Nova Scotia
Math League

2010–2011

Game Two

SOLUTIONS

Team Question Solutions

1. By inspection (or otherwise) we see that $y = 2x + 3$ and $y = 4x + 5$ intersect at $(-1, 1)$. For $y = x + b$ to pass through this point we require $b = 2$.

2. There are 2^5 subsets of $\{1, 2, 3, 4, 5\}$ in total. Such a subset S does not contain 1 or 5 if and only if it is $S \subseteq \{2, 3, 4\}$. Since there are 2^3 such sets S, the desired answer is $2^5 - 2^3 = 24$.

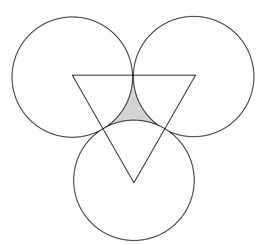
3. For simplicity, say the tank has a capacity of 1 litre, and let t be the time (in hours) that it takes for the tank to become half full. In that time, $t/3$ litres flow in to the tank and $t/5$ litres flow out. So we solve $\frac{t}{3} - \frac{t}{5} = \frac{1}{2}$ to get $t = \frac{15}{4}$.

4. Notice that $x = a$ is a root of $f(1 + 2x) = 0$ if and only if $x = 1 + 2a$ is a root of $f(x) = 0$. Since $f(1 + 2x) = 3x + 4x^2 = x(3 + 4x)$, the roots of $f(1 + 2x) = 0$ are $x = 0$ and $x = -\frac{3}{4}$. Hence the roots of $f(x) = 0$ are $x = 1 + 2 \cdot 0 = 1$ and $x = 1 + 2(-\frac{3}{4}) = -\frac{1}{2}$. Their sum is $\frac{1}{2}$.

   Note: This idea works for any polynomial f, not just quadratics. Suppose $f(1 + 2x) = 0$ has roots $a_1, \ldots, a_k$ that sum to S. Then $f(x) = 0$ would have roots $1 + 2a_1, \ldots, 1 + 2a_k$, and these sum to $k + 2S$. And remember that S itself is easily determined from the coefficients of $f(1 + 2x)$. For instance, if $f(1 + 2x) = 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5$, then the sum of the roots of $f(x) = 0$ is $-\frac{6}{7}$, and the sum of the roots of $f(x) = 0$ is $5 + 2(-\frac{6}{7})$.

5. There are $\binom{9}{2} = \frac{1}{2} \cdot 9 \cdot 8 = 36$ ways of choosing 2 balls from 9. Of these, there are 8 consecutive pairs, namely $\{1, 2\}, \{2, 3\}, \ldots, \{8, 9\}$. Hence the probability of obtaining a consecutive pair is $\frac{8}{36} = \frac{2}{9}$, meaning the chance of not doing so is $\frac{7}{9}$.

6. Draw the triangle with vertices at the centres of the circles, as shown in the diagram below. Then the shaded area is simply the area of this triangle less the area of the three circular sectors inside it. The triangle is equilateral side length 4 and hence has area $(\frac{4}{2})^2\sqrt{3} = 4\sqrt{3}$. The circular sectors are each $\frac{1}{6}$ of a circle with radius 2, so together they comprise half of such a circle, with total area $\frac{1}{2}\pi(2)^2 = 2\pi$. Thus the shaded area is $4\sqrt{3} - 2\pi$.



7. The first 4 terms contribute $i + 2i^2 + 3i^3 + 4i^4 = i - 2 + 3i + 4 = 2 - 2i$ to the sum. Note that the next 4 terms give precisely the same thing, since

$$5i^5 + 6i^6 + 7i^7 + 8i^8 = 5i - 6 - 7i + 8 = 2 - 2i.$$

   This is no coincidence. This pattern repeats, with each consecutive block of 4 terms summing to $2 - 2i$. There are 250 such blocks in our sum, along with the extraneous term $1001i^{1001} = 1001i$. Thus the desired sum is $250(2 - 2i) + 1001i = 500 + 501i$.

Note: The pattern above occurs because we have

$$i^n = \begin{cases} 1 & \text{if } n = 4k \text{ for some } k \\ i & \text{if } n = 4k+1 \text{ for some } k \\ -1 & \text{if } n = 4k+2 \text{ for some } k \\ -i & \text{if } n = 4k+3 \text{ for some } k. \end{cases}$$

Thus the block $(4k+1)i^{4k+1} + (4k+2)i^{4k+2} + (4k+3)i^{4k+3} + (4k+4)i^{4k+4}$ evaluates to $(4k+1)i - (4k+2) - (4k+3)i + (4k+4) = 2 - 2i$.

8. Let $|AD| = |AE| = |AF| = 1$. Since $AECF$ is a parallelogram, we have $|FC| = |AE| = 1$. And Pythagorean theorem on right triangle $\triangle DAF$ gives $|DF| = \sqrt{2}$, so $|DC| = 1 + \sqrt{2}$.

   Note that parallelograms $ABCD$ and $AECF$ both have the same height relative to the "base" line DC. Thus their areas will be in the same proportion as their bases, this being

$$\frac{|FC|}{|DC|} = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1.$$

   Since the area of $ABCD$ is 4, the area of $AECF$ is $4(\sqrt{2} - 1)$.

9. Consider instead the string 0000 0001 0002 0003 0004 $\cdots$ 2010 2011, consisting of 2012 blocks of 4 digits each. Clearly the sum of the digits in this string is the same as that in the number we are given.

