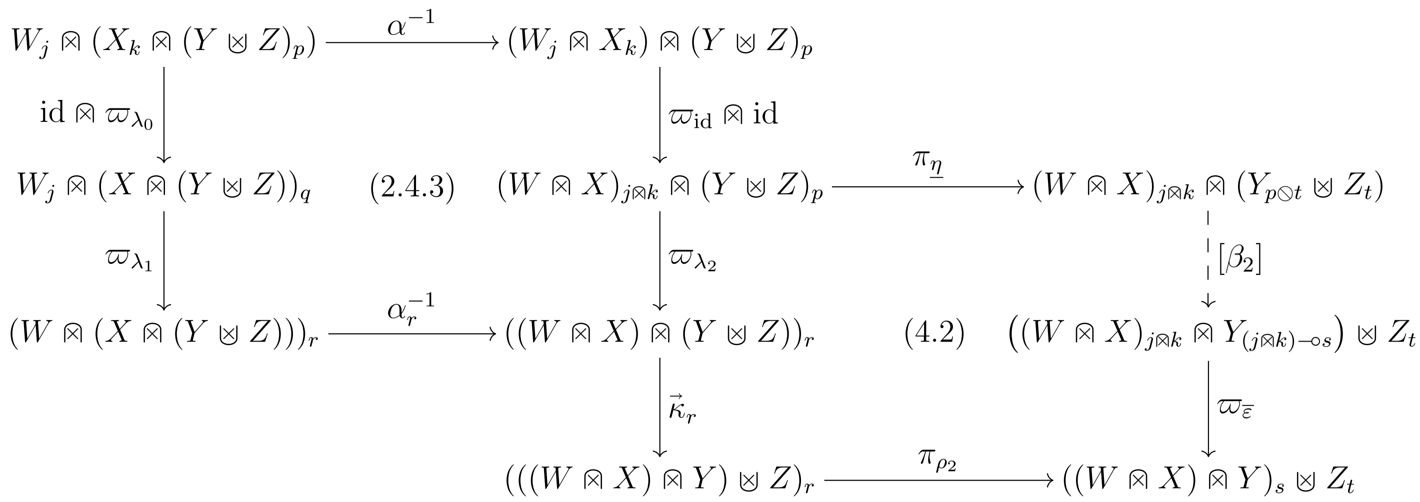
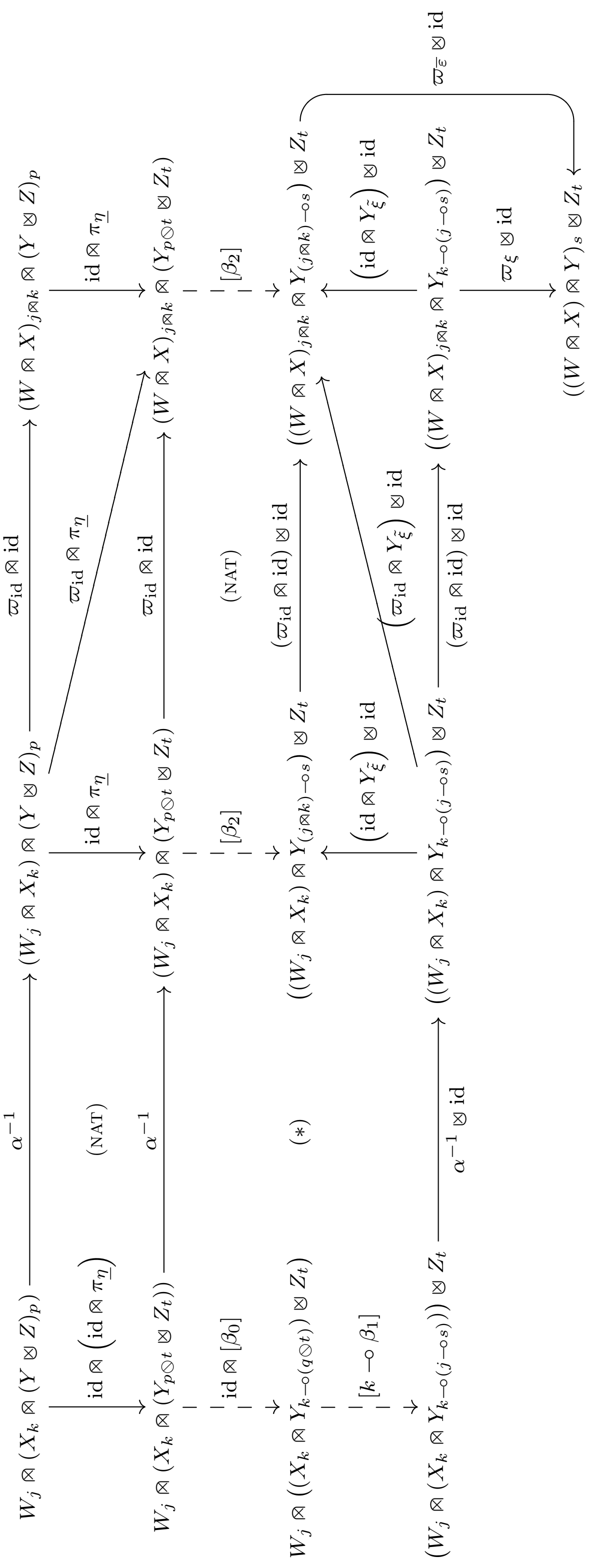


Figure A.1:



$W_j \boxtimes (X_k \boxtimes (Y \boxtimes Z)_p)$
α^{-1}
$(W_j \boxtimes X_k) \boxtimes (Y \boxtimes Z)_p$
$\mathrm{id} \boxtimes \varpi_{\lambda_0}$
$\varpi_{\mathrm{id}} \boxtimes \mathrm{id}$
$W_j \boxtimes (X \boxtimes (Y \boxtimes Z))_q$
(2.4.3)
$(W \boxtimes X)_{j \boxtimes k} \boxtimes (Y \boxtimes Z)_p$
$\pi_{\underline{\eta}}$
$(W \boxtimes X)_{j \boxtimes k} \boxtimes (Y_{p \oslash t} \boxtimes Z_t)$
ϖ_{λ_1}
ϖ_{λ_2}
$[\beta_2]$
$(W \boxtimes (X \boxtimes (Y \boxtimes Z)))_r$
α_r^{-1}
$((W \boxtimes X) \boxtimes (Y \boxtimes Z))_r$
(4.2)
$\big((W \boxtimes X)_{j \boxtimes k} \boxtimes Y_{(j \boxtimes k) \multimap s}\big) \boxtimes Z_t$
