Nova Scotia

Math League

2011–2012

Game Three

Problems

Team Questions

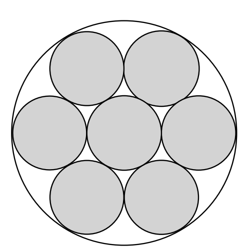
1. A man standing in line at the bank notices that 5/6 of the people in the line are ahead of him and 1/7 are behind him. How many people are in the line?

2. A ladder leans up against a vertical wall so that the top of the ladder is 12 feet above ground. If the bottom of the ladder is pulled 8 feet further away from the wall then the top of the ladder will rest at the foot of the wall. How many feet long is the ladder?

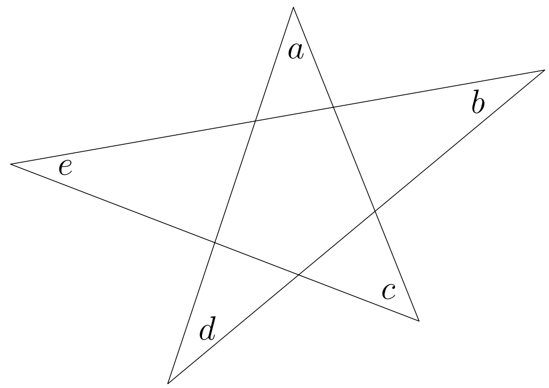
3. A positive integer is a **palindrome** if its digits read the same forward and backward. For example, 13531 is a palindrome.

   How many palindromes are there between 1 and 10000, inclusive?

4. The circles in the diagram below are tangent to one another. Find the ratio of the shaded area to the area of the large circle.



5. Let a, b, c, d, e be the angles indicated in the diagram below, measured in degrees. Find the sum $a + b + c + d + e$.



6. At sunrise, Alice and Bob begin hiking on a trail that is in the shape of a large loop. They start at the same point, but traverse the trail in opposite directions, each walking at constant speed.

   Alice and Bob pass by each other at 11am. Alice completes the loop at 3pm but Bob doesn't finish until 8pm.

   What time was sunrise on this day?

7. Let $a_1, a_2, \ldots, a_{2012}$ be an arithmetic sequence with common difference 2. Given that

$$a_1 + a_2 + a_3 + \cdots + a_{2012} = 10000,$$

find the value of

$$a_2 + a_4 + a_6 + \cdots + a_{2012}.$$

8. Three distinct numbers are selected at random from the set $\{1, 2, 3, \ldots, 10\}$. What is the probability that the product of these numbers is even?

9. For any set S of integers let $f(S)$ be the sum of all elements of S. For example, $f(\{1, 4, 5\}) = 10$ and $f(\emptyset) = 0$.

   Suppose we evaluate $f(A)$ for every subset A of the set $\{2, 3, 5, 7, 11\}$, and then add together all of the resulting values. What is the final result?

   Caution: Don't forget that $\{2, 3, 5, 7, 11\}$ is a subset of $\{2, 3, 5, 7, 11\}$.

10. A friend proposes the following guessing game: He chooses an integer between 1 and 100, inclusive, and you repeatedly try to guess his number. He tells you whether each incorrect guess is higher or lower than his chosen number, but you are allowed **at most one high guess** overall.

    You win the game when you guess his number correctly. You lose the game the instant you make a second high guess.

    What is the minimum number of guesses in which you can **guarantee** you will win the game?

Pairs Relay

P-A. Let $\texttt{A}$ be the number of distinct rearrangements of the letters of the word "TEST" such that the two T's are **not** adjacent.

$\boxed{\text{Pass on } \texttt{A}}$

P-B. You will receive $\texttt{A}$.

Let $\texttt{B}$ be the number of pairs (x, y) of **nonnegative** integers such that $3x + 4y = 6\texttt{A}$.

$\boxed{\text{Pass on } \texttt{B}}$

P-C. You will receive $\texttt{B}$.

Suppose positive numbers x, y and z satisfy

$$xy = 1$$
$$yz = 9$$
$$zx = \texttt{B}$$

Let $\texttt{C} = xyz$.

$\boxed{\text{Pass on } \texttt{C}}$

P-D. You will receive $\texttt{C}$.

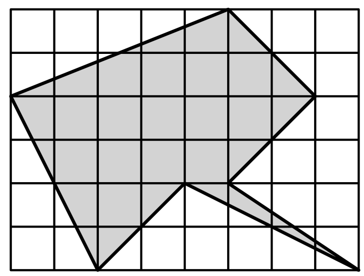
Let $\texttt{D}$ be the unique positive number such that

$$\frac{\texttt{D}}{1 + \frac{1}{\texttt{D}}} = \frac{1}{\texttt{C}(\texttt{C} - 1)}.$$

$\boxed{\text{Done!}}$

Individual Relay

I-A. Each small square in the diagram below has an area of one square unit. Let $\mathtt{A}$ be the area (in square units) of the shaded region.



$\boxed{\text{Pass on } \mathtt{A}}$

I-B. You will receive $\mathtt{A}$.

Let x be the average of the numbers $\{1, 7, 11, 19, \mathtt{A}\}$. Let $\mathtt{B}$ be the average of the numbers

$$\{1, 7, 11, 19, \mathtt{A}, x-1, x, x+1\}.$$

$\boxed{\text{Pass on } \mathtt{B}}$

I-C. You will receive $\mathtt{B}$.

Let $\mathtt{C}$ be the minimum possible value of $x^2 + 6x + \mathtt{B}$ (for $x \in \mathbb{R}$).

$\boxed{\text{Pass on } \mathtt{C}}$

I-D. You will receive $\mathtt{C}$.

Let

$$\mathtt{D} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\cdots\left(1 + \frac{1}{2\mathtt{C}}\right).$$

$\boxed{\text{Done!}}$