Nova Scotia Math League

2013–2014

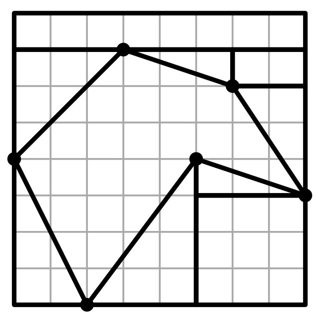
Game Two

SOLUTIONS

Team Question Solutions

1. Notice that the final digits of $9^1, 9^2, 9^3, 9^4, \ldots$ are $9, 1, 9, 1, \ldots$. In general, the final digit of 9^n is 9 if n is odd, and 1 if n is even. Since 9^9 is odd, the final digit of 9^{9^9} is 9.

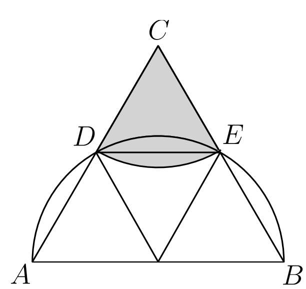
2. The area of the entire square grid is $8^2 = 64$. The area *outside* the hexagon is readily found to be 19.5 by splitting it into rectangles and triangles (for instance, as depicted below). Thus the hexagon has area $64 - 19.5 = 24.5$.



   Alternative Solution: Use Pick's Theorem to calculate the area by counting lattice points. There are 21 lattice points inside the hexagon, and 9 on its boundary. Thus the area of the hexagon is $21 + \frac{9}{2} - 1 = 24.5$.

3. Clearly there is only 1 cube with all sides painted red, and only 1 cube with exactly one side painted red. There are 2 cubes with exactly two sides painted red: one in which the red sides are adjacent, and one in which the red sides are opposite. There are also 2 cubes with 3 sides painted red: one in which the three sides share a corner, and one in which two are opposite and a third is adjacent to both of these. By symmetry, the number of cubes with 4, 5, or 6 sides painted red equal the number of cubes with 2, 1, or 0 sides painted red. Thus the total number of cubes is $2(1 + 1 + 2) + 2 = 10$.

4. Divide the triangle into smaller equilateral triangles with sides of length 1, and notice by symmetry that the desired area is simply the shaded area shown below. But this is just one sixth of the area of the unit circle, which is $\frac{\pi}{6}$.



5. When Alice chooses either 1 or 10, Mary has only 3 possible choices so as to stay within two of Alice. When Alice chooses either 2 or 9, Mary has 4 choices available. When Alice chooses any of 3 through 8, Mary has 5 choices. So there are $2 \cdot 3 + 2 \cdot 4 + 6 \cdot 5 = 44$ choices of numbers that differ by at most 2, and there are $10 \cdot 10 = 100$ choices in total, giving a probability of $\frac{44}{100} = \frac{11}{25}$.

6. The list begins with the $3 \cdot 4! = 72$ words that start with A, E, or G. Next we have the $3! = 6$ words that start with MA, followed by the 2 words that start with MEA, and then finally we arrive at MEGAN. So her name appears at position $72 + 6 + 2 + 1 = 81$ in the list.

7. Subtract the two equations to get $3x + y = -1$. Each intersection point (x, y) must satisfy this equation, which is to say that the intersection points lie on the line with equation $y = -3x - 1$. Thus the desired slope is -3.

8. Let the intersection points be $A = (x_1, y_1)$ and $B = (x_2, y_2)$, and let $\Delta x = x_1 - x_2$ and $\Delta y = y_1 - y_2$. The desired length is Since A and B lie on $y = 2x + 6$ we have $\Delta y = 2\Delta x$, and thus the desired length is

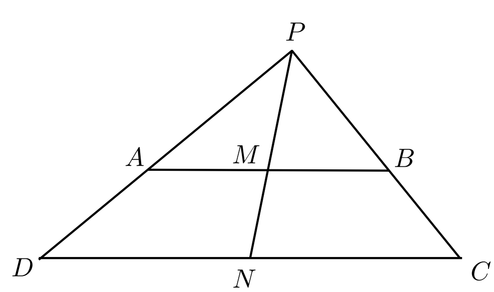
$$|AB| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{5(\Delta x)^2}.$$

But

$$(\Delta x)^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2.$$

Setting $2x + 6 = 10 - x^2$, we find that x_1 and x_2 are the roots of $x^2 + 2x - 4 = 0$. From the coefficients of this quadratic, we deduce $x_1 + x_2 = -2$ and $x_1x_2 = -4$. Therefore $|AB| = \sqrt{5((-2)^2 - 4(-4))} = \sqrt{100} = 10$.

9. Extend sides AD and BC to meet at P. Since AB is parallel to CD, the median from P of $\triangle DPC$ passes through the midpoints M and N of AB and CD, respectively. Moreover, $\triangle DPC$ is right-angled at P, since $\angle PDC + \angle PDC = 40^\circ + 50^\circ = 90^\circ$. This implies $|PN| = |NC| = |ND| = \frac{10}{2} = 5$.



But $\triangle PAB$ and $\triangle PDC$ are similar, so $\frac{|PM|}{|PN|} = \frac{|AB|}{|DC|} = \frac{6}{10}$. With $|PN| = 5$ we conclude that $|PM| = 3$, and therefore $|MN| = |PN| - |PM| = 5 - 3 = 2$.

