$*$-autonomous functor categories

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Fact 1 (J.M.E., CT'06)

Frobenius monoid in **Sup** (w.r.t. non-trivial l.d. structure) = quantale with (not necessarily cyclic) dualising element.

Question 1 (D.K., CT'06)

Does theory of Frobenius monoids apply to Girard quouples?

(a) Is there some l.d. category whose Frobenius monoids are quouples with a (not necessarily cyclic) dualising element?

(b) Is cyclicity equivalent to some axiom on Frobenius monoids?

Observation 1

$$\text{quouple} = [\text{monoidal functor } (\bullet \to \bullet) \longrightarrow \mathbf{Sup}].$$

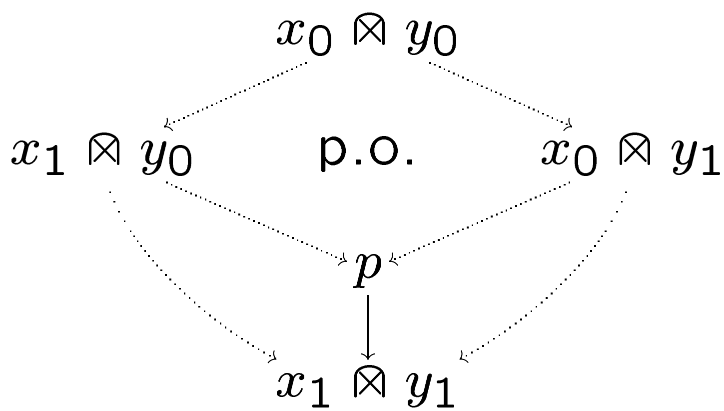
Observation 2 (folklore?)

$$[\text{monoidal functor } (\bullet \to \bullet) \longrightarrow \mathbf{Sup}] = \text{monoid in } \mathbf{Sup}^{(\bullet \to \bullet)},$$

when latter equipped with “convolution tensor product”.

Definition 1 ("Convolution tensor product")

$$\begin{array}{c} x_0 \\ \downarrow \\ x_1 \end{array} \boxtimes \begin{array}{c} y_0 \\ \downarrow \\ y_1 \end{array} :=$$



Arrows $\begin{array}{c} x_0 \\ \downarrow \\ x_1 \end{array} \boxtimes \begin{array}{c} y_0 \\ \downarrow \\ y_1 \end{array} \to \begin{array}{c} z_0 \\ \downarrow \\ z_1 \end{array}$ correspond bijectively to n.t.s

$$x_j \boxtimes y_k \longrightarrow z_{j \wedge k}.$$

Tensor unit is the unique sup-homomorphism $\mathbf{0} \to \mathbf{2}$.

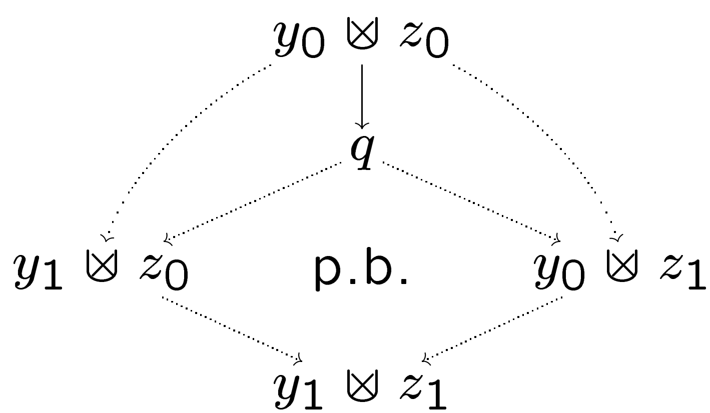
Observation 3

$(\mathbf{Sup}^{(\bullet\to\bullet)}, \otimes, 0 \to 2)$ is closed, and $2 \to 0$ is dualising. Duality is

$$\begin{array}{c} x_0 \\ \downarrow \\ x_1 \end{array} \mapsto \begin{array}{c} {x_1}^* \\ \downarrow \\ {x_0}^* \end{array} \qquad [\textit{i.e.}, \ (x^*)_j = (x_{\neg j})^*]$$

—therefore, dual tensor product is "co-convolution": arrows

$$\begin{array}{c} x_0 \\ \downarrow \\ x_1 \end{array} \to \begin{array}{c} y_0 \\ \downarrow \\ y_1 \end{array} \boxtimes \begin{array}{c} z_0 \\ \downarrow \\ z_1 \end{array} \cong$$



correspond bijectively to n.t.s $x_{j\vee k} \longrightarrow y_j \boxtimes z_k$.

Observation 4

$$
\begin{aligned}
&[\text{linear functors } (\bullet \to \bullet) \longrightarrow \mathbf{Sup}] \\
&\quad = \ [\text{linear functors } 1 \longrightarrow \mathbf{Sup}^{(\bullet\to\bullet)}] \\
&\quad = \ \text{“cyclic nuclear monoids” in } \mathbf{Sup}^{(\bullet\to\bullet)}
\end{aligned}
$$

Theorem 1

Frobenius monoids in $\mathbf{Sup}^{(\bullet\to\bullet)}$, w.r.t. $\cap\!\!\!\!\times$ and $\cup\!\!\!\!\times$ defined above, are indeed quouples equipped with a dualising element.

[Proof contained in appendix of . . .]

Observation 5

Cyclicity of dualising elements, for both quantales and quouples, is equivalent to the commutativity of

$$
\begin{array}{ccccc}
q \boxtimes q & \xrightarrow{\mu} & q & \xrightarrow{\varepsilon} & \mathbf{2} \\
{\scriptstyle \chi}\big\downarrow & & & & \big\| \\
q \boxtimes q & \xrightarrow{\mu} & q & \xrightarrow{\varepsilon} & \mathbf{2}.
\end{array}
$$

—a condition which, in the context of Frobenius algebras, is called *symmetry*.