$\vec{\kappa}_r$
$\varpi_{\overline{\varepsilon}}$
$(((W \boxtimes X) \boxtimes Y) \boxtimes Z)_r$
π_{ρ_2}
$((W \boxtimes X) \boxtimes Y)_s \boxtimes Z_t$

Figure A.2:



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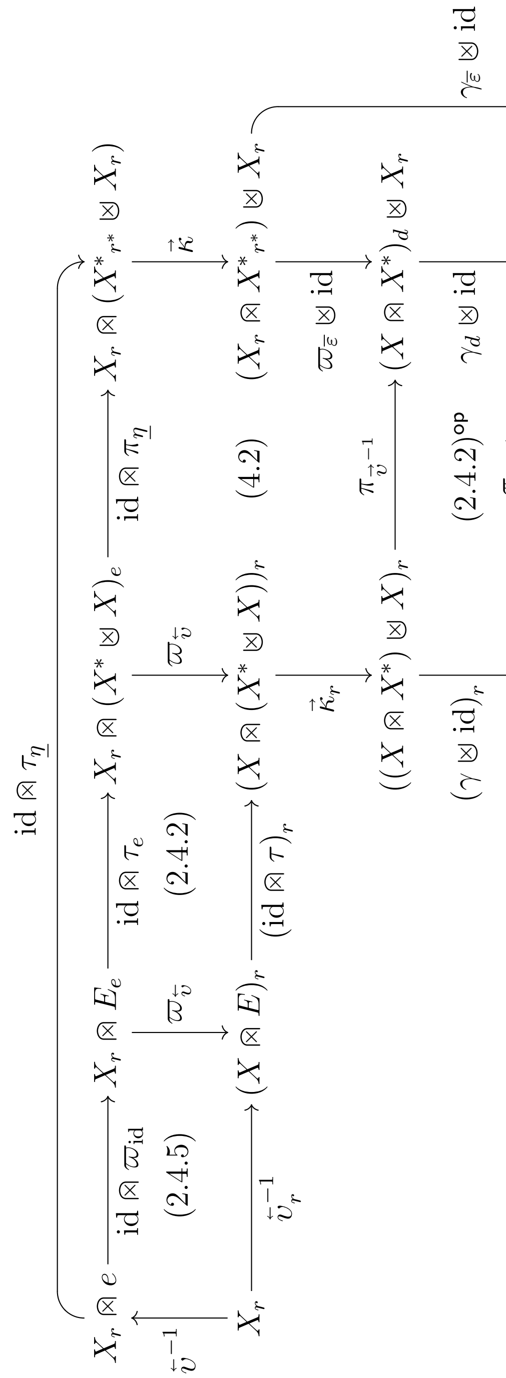

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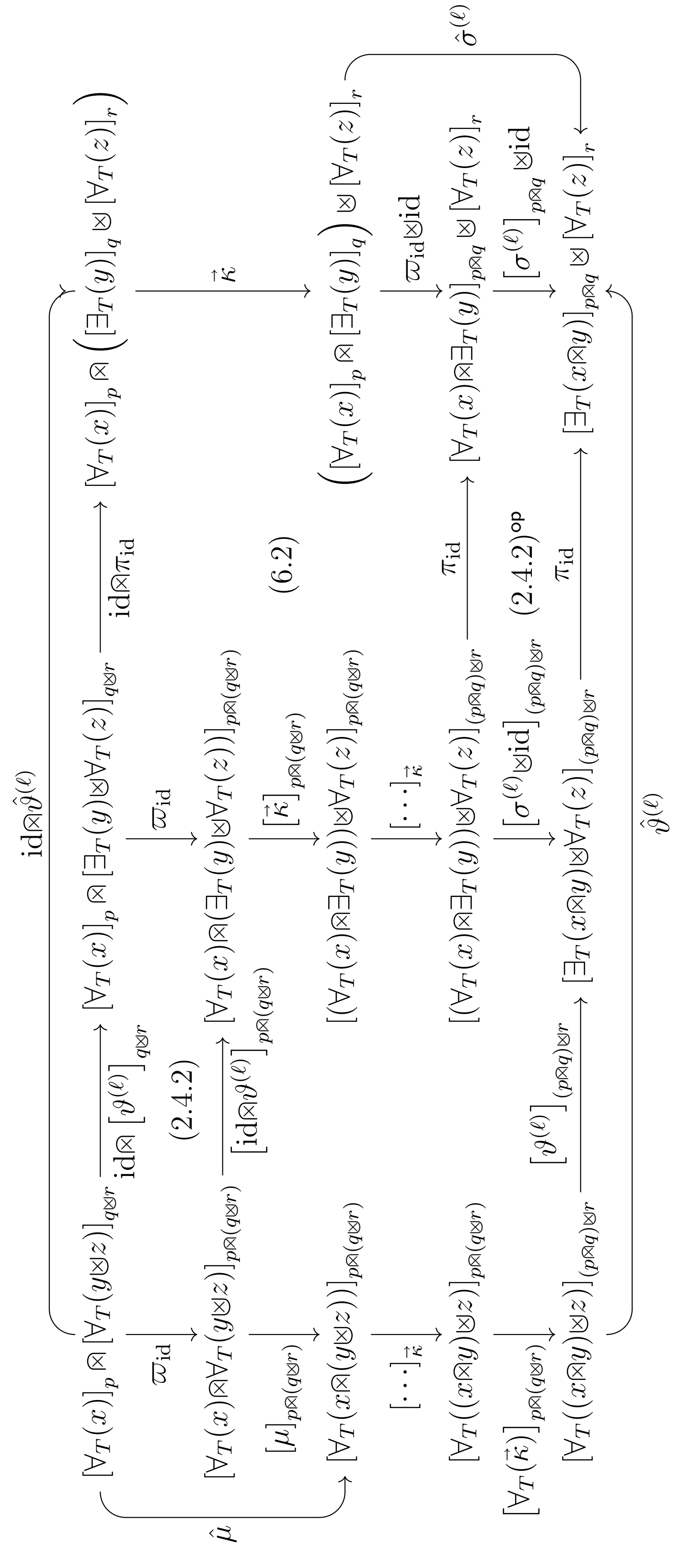

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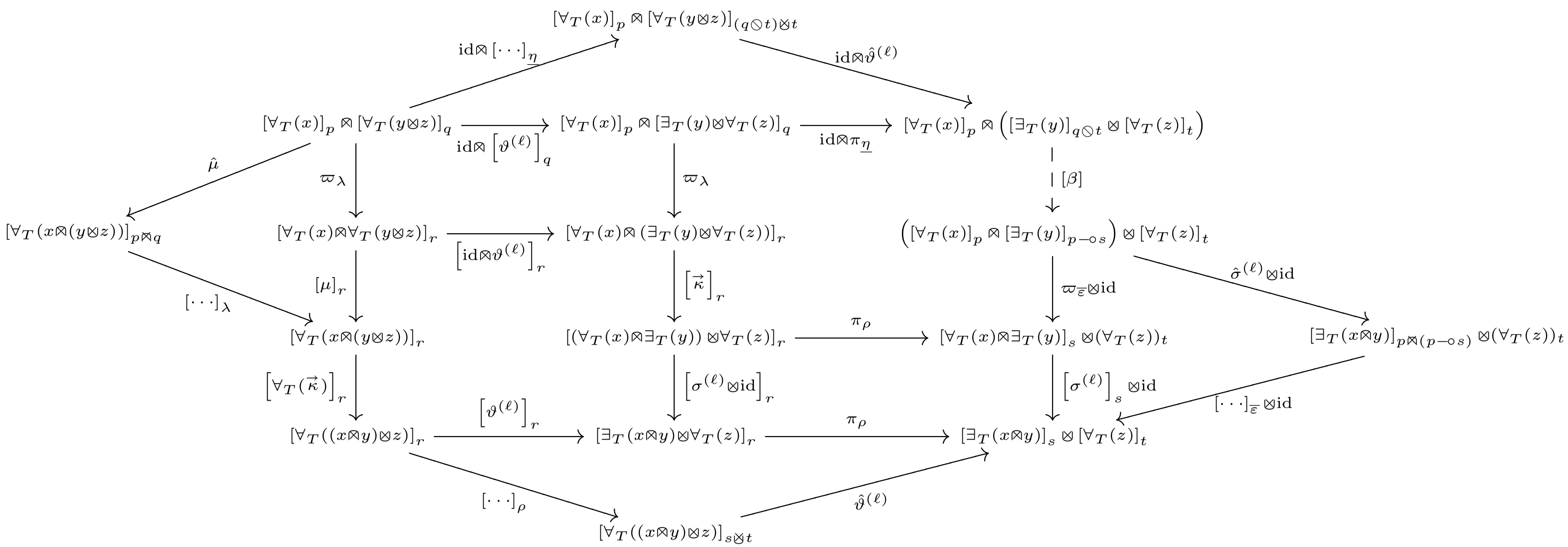
$[\forall_T(x)]_p \ltimes [\forall_T(y \rtimes z)]_{(q \oslash t) \rtimes t}$
$\mathrm{id} \ltimes [\cdots]_{\underline{\eta}}$
$\mathrm{id} \ltimes \hat{\vartheta}^{(\ell)}$
$[\forall_T(x)]_p \ltimes [\forall_T(y \rtimes z)]_q$
