Nova Scotia

Math League

2012–2013

Game Three

PROBLEMS

Team Questions

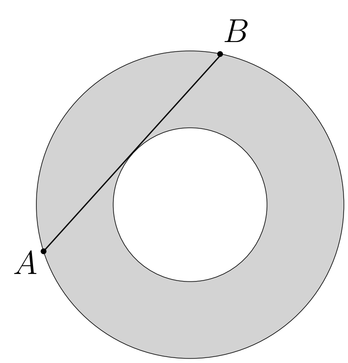
1. Define the sequence $\{a_n\}$ by $a_0 = 1$, $a_1 = 2$, and

$$a_n = 2a_{n-1} - a_{n-2} + 2 \quad \text{for } n \geq 2.$$

   Evaluate a_{100}.

   Hint: Generate some data and look for a pattern!

2. The two circles in the diagram below share a common centre, and chord AB of the larger circle is tangent to the smaller circle. Given that $|AB| = 10$, find the area of the shaded annular region.

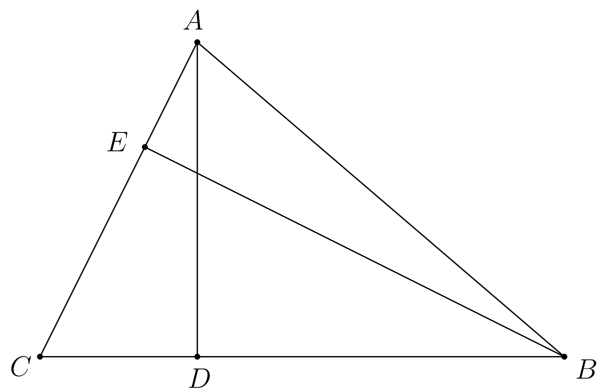


3. Jack and Zoë go skating on the Halifax oval. Both kids skate at constant speed and, when they skate in the same direction, Jack passes by Zoë at regular intervals. But when Zoë switches direction she finds that she and Jack pass by each other 9 times more frequently.

   Find the ratio of Jack's speed to Zoë's speed.

4. Find the minimum value of $\dfrac{3^{x^2+8}}{27^{2x+1}}$ over all real numbers x.

5. In $\triangle ABC$, altitudes AD and BE are perpendicular to sides BC and AB, respectively, with $|AE| = 2$, $|EC| = 4$, and $|CD| = 3$. Find $|DB|$.



6. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For instance, $\lfloor 3.4 \rfloor = 3$, $\lfloor 8.9 \rfloor = 8$ and $\lfloor 10 \rfloor = 10$.

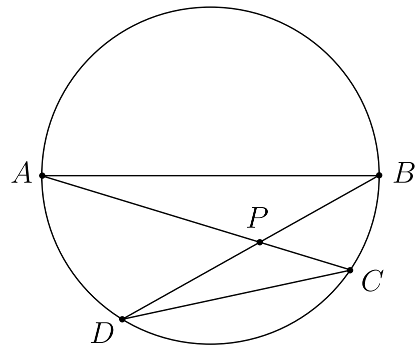
   Evaluate the following:

$$\left\lfloor \sqrt{1} \right\rfloor + \left\lfloor \sqrt{2} \right\rfloor + \left\lfloor \sqrt{3} \right\rfloor + \cdots + \left\lfloor \sqrt{100} \right\rfloor .$$

7. Three distinct vertices of a regular hexagon are chosen at random and they are joined to form a triangle.

   What is the probability that this triangle is right-angled?

8. Points A, B, C and D are chosen on the circumference of a circle such that AB is a diameter, and chords AC and BD intersect at a point P inside the circle. (See the diagram below.) Suppose $\angle APB = 150^\circ$. Find the ratio of the area of $\triangle APB$ to that of $\triangle CPD$

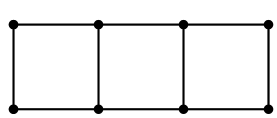


9. Let

$$S = 9 + 99 + 999 + 9999 + 99999 + \cdots + \overbrace{99999\cdots999}^{2013 \text{ nines}}.$$

Find the sum of the digits of S.

10. Each of the 10 line segments in the figure below is to be coloured either red or green so that all 3 squares have two red sides and two green sides apiece. In how many ways can this be done?



Pairs Relay

P-A. Let $\mathtt{A}$ be the number of **different** possible values of the expression

$$\frac{x}{y+1}+\frac{y}{z+1}+\frac{z}{x+1}$$

where each of x, y, and z is known to be either 0 or 1.

$\boxed{\text{Pass on } \mathtt{A}}$

P-B. A line with slope 2 and a line with slope 3 both pass through the point $(10, \mathtt{A})$. Let $\mathtt{B}$ be the distance between the x-intercepts of these lines.

$\boxed{\text{Pass on } \mathtt{B}}$

P-C. Let $k = 1 + \mathtt{B}^{-1}$. This should be an integer!

Let $\mathtt{C}$ be the number of positive divisors of 4^k.

Hint: Don't forget that 1 and n are both divisors of n. For instance, 6 has four positive divisors, namely 1, 2, 3, and 6.

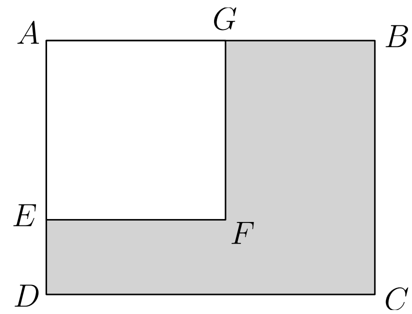
$\boxed{\text{Pass on } \mathtt{C}}$

P-D. An isosceles triangle has two equal sides of length 15 and an area of 108. Point P lies inside the triangle such that the distance from P to either of the two equal sides is 3, while its distance from the remaining side is $\mathtt{C}$.

Let $\mathtt{D}$ be length of the remaining side of the triangle. $\boxed{\text{Done!}}$

Individual Relay

I-A. In the diagram below, the square $AGFE$ has been removed from the rectangle $ABCD$ to leave the shaded region $BCDEFG$. The area of the shaded region is 230 square units, and $|DE| = 5$ and $|GB| = 10$.



Let $\mathtt{A}$ be the side length of the removed square.

Pass on $\mathtt{A}$

I-B. Let $\mathtt{B}$ be the largest integer such that $2^{\mathtt{B}}$ divides into $\mathtt{A}^{\mathtt{A}}$

Pass on $\mathtt{B}$

I-C. Let r and s be the solutions of $x^2 - \sqrt{\mathtt{B}}x + 2 = 0$ and let $\mathtt{C} = r^2 + s^2$.

Warning: Don't be alarmed if $\mathtt{B}$ is not a perfect square. It shouldn't be!

Pass on $\mathtt{C}$

I-D. A $2 \times 2 \times \mathtt{C}$ tunnel is bored through the middle of a $\mathtt{C} \times \mathtt{C} \times \mathtt{C}$ cube, from one face straight to the opposite face. Let $\mathtt{D}$ be the **increase** in surface area from the original cube to the final solid.

Done!