[This condition is discussed further in section 5 of my CT'06 paper.]

Question 2

How far can Observations 3 and 4 be generalised?

Theorem 2

Suppose $\mathcal{J}$ and $\mathcal{K}$ are (not necessarily symmetric) $*$-autonomous categories, such that $\mathcal{K}$ has "(co)limits of size $\mathcal{J}$" (*e.g.* finite (co)limits, if $\mathcal{J}$ is finite). Then $\mathcal{K}^{\mathcal{J}}$ is also $*$-autonomous, w.r.t. the "natural operations":

$$
\begin{aligned}
(X \otimes Y)_r &= \operatorname*{colim}_{p \otimes q \to r} X_p \otimes Y_q, \\
(X \parr Y)_r &= \lim_{r \to s \parr t} X_s \parr Y_t, \\
(X^*)_r &= (X_{{}^*r})^*, \text{ and} \\
({}^*X)_r &= {}^*(X_{r^*}).
\end{aligned}
$$

Sketch of Proof, Part 1

It suffices to construct linear distributions

$$X \otimes (Y \oplus Z) \longrightarrow (X \otimes Y) \oplus Z$$
$$(X \oplus Y) \otimes Z \longrightarrow X \oplus (Y \otimes Z)$$

and linear adjoints

$$E \longrightarrow X^* \oplus X \qquad X \otimes X^* \longrightarrow D$$
$$E \longrightarrow X \oplus {}^*X \qquad {}^*X \otimes X \longrightarrow D$$

But to define a map

$$\begin{array}{ccc} [X \otimes (Y \oplus Z)]_r & \longrightarrow & [(X \otimes Y) \oplus Z]_r \\ \| & & \| \\ \underset{p \otimes q \to r}{\mathrm{colim}} \left[X_p \otimes \left(\underset{q \to s' \oplus t'}{\lim} Y_{s'} \oplus Z_{t'} \right) \right] & & \underset{r \to s \oplus t}{\lim} \left[\left(\underset{p' \otimes q' \to s}{\mathrm{colim}} X_{p'} \otimes Y_{q'} \right) \oplus Z_t \right] \end{array}$$

Sketch of Proof, Part 2

it suffices to define natural maps

$$X_p \otimes \left(\lim_{q \to s' \parr t'} Y_{s'} \parr Z_{t'} \right) \longrightarrow \left(\operatorname{colim}_{p' \otimes q' \to s} X_{p'} \otimes Y_{q'} \right) \parr Z_t$$

indexed by arrows $p \otimes q \longrightarrow r \longrightarrow s \parr t$. Given such maps, we can transpose their composite into a map $q \oslash t \longrightarrow p \multimap s$. [Here, $\oslash$ denotes the functor for which $(\ ) \oslash t$ is left adjoint to $(\ ) \parr t$.]

Therefore, we can consider the composites

$$\begin{array}{ccccc}
X_p \otimes \left(\lim_{q \to s' \parr t'} Y_{s'} \parr Z_{t'} \right) & & & & \left(\operatorname{colim}_{p' \otimes q' \to s} X_{p'} \otimes Y_{q'} \right) \parr Z_t \\
\downarrow & & & & \uparrow \\
X_p \otimes (Y_{q \oslash t} \parr Z_t) & \longrightarrow & (X_p \otimes Y_{q \oslash t}) \parr Z_t & \longrightarrow & (X_p \otimes Y_{p \multimap s}) \parr Z_t
\end{array}$$

and these have the desired properties.

Sketch of Proof, Part 3

Similarly, $E \longrightarrow X^* \boxtimes X$ is essentially defined by the process

$$\frac{\dfrac{\dfrac{e \longrightarrow s \boxtimes t}{{}^*s \longrightarrow t}}{X_{{}^*s} \longrightarrow X_t}}{e \longrightarrow (X_{{}^*s})^* \boxtimes X_t.}$$

which allows us to construct a map

$$e \longrightarrow (X^* \boxtimes X)_e = \lim_{e \to s \boxtimes t} (X^*)_s \boxtimes X_t$$

which, in turn, defines a n.t. $E \longrightarrow X^* \boxtimes X$.

Scholium

The construction of linear distributions for $\mathcal{K}^{\mathcal{J}}$ does not require the full $*$-autonomous structure of either $\mathcal{J}$ or $\mathcal{K}$. It suffices that

(a) $\mathcal{J}$ and $\mathcal{K}$ are l.d.,

(b) $\mathcal{K}$ has distributive limits and colimits of size $\mathcal{J}$, and

(c) $\mathcal{J}$ is bilinear—*i.e.*, $\otimes$ is closed and $\oplus$ is coclosed.

Theorem 3

Under the same hypotheses,

$$\begin{aligned}[\text{linear functors } \mathcal{J} \longrightarrow \mathcal{K}] &= [\text{linear functors } 1 \longrightarrow \mathcal{K}^{\mathcal{J}}] \\ &= \text{“cyclic nuclear monoids” in } \mathcal{K}^{\mathcal{J}}\end{aligned}$$

Pre-prints:

J.M. Egger, *The Frobenius relations meet linear distributivity*.

J.M. Egger, $*$*-autonomous functor categories*.

available at http://www.mscs.dal.ca/~jegger/

Coming soon:

J.M. Egger and D. Kruml, *On the linear distributive structure of* $\mathbf{Sup}$.

J.M. Egger and D. Kruml, *Girard couples in quantale theory*.