$\mathrm{id} \ltimes \left[\vartheta^{(\ell)}\right]_q$
$[\forall_T(x)]_p \ltimes [\exists_T(y) \rtimes \forall_T(z)]_q$
$\mathrm{id} \ltimes \pi_{\underline{\eta}}$
$[\forall_T(x)]_p \ltimes \left([\exists_T(y)]_{q \oslash t} \rtimes [\forall_T(z)]_t\right)$
$\hat{\mu}$
ϖ_λ
$[\beta]$
$[\forall_T(x \ltimes (y \rtimes z))]_{p \ltimes q}$
$[\forall_T(x) \ltimes \forall_T(y \rtimes z)]_r$
$\left[\mathrm{id} \ltimes \vartheta^{(\ell)}\right]_r$
$[\forall_T(x) \ltimes (\exists_T(y) \rtimes \forall_T(z))]_r$
$\left([\forall_T(x)]_p \ltimes [\exists_T(y)]_{p \multimap s}\right) \rtimes [\forall_T(z)]_t$
$[\cdots]_\lambda$
$[\mu]_r$
$\left[\vec{\kappa}\right]_r$
$\varpi_{\overline{\varepsilon}} \rtimes \mathrm{id}$
$\hat{\sigma}^{(\ell)} \rtimes \mathrm{id}$
