Nova Scotia

Math League

2011–2012

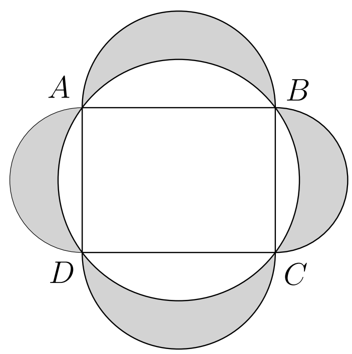
Game One

PROBLEMS

Team Questions

1. Which number is bigger: 2010^{2011} or 2011^{2010}?

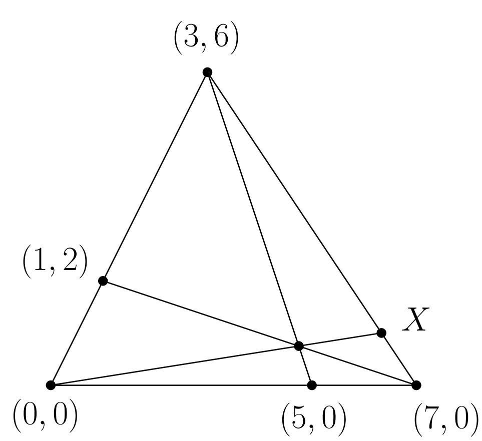
2. In the figure below, a semicircle is attached to each side of rectangle $ABCD$ and the circle through the corners of the rectangle is drawn. Given that $|AB| = 4$ and $|BC| = 3$, find the shaded area.



3. Find the number of real solutions to the equation

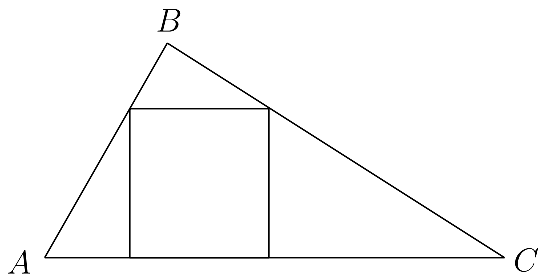
$$x^{x^2+2011x+2012} = 1.$$

4. Find the coordinates of the point X.



5. The cubic equation $2x^3 + 2x^2 - 3x - 1 = 0$ has roots α, β, and γ. Find a cubic equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$.

6. A square fits inside a triangle as depicted below. Suppose $\angle B = 90^\circ$ and $|AB| = 1$ and $|BC| = 2$. What is the area of the square?



7. The numbers $\{17, 29, 47, 59, 71, 89, 101, 113\}$ can be entered into the eight empty cells of the 3×3 grid below to form a "magic square", in which the sums of the entries along every row, column, and diagonal are the same.

	5	

What number goes in the centre cell?

8. Jenny and her brother Mike are lucky enough to meet Santa as he pops out of their chimney. It's Santa's last stop for the night, so he has only 8 gifts left in his sack, 4 for Jenny and 4 for Mike.

If Santa hands out the gifts one at a time, pulling them from his sack at random, what is the probability that Jenny will always have at least as many gifts as Mike throughout this process?

9. The lock on the door to the office supply room has 5 buttons labeled A through E. To open the door I must enter the correct pass-code, which I do not know because it has recently been changed.

   I do know that this type of lock allows either one or two buttons to be pressed at a time, and that all pass-codes involve pressing each button exactly once overall.

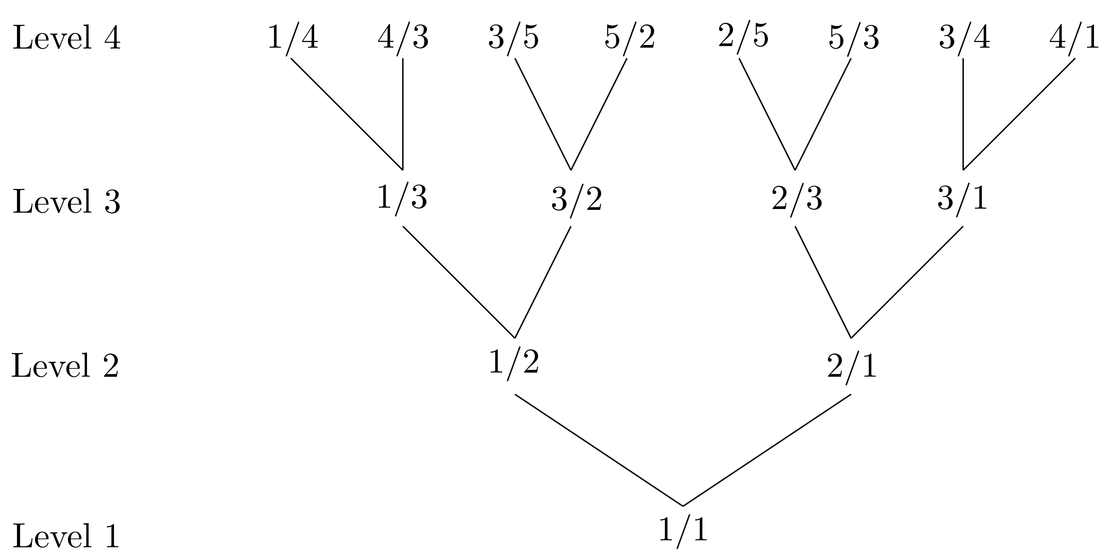
   For instance, either of A–C–B–E–D or AC–E–BD might be the correct combination, but I know A–CD–B isn't right (since E is unused) and neither is A–CD–AE–B (since A is used twice).

