Nova Scotia

Math League

2013–2014

Game One

SOLUTIONS

Team Question Solutions

1. One can evaluate this expression directly, working from the inside to the outside. This is tedious but easily manageable, requiring 12 multiplications by 2. But it is better to look for structure. Note that expanding the sum with the distributive law yields

$$2^{11} + 2^{10} + 2^9 + \cdots + 2^3 + 2^2 + 2 + 1.$$

This is readily seen to equal $\frac{2^{12}-1}{2-1} = 4095$, as it is is a finite geometric series with common ratio 2. (Alternatively, its binary representation is $(11111111111)_2$, which is one less than $(1000000000000)_2 = 2^{12}$.)

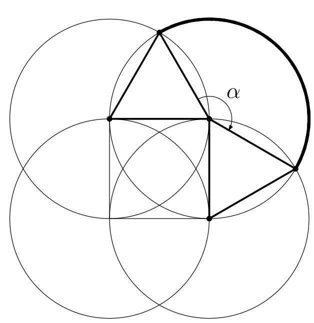
Alternative Solution: Notice that $2 + 1 = 3 = 2^2 - 1$. Then we have:

$$\begin{aligned} 2(2+1)+1 &= 2(2^2-1)+1 = 2^3-1 \\ 2(2(2+1)+1)+1 &= 2(2^3-1)+1 = 2^4-1 \\ 2(2(2(2+1)+1)+1)+1 &= 2(2^4-1)+1 = 2^5-1, \end{aligned}$$

and so on. Thus the given expression is $2^{12} - 1 = 4095$.

2. Let s and f be the speeds (in m/s) of the slower and faster trains, respectively. When the trains are moving in the same direction their relative speed is $f - s$ m/s, and the trains completely pass one another once $200 + 220 = 420$ metres have been traversed at this speed. Thus $420 = 30(f - s)$, which gives $14 = f - s$. Similarly, when the trains are moving in the same direction their relative speed is $f + s$ m/s, and since they pass each other in 7 seconds we have $420 = 7(f + s)$, giving $60 = f + s$. Solving for s gives $s = 23$.

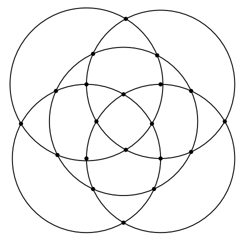
3. Consider one lobe of the figure, which is the arc of a unit circle subtended by some angle α, as indicated in the diagram below.



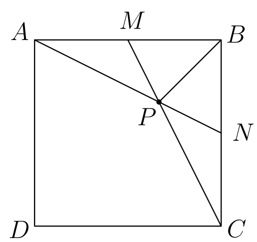
Note that the two triangles drawn in the figure are equilateral since every side is the radius of a unit circle. Thus $\alpha = 360° - 60° - 90° - 60° = 150°$. The length of the arc subtended by α is therefore $\frac{150}{360} \cdot 2\pi = \frac{5}{6}\pi$. The perimeter of the entire figure is therefore $4 \cdot \frac{5}{6}\pi = \frac{10}{3}\pi$.

4. Adding all the equations results in $3(x+y+z+w) = 10$, so $x+y+z+w = \frac{10}{3}$. Subtracting $y+z+w=2$ from this equation gives $x = \frac{4}{3}$.

5. Two circles can intersect in at most 2 points. So the maximum number of intersection points is twice the number of pairs of circles; that is, $2\binom{5}{2} = 20$. It is not difficult to come up with a configuration that attains this maximum. One such configuration is shown below.



6. Draw line PB, and let M and N be the midpoints of AB and BC, respectively.



The side of the square is 6, so we have $|AM| = |MB| = |BN| = |NC| = 3$. Note that Area$(\triangle AMP)$ = Area$(\triangle MBP)$, since both triangles have base 3 and share the same altitude to P. Let a be the common area of these triangles. Then by symmetry we also have Area$(\triangle BNP)$ = Area$(\triangle NCP) = a$. It follows that Area$(\triangle ABN) = 3a$ and Area$(APCD) = 36 - 4a$. But clearly Area$(\triangle ABN) = \frac{1}{2}|AB| \cdot |BN| = 9$, so $a = 3$ and Area$(APCD) = 12$.

Alternative Solution: There are many ways to solve this problem. The "brute force" attack is to coordinatize the problem, say by letting $D = (0,0)$ and $B = (6,6)$. Then lines AN and MC (see diagram above) have equations $y = 6 - \frac{1}{2}x$ and $y = 12 - 2x$, and they intersect at $P = (4,4)$. It is now straightforward to find the area of $APCD$.

Yet another approach is to note that AN and CM are medians of triangle ABC. Thus P divides AN in the ratio $2:1$, hence Area$(\triangle APB) = \frac{2}{3}$Area$(\triangle ANB) = 6$. Similarly Area$(\triangle CPB) = 6$, leaving the area of $ABCD$ to be $36 - 6 - 6 = 24$.