$[\forall_T(x \ltimes (y \rtimes z))]_r$
$[(\forall_T(x) \ltimes \exists_T(y)) \rtimes \forall_T(z)]_r$
π_ρ
$[\forall_T(x) \ltimes \exists_T(y)]_s \rtimes (\forall_T(z))_t$
$[\exists_T(x \ltimes y)]_{p \ltimes (p \multimap s)} \rtimes (\forall_T(z))_t$
$\left[\forall_T(\vec{\kappa})\right]_r$
$\left[\sigma^{(\ell)} \rtimes \mathrm{id}\right]_r$
$\left[\sigma^{(\ell)}\right]_s \rtimes \mathrm{id}$
$[\cdots]_{\overline{\varepsilon}} \rtimes \mathrm{id}$
$\left[\vartheta^{(\ell)}\right]_r$
$[\forall_T((x \ltimes y) \rtimes z)]_r$
$[\exists_T(x \ltimes y) \rtimes \forall_T(z)]_r$
$[\exists_T(x \ltimes y)]_s \rtimes [\forall_T(z)]_t$
$[\cdots]_\rho$
$\hat{\vartheta}^{(\ell)}$
$[\forall_T((x \ltimes y) \rtimes z)]_{s \rtimes t}$

Figure A.6:

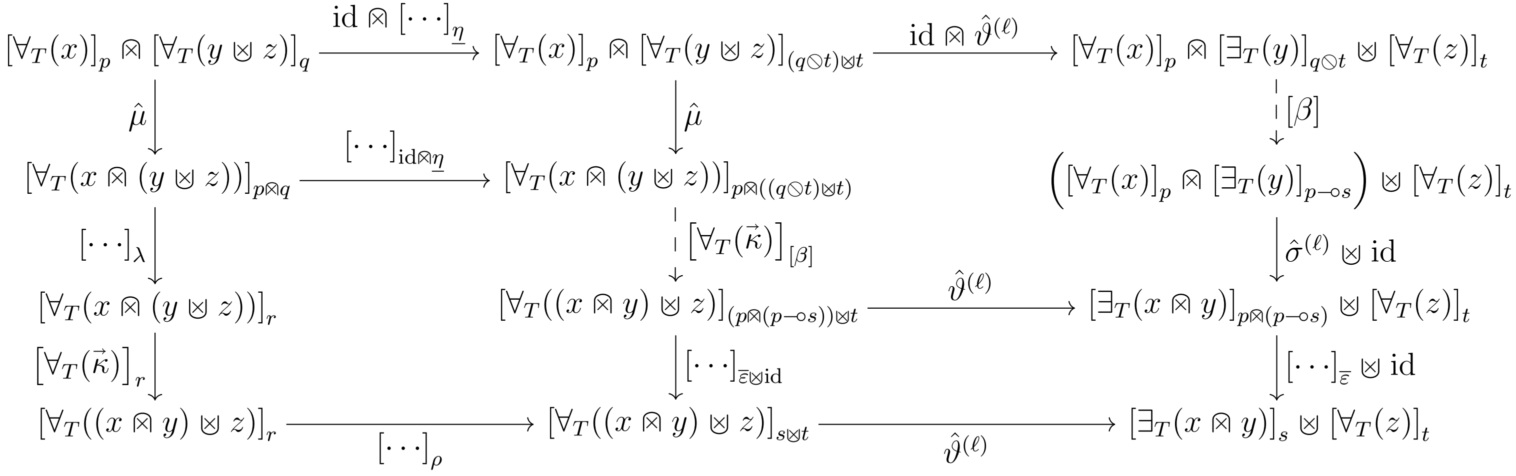
$[\forall_T(x)]_p \otimes [\forall_T(y \boxtimes z)]_q$
$\mathrm{id} \otimes [\cdots]_{\underline{\eta}}$
$[\forall_T(x)]_p \otimes [\forall_T(y \boxtimes z)]_{(q \oslash t) \boxtimes t}$
$\mathrm{id} \otimes \hat{\vartheta}^{(\ell)}$
$[\forall_T(x)]_p \otimes [\exists_T(y)]_{q \oslash t} \boxtimes [\forall_T(z)]_t$
$\hat{\mu}$
$\hat{\mu}$
$[\beta]$
$[\forall_T(x \otimes (y \boxtimes z))]_{p \otimes q}$
$[\cdots]_{\mathrm{id} \otimes \underline{\eta}}$
$[\forall_T(x \otimes (y \boxtimes z))]_{p \otimes ((q \oslash t) \boxtimes t)}$
$\left([\forall_T(x)]_p \otimes [\exists_T(y)]_{p \multimap s}\right) \boxtimes [\forall_T(z)]_t$
$[\cdots]_\lambda$
$[\forall_T(\vec{\kappa})]_{[\beta]}$
$\hat{\sigma}^{(\ell)} \boxtimes \mathrm{id}$
$[\forall_T(x \otimes (y \boxtimes z))]_r$
$[\forall_T((x \otimes y) \boxtimes z)]_{(p \otimes (p \multimap s)) \boxtimes t}$
$\hat{\vartheta}^{(\ell)}$
$[\exists_T(x \otimes y)]_{p \otimes (p \multimap s)} \boxtimes [\forall_T(z)]_t$
$[\forall_T(\vec{\kappa})]_r$
$[\cdots]_{\overline{\varepsilon} \boxtimes \mathrm{id}}$
$[\cdots]_{\overline{\varepsilon}} \boxtimes \mathrm{id}$
$[\forall_T((x \otimes y) \boxtimes z)]_r$
$[\cdots]_\rho$
$[\forall_T((x \otimes y) \boxtimes z)]_{s \boxtimes t}$
$\hat{\vartheta}^{(\ell)}$
$[\exists_T(x \otimes y)]_s \boxtimes [\forall_T(z)]_t$

Figure A.7:

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