10. Let x be the number of students at the assembly. Then $\frac{4}{7}x$, $\frac{2}{5}x$ and $\frac{1}{4}x$ must be whole numbers, which occurs only when x is divisible by 7, 5, and 4. This is the same as saying x is divisible by $7 \cdot 5 \cdot 4 = 140$, so we conclude that $x = 140, 280$, or 420. However, $442 - x$ students skipped the assembly, and $\frac{2}{3}$ of these students must be boys. Thus $442 - x$ must be divisible by 3. A quick check shows $x = 280$ is the only possibility.

Pairs Relay Solutions

P-A. Let $x = \mathtt{A}/100$. Then we have $(1+x)(1-x)500 = 495$, which gives $x^2 = \frac{1}{100}$ and thus $x = \frac{1}{10}$. Therefore $\mathtt{A} = 10$.

P-B. Each loop is $\mathtt{A}/2$ kilometres. It takes the man $\frac{\mathtt{A}}{2}/10 = \frac{\mathtt{A}}{20}$ hours to run the first loop, and $\frac{\mathtt{A}}{2}/\mathtt{B} = \frac{\mathtt{A}}{2\mathtt{B}}$ hours to run the second loop. Since he runs a total of $\mathtt{A}$ kilometers, his average speed over the entire run is

$$\frac{\text{distance}}{\text{time}} = \frac{\mathtt{A}}{\frac{\mathtt{A}}{20} + \frac{\mathtt{A}}{2\mathtt{B}}} = \frac{20\mathtt{B}}{\mathtt{B}+10}.$$

Setting this fraction equal to 12 and solving for $\mathtt{B}$ gives $\mathtt{B} = 15$.

Note: The answer does not depend on $\mathtt{A}$! In general, the answer will depend only on the ratio between the distances around the first and second loops, not the distances themselves. (In our case the distances are the same.) Can you explain this?

P-C. We are told that removal of $\mathtt{C}$ from $\{1, \ldots, \mathtt{B}\}$ results in a set whose average is $\mathtt{C}$. The only way this can happen is if $\mathtt{C}$ is the average of the original set. But the average of $\{1, \ldots, \mathtt{B}\}$ is clearly $\frac{\mathtt{B}+1}{2}$, so we have $\mathtt{C} = \frac{\mathtt{B}+1}{2}$. With $\mathtt{B} = 15$ we get $\mathtt{C} = 8$.

P-D. Let r_1 and r_2 be the radii of the small and medium circles, respectively. Then the diameters of the small and medium circles sum to the diameter of the large circle, so we have $2r_1 + 2r_2 = \mathtt{C}$. Also, since the medium circle has area 9 times that of the small circle, we have $r_2 = 3r_1$. Therefore $r_1 = \frac{\mathtt{C}}{8}$, and the shaded area is $\mathtt{D} = \pi(\frac{\mathtt{C}}{2})^2 - \pi(\frac{\mathtt{C}}{8})^2 - 9\pi(\frac{\mathtt{C}}{8})^2 = \frac{3\pi\mathtt{C}^2}{32}$. With $\mathtt{C} = 8$ we have $\mathtt{D} = 6\pi$.

Individual Relay Solutions

I-A. The profit per widget sold is $\frac{5}{2} - \frac{5}{3} = \frac{5}{6}$. For a profit of \$50, Bob must sell $\mathtt{A} = 50/\frac{5}{6} = 60$ widgets.

I-B. Each of the three men (Jack and his two sons) will age $\mathtt{A} - \mathtt{B}$ years between today and Jack's $\mathtt{A}$-th birthday. Since the sum of their ages is currently $\mathtt{A}$, the sum of their ages on Jack's $\mathtt{A}$-th birthday will be $\mathtt{A} + 3(\mathtt{A} - \mathtt{B}) = 4\mathtt{A} - 3\mathtt{B}$. Setting this equal to $2\mathtt{A}$ yields $\mathtt{B} = \frac{2}{3}\mathtt{A}$. With $\mathtt{A} = 60$, we get $\mathtt{B} = 40$.

I-C. The rate of digging, measured in metres per man-hour, is $\frac{100}{15 \cdot 6} = \frac{10}{9}$. Thus we have $\frac{\mathtt{B}}{12\mathtt{C}} = \frac{10}{9}$, or simply $\mathtt{C} = \frac{3\mathtt{B}}{40}$. With $\mathtt{B} = 40$, we get $\mathtt{C} = 3$.

I-D. Notice that $x \star y = (x+1)(y+1) - 1$. It follows that $\mathtt{D} = \mathtt{C} \star (\mathtt{C} \star (\mathtt{C} \star \mathtt{C})) = (\mathtt{C}+1)^4 - 1$. With $\mathtt{C} = 3$ we get $\mathtt{D} = 4^4 - 1 = 255$.

Note: It is also straightforward to compute $\mathtt{D}$, once $\mathtt{C}$ is known, by repeatedly applying the rule for $x \star y$. We have $3 \star 3 = 9 + 3 + 3 = 15$, and then $3 \star (3 \star 3) = 3 \star 15 = 45 + 3 + 15 = 63$, etc.