   Now pair each block $0X_1X_2X_3$ with the block $1Y_1Y_2Y_3$, where $Y_i = 9 - X_i$. For instance, 0437 is paired with 1562. Note that the sum of all digits amongst any pair is $1 + 9 + 9 + 9 = 28$.

   Our string can be decomposed into 1000 such pairs, ranging from $(0000, 1999)$ to $(0999, 1000)$, along with the remainder $20002001 \cdots 2011$. Thus the sum of all digits in the string is $1000 \cdot 28 + 2 \cdot 12 + (1 + 2 + \cdots + 9 + 1 + 1 + 1) = 28072$.

10. From the given formula for g we deduce that $g(\sqrt{x+y}) = g(\sqrt{x})g(\sqrt{y})$ for all $x, y > 0$. Then

$$\begin{aligned} g(3) = g(\sqrt{9}) = g(\sqrt{8+1}) &= g(\sqrt{8})g(1) \\ &= g(\sqrt{4+4})g(1) \qquad = g(\sqrt{4})g(\sqrt{4})g(1) = g(2)g(2)g(1). \end{aligned}$$

   Since $g(2) = 16$ and $g(1) = 2$ we get $g(3) = 512$.

   Note: In fact, it isn't hard to prove by induction that $g(x)g(y) = g(\sqrt{x^2 + y^2})$ implies

$$g(x_1)g(x_2)\cdots g(x_k) = g(\sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}).$$

   Setting all $x_i = 1$ gives $g(1)^k = g(\sqrt{k})$. Hence $g(n) = g(1)^{n^2}$ for all n.

Pairs Relay Solutions

A. In year n, John makes $10(n+1)$ thousand dollars, while Andrew makes 2^n thousand dollars. A quick calculation shows that $n = 7$ is the first value of n for which 2^n is larger than $10(n+1)$. Hence $\mathtt{A} = 7$.

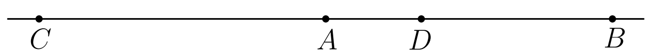
B. Three given lengths can form the sides of a triangle precisely when the sum of any two lengths exceeds the third. The straws of length 2,3, and 6 alone therefore don't form a triangle, since $2+3 \not> 6$.

But adding a straw of length $\mathtt{A} = 7$ into the mix allows for triangles with side lengths $(2,6,7)$ and $(3,6,7)$. Note that $(2,3,7)$ is disallowed. Hence there are only two possible triangles, and $\mathtt{B} = 2$.

C. The number of pairs of positive integers (n, m) such that $nm = 50\mathtt{B}$ is equal to the number of divisors of $50\mathtt{B}$, since if we let n be any divisor then we can set $m = 50\mathtt{B}/n$ and achieve $nm = 50\mathtt{B}$.

Since $\mathtt{B} = 2$ we are looking for divisors of 100. There are nine of these, namely 1, 2, 4, 5, 10, 20, 25, 50, and 100. Hence $\mathtt{C} = 9$.

D. The situation is possible if, for instance, Doug is "between" Alice and Betty, while Doug is on the opposite side of Alice from Betty, as depicted below:



Here we have $|DB| = 2|AD|$ and $|DB| + |AD| = \mathtt{C}$, so that $|AD| = \frac{1}{3}\mathtt{C}$. We also have $|CB| = 2|CA|$ and $|CB| = |CA| + \mathtt{C}$, so that $|CA| = \mathtt{C}$.

The distance between Doug and Charlie is then $\mathtt{D} = |CA| + |AD| = \mathtt{C} + \frac{1}{3}\mathtt{C} = \frac{4}{3}\mathtt{C}$. With $\mathtt{C} = 9$ this yields $\mathtt{D} = 12$.

Individual Relay Solutions

A. When expanding $(1+7x-10x^2-7x^3)(1-7x-10x^2-7x^3)$, the final x^4 contribution will be as $(-10x^2)(-10x^2)+(7x)(-7x^3)+(-7x^3)(7x)=2x^4$. So $\mathtt{A}=2$.

B. By definition of $\circ$ we have

$$1\circ \mathtt{A}=\mathtt{A}\circ 1=\frac{1}{1^{-1}+\mathtt{A}^{-1}}=\frac{\mathtt{A}}{\mathtt{A}+1}.$$

Thus

$$\begin{aligned}\mathtt{B}=\frac{1}{(1\circ \mathtt{A})\circ(\mathtt{A}\circ 1)}&=\frac{1}{1\circ \mathtt{A}}+\frac{1}{\mathtt{A}\circ 1}\\&=\frac{\mathtt{A}+1}{\mathtt{A}}+\frac{\mathtt{A}+1}{\mathtt{A}}\\&=2+\frac{2}{\mathtt{A}}.\end{aligned}$$

Since $\mathtt{A}=2$, we have $\mathtt{B}=3$.

C. The line $y=\mathtt{B}x-\mathtt{C}$ will intersect the parabola $y=x^2-x$ exactly once if and only i the equation $\mathtt{B}x-\mathtt{C}=x^2-x$ has exactly one real solution x. Set $\mathtt{B}=3$ and rearrange to get $x^2-4x+\mathtt{C}=0$. The discriminant of this quadratic is $16-4\mathtt{C}$, and we need this to be 0 to obtain precisely one solution. Thus $\mathtt{C}=4$.

D. A number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. The given number "38C69D1" produces an alternating sum of $3-8+\mathtt{C}-6+9-\mathtt{D}+1=\mathtt{C}-\mathtt{D}-1$. Setting $\mathtt{C}=4$ yields a sum of $3-\mathtt{D}$, and the only way this can be divisible by 11 is if $\mathtt{D}=3$.