7. Since the punch is initially 40% juice, let there originally be 5 units of liquid in bowl, including 2 units of juice. Suppose the ladle has a volume of ℓ units.

Then the conditions stated in the problem give the following equality

$$\frac{p}{100} = \frac{2 - \frac{2}{5}\ell + \ell}{5} = \frac{2 + \frac{3}{5}\ell}{5}$$

and also

$$\frac{p}{100} = \frac{2 + 3\ell}{5 + 3\ell}.$$

One can now set these expressions equal, solve for ℓ and then back- substitute to get p. But it is simpler to use the following rule of ratios: If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c-a}{d-b}$. Applying this to the above equations gives

$$\frac{p}{100} = \frac{(2 + 3\ell) - (2 + \frac{3}{5}\ell)}{(5 + 3\ell - 5} = \frac{3\ell - \frac{3}{5}\ell}{3\ell} = \frac{4}{5}.$$

That is, $p = 80$.

Note: Solving the equation for ℓ will yield two solutions, one of them being $\ell = 0$. This makes sense, since of course the concentration doesn't change if the ladle has zero volume. But this is a silly solution and is dismissed.

8. Let p be the probability that the first player to roll is also the first to roll a 6. Either the first player wins on his first roll, with probability $\frac{1}{6}$, or he does not, with probability $\frac{5}{6}$. But in the latter case the game begins anew with the original second player now being the first player! The original first player only wins if the *new* first player *loses*, which happens with probability $1 - p$. Thus $p = \frac{1}{6} + \frac{5}{6}(1 - p)$, which yields $p = \frac{6}{11}$.

   Alternative Solution: Andy wins whenever a 6 is first rolled on an *odd* numbered turn. Thus he wins with probability $\frac{1}{6} + (\frac{5}{6})^2 \cdot \frac{1}{6} + (\frac{5}{6})^4 \cdot \frac{1}{6} + (\frac{5}{6})^6 \cdot \frac{1}{6} + \cdots$ This is an infinite geometric series with initial term $\frac{1}{6}$ and common ratio $(\frac{5}{6})^2 = \frac{25}{36}$. Its sum is therefore $\frac{1}{6}(1 - \frac{25}{36})^{-1} = \frac{6}{11}$.

9. Let the roots of $y = x^2 - 14x + 40$ be α and β. Then, by the secant-tangent theorem (a.k.a. power-of-the-point) we have $|OP|^2 = \alpha \cdot \beta = 40$. Thus $|OP| = \sqrt{40} = 2\sqrt{10}$.

   Alternative Solution: The x-intercepts of $y = x^2 - 14x + 40$ are $(10, 0)$ and $(4, 0)$, and its vertex is $(7, -9)$. Let C be the centre of the circle passing through these three points. By symmetry, $C = (7, a)$ for some a. Since C is equidistant from $(4, 0)$ and $(7, -9)$ we have $a^2 + (4 - 7)^2 = (7 - 7)^2 + (a + 9)^2$, which yields $a = -4$. Thus $C = (7, -4)$ and the circle has radius 5. Since the tangent OP is perpendicular to CP, Pythagorean theorem yields $|OP|^2 + |CP|^2 = |OC|^2$. That is $|OP|^2 + 5^2 = 7^2 + (-4)^2$, and thus $|OP| = \sqrt{40}$.

10. There are 9 lines in the figure, and each of these lines intersects every other. Any three mutually intersecting lines define a triangle, *except* in the case where the lines are

concurrent (*i.e.* pass through a common point). We can select 3 lines from the 9 in our figure in $\binom{9}{3}$ ways. These lines will be concurrent only when all three pass through one of the two bottom vertices of the large triangle. Each such vertex lies on 5 lines, so $2 \cdot \binom{5}{3}$ of the $\binom{9}{3}$ possible choices of lines do *not* yield a triangle. Thus the total number of triangles is $\binom{9}{3} - 2\binom{5}{3} = 84 - 2 \cdot 10 = 64$.

Alternative Solution: There are many other ways to count the triangles. As usual, the key is to be organized. For example, it is easy to see that every triangle must involve at least one of the two bottom vertices of the large triangle, so it is sensible to first count all triangles that involve only one of these vertices; then multiply by 2 and subtract the triangles that involve both bottom vertices to avoid over-counting.

Note: The fact that the answer is a perfect cube is not a coincidence. If instead we start with a large triangle, choose two vertices, and draw k lines from each to the opposite side, then the resulting figure will contain $(k+1)^3$ triangles. The problem presented here is the case $k = 2$. Can you prove the general result?

Pairs Relay Solutions

P-A. Everyone must vote in favour of one motion, both motions, or neither motion, so

$$\begin{aligned}1000 = {} & (\text{\# in favour of motion 1}) + (\text{\# in favour of motion 2}) \\ & - (\text{\# in favour of both motions}) + (\text{\# in favour of neither motion}).\end{aligned}$$

Therefore $1000 = 744 + 526 - \mathtt{A} + 32$, giving $\mathtt{A} = 302$.