   How many possible pass-codes are there?

10. The *tree of fractions* is defined recursively as follows:

    - Level 1 contains only the fraction $1/1$.
    - Each fraction i/j at level n spawns two fractions at level $n+1$, namely $i/(i+j)$ and $(i+j)/j$, branching up to the left and right, respectively.

    The first four levels of the tree of fractions are depicted below:



    At what level in the tree do we find the fraction $\dfrac{1001}{2011}$?

Pairs Relay

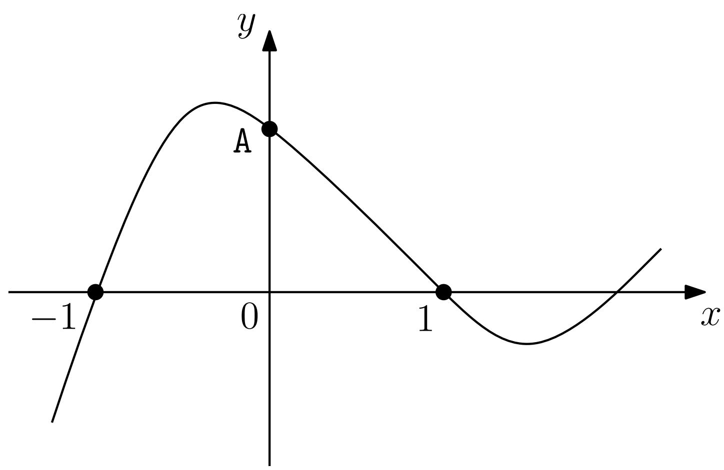
P-A. Suppose $\sqrt{10+\sqrt{84}} = \sqrt{x} + \sqrt{y}$ for some integers x and y.

Let $\mathtt{A} = |x-y|$.

$\boxed{\text{Pass on } \mathtt{A}}$

P-B. You will receive $\mathtt{A}$.

Part of the graph of $y = ax^3 + bx^2 + cx + d$ is shown below.



Let $\mathtt{B} = b$.

$\boxed{\text{Pass on } \mathtt{B}}$

P-C. You will receive $\mathtt{B}$.

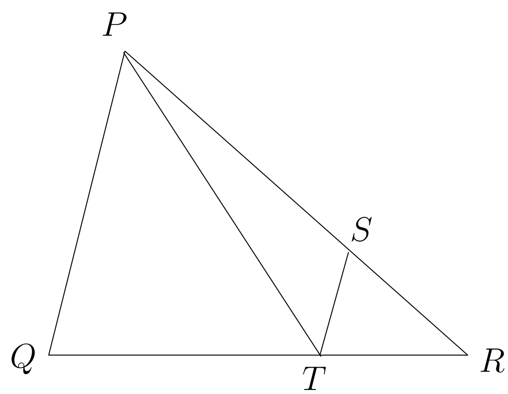
Lines $y = \mathtt{B}(x+n)$ and $x = \mathtt{B}(y+m)$ intersect at $(x, y) = (\mathtt{B}+1, \mathtt{B}-1)$.

Let $\mathtt{C} = m + n$.

$\boxed{\text{Pass on } \mathtt{C}}$

P-D. You will receive $\texttt{C}$.

In the figure below, ST is parallel to PQ, with $|PQ| = \texttt{C}$ and $|ST| = \dfrac{\texttt{C}}{3}$. Furthermore, segment PT bisects angle $\angle QTS$.



Let $\texttt{D} = |RT|$. $\boxed{\text{Done!}}$

Individual Relay

I-A. The polynomial $3x^3 - 9x^2 + \mathtt{A}x - 15$ has $x - 3$ as a factor.

$\boxed{\text{Pass on } \mathtt{A}}$

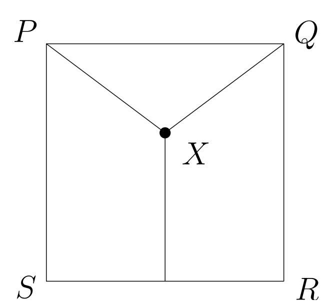
I-B. You will receive $\mathtt{A}$.

Let $\mathtt{B}$ be the units digit of $123^{\mathtt{A}} + \mathtt{A}^{123}$.

$\boxed{\text{Pass on } \mathtt{B}}$

I-C. You will receive $\mathtt{B}$.

Point X lies inside a square $PQRS$ of side length $\mathtt{B}$ so that it is equidistant from vertices P and Q and from side RS. Let $\mathtt{C}$ be this common distance.



$\boxed{\text{Pass on } \mathtt{C}}$

I-D. You will receive $\mathtt{C}$.

Integers p and q are to be selected from $\{1, 2, \ldots, \mathtt{C}\}$ so that the equation $x^2 + px + q = 0$ has real roots.

Let $\mathtt{D}$ be the number of possible choices for p and q. $\boxed{\text{Done!}}$