P-B. Short division gives $\frac{1}{7} = 0.\overline{142857}$. Thus the decimal expansion repeats with a period of length 6, so the $\mathtt{A}$-th digit after the decimal point will be determined by the remainder left upon dividing $\mathtt{A}$ by 6.

Since $\mathtt{A} = 302$ leaves remainder 2 upon division by 6, the $\mathtt{A}$-th digit will be the same as the 2nd digit. Thus $\mathtt{B} = 4$.

P-C. While we could wait for $\mathtt{B}$ and then determine $\mathtt{C}$ by a sequence of trial divisions (at most 10 of them), it is much more efficient to use the fact that a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Thus $\mathtt{1C2C3C4C5B}$ will be divisible by 9 precisely when $(1+2+3+4+5) + 4\mathtt{C} + \mathtt{B} = 15 + 4\mathtt{C} + \mathtt{B}$ is divisible by 9.

So with $\mathtt{B} = 4$, we are looking for the digit $\mathtt{C}$ such that $19 + 4\mathtt{C}$ is divisible by 9. The answer $\mathtt{C} = 2$ is quickly found by inspection.

Note: The hunt for $\mathtt{C}$ can be automated by employing modular arithmetic. For those familiar with such methods, the relevant calculation is as follows:

$$\begin{aligned}15 + 4\mathtt{C} + \mathtt{B} \equiv 0 \pmod 9 \quad & \iff \quad 4\mathtt{C} \equiv 3 - \mathtt{B} \pmod 9 \\ & \iff \quad \mathtt{C} \equiv 2\mathtt{B} - 6 \pmod 9\end{aligned}$$

The latter congruence instantly gives $\mathtt{C} = 2$ when $\mathtt{B} = 4$.

P-D. Although not necessary, it is perhaps easiest to first replace $x + \mathtt{C}$ by u to simplify computations. Then the equation is solved as follows:

$$\begin{aligned}\frac{u}{u+1} - \frac{u-1}{u} = \frac{1}{(u-2\mathtt{C})^2} \quad & \implies \quad \frac{u^2 - (u^2-1)}{u(u+1)} = \frac{1}{(u-2\mathtt{C})^2} \\ & \implies \quad \frac{1}{u(u+1)} = \frac{1}{(u-2\mathtt{C})^2} \\ & \implies \quad u(u+1) = (u-2\mathtt{C})^2 \\ & \implies \quad u^2 + u = u^2 - 4\mathtt{C}u + 4\mathtt{C}^2 \\ & \implies \quad u = \frac{4\mathtt{C}^2}{1+4\mathtt{C}}.\end{aligned}$$

With $\mathtt{C} = 2$ we get $u = \frac{16}{9}$, and therefore $\mathtt{D} = x = u - \mathtt{C} = -\frac{2}{9}$.

Individual Relay Solutions

I-A. Begin with 3 and repeatedly multiply by 3, keeping track only of the units digit. This yields the repeating sequence sequence $3, 9, 7, 1, 3, 9, 7, 1, \ldots$ with period 4. Thus the units digit of 2013^{2014} is determined by the remainder left when 2014 is divided by 4. Since this remainder is 2, the units digit of 2013^{2014} is 9.

Similarly, starting with 4 and repeatedly multiplying by 4 yields the sequence $4, 6, 4, 6, \ldots$ of units digits. Thus the units digit of 2014^{2013} will be 4, since 2013 is odd.

It follows that $\mathtt{A} = 9 - 4 = 5$.

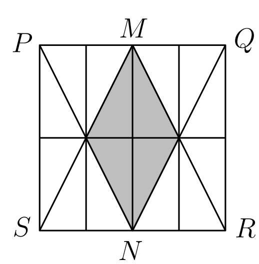
I-B. From the given information, first note that $a_2 = 4a_1 - 3a_0 = 4\mathtt{A} - 3\mathtt{A} = \mathtt{A}$. But then we similarly have $a_3 = 4a_2 - 3a_1 = 4\mathtt{A} - 3\mathtt{A} = \mathtt{A}$, and in fact repeating this argument shows that we have $a_n = \mathtt{A}$ for *every* n.

Thus $\mathtt{B} = \mathtt{A} = 5$.

I-C. We know α and β are the roots of the quadratic $x^2 - \mathtt{B}x + (\mathtt{B} - 1) = 0$. Thus $\alpha + \beta = \mathtt{B}$ and $\alpha\beta = \mathtt{B} - 1$. It follows that $\mathtt{C} = \alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = \mathtt{B}(\mathtt{B} - 1)$.

With $\mathtt{B} = 5$ we get $\mathtt{C} = 5 \cdot 4 = 20$.

I-D. Draw a few extra lines in the diagram to obtain a dissection of the square into 16 congruent triangles as indicated below.



Since the shaded region is comprised of 4 triangles, its area is $\frac{1}{4}$ that of the entire square.

Given that the perimeter of the square is $\mathtt{C} = 20$, we know its side is 5 and hence its area is 25. Thus $\mathtt{D} = \frac{25}{4}$.