Nova Scotia Math League

2017–2018

Game Two

PROBLEMS AND SOLUTIONS

Team Questions

1. Emily numbers the pages of her math course notes, starting as usual with $1, 2, 3, \ldots$. In doing so, she writes exactly 999 digits.

   How many pages of notes does Emily have?

   Solution: Let x be the number of 3-digit pages. Then $9 + 2(90) + 3x = 999 \implies x = 270$. So there are $x + 90 + 9 = 369$ pages in total.

   $\square$

2. Amy and Sarah run a race. Amy runs 25% faster than Sarah and finishes the race 15 seconds ahead. How long does it take Amy to finish the race?

   Solution: Say it takes Amy t seconds to finish the race. Then Sarah's time is $t + 15$, and also $1.25t$. Solve $1.25t = t + 15$ to get $t = 60$ seconds.

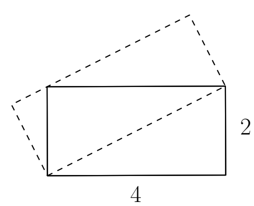
   $\square$

3. Today is both my and my son's birthday. I am now twice as old as my father was on the day I was born. Coincidentally, on the day my son was born, my father was twice my age! My father is now 78 years old. How old is my son?

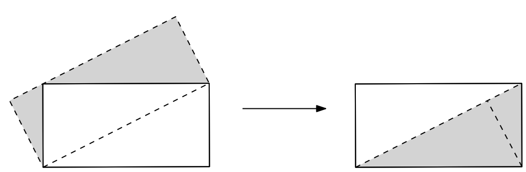
   Solution: Let x, y, z be the current ages of me, my father, and my son, respectively. Then $x = 2(y - x) \implies 2y = 3x$, and $y - z = 2(x - z) \implies z = 2x - y = y/3$. With $y = 78$ get $z = 26$.

   $\square$

4. Find the area of the dashed rectangle.



   Solution: Move pieces around as shown below. Thus the dashed rectangle has the same area as the original, namely $4 \cdot 2 = 8$.



   $\square$

5. I recently bought some bitcoin. Over the first week, I lost X% of my investment. But bitcoin then gained $2X$% in the following week, bringing my two week profit to 8%.

   Find all possible values of X.

   Solution: Let $x = X/100$. Then $(1 - x)(1 + 2x) = \frac{108}{100}$. Get $2x^2 - x + \frac{2}{25} = 0$, so that $0 = 50x^2 - 25x + 2 = (10x - 1)(5x - 2)$. Thus $x = \frac{2}{5}$ or $x = \frac{1}{10}$, giving $X = 40$ or $X = 10$. $\square$

6. Find the number of digits in the decimal expansion of 20^{30}.

Solution: This is $2^{30} \cdot 10^{30}$. Now $2^{30} = (2^{10})^3 = (1024)^3 \approx (10^3)^3$ has 10 digits, so 20^{20} has $30 + 10 = 40$ digits. □

7. You begin with a 1L jug of wine. One cup (250mL) of wine is removed and replaced with water, and the container is thoroughly mixed. This process is repeated twice more. What is the final ratio of water to wine in the jug?

Solution: Let α be the current amount of wine in the jug (in litres). Removing $\frac{1}{4}$L of the solution removes $\frac{1}{4}$ of the wine, leaving $\alpha - \frac{1}{4}\alpha = \frac{3}{4}\alpha$ litres of wine. Iterate three times to get $(\frac{3}{4})^3 = \frac{27}{64}$L of wine. Thus the water to wine ratio is $\frac{64-27}{27} = \frac{37}{27}$.

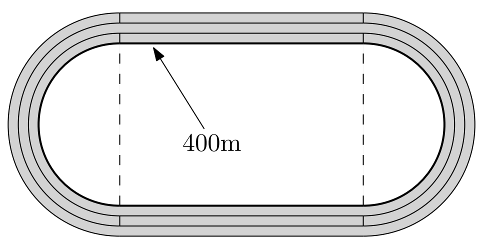
□

8. The faces of a $4 \times 4 \times 4$ cube are painted red. The cube is then sliced into 64 unit cubes. One of these cubes is selected at random and thrown like a die. Determine the probability that the face showing is painted.

Solution: There are $4^3 \cdot 6$ faces, with $4^2 \cdot 6$ of them painted. Each is equally likely to end up showing after the roll, so the desired probability is $(4^2 \cdot 6)/(4^3 \cdot 6) = 1/4$.

□

9. A running track is composed of three adjacent lanes, each 1m wide, with parallel straightaways and semicircular curves. The distance around the **inside** of the innermost lane is precisely 400m. Find the area of the track.



Solution: Let the radius of the inner circle be r and the straightaway be x. Then $400 = 2\pi r + 2x$. So the desired area is $\pi(r+3)^2 + 2(r+3)x - (\pi r^2 + 2rx) = \pi(6r+9) + 6x = 3(2\pi r + 2x) + 9\pi = 3(400) + 9\pi = 1200 + 9\pi$.

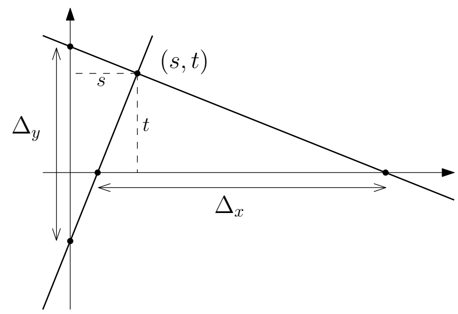
Note: This computation can be simplified by noting that the nature of the question suggests the answer cannot depend on x. So we might as well take x to be a convenient numbers, such as $x = 0$ (in which case the track is just a circle), or $x = 200$ (in which case the track is a single straight line).

More generally, one can show that is a path of width w is created around a convex figure of perimeter P, then the area of the path is $Pw + \pi w^2$.

□

10. Two perpendicular lines pass through the point $(3,7)$. The distance between their x-intercepts is 15. Find the distance between their y-intercepts.

Solution: The diagram below shows perpendicular lines through a generic point (s,t), with distances Δ_x and Δ_y between the x- and y-intercepts, respectively. Observe that similar triangles $\Delta_y/s = \Delta_x/t$. Taking $(s,t) = (3,7)$ and $\Delta_x = 15$ gives $\Delta_y = \frac{3 \cdot 15}{7} = \frac{45}{7}$.



Note: The above solution presupposes the existence of the configuration. It is easy to show that the minimum distance between x-intercepts of a pair of perpendicular lines through (s,t) is in fact $2t$. (This is manifestation of the arithmetic-geometric mean inequality.) Since $15 > 2(7)$, the configuration in this problem does indeed exist.

One could instead follow an algebraic approach. Let the lines be $x - s = m(y-t)$ and $x - s = -\frac{1}{m}(y-t)$, with x-intercepts $s - mt$ and $s + \frac{1}{m}t$, respectively. The difference between these intercepts is $\Delta_x = (m + \frac{1}{m})t$, and since $m + \frac{1}{m} \geq 2$ for real values of m, we witness the fact that $\Delta_x \geq 2t$, as claimed.

One can continue the above line of argument to solve the problem at hand. The y-intercepts of our lines are $t - \frac{1}{m}s$ and $t + ms$, with difference $\Delta_y = (m + \frac{1}{m})s$. Thus $\Delta_y/\Delta_x = s/t$, as before.

$\square$

Pairs Relay

P-A. Increasing the length of each side of a cube by 1 cm increases its surface area by 66 cm^2.

Let A be the corresponding increase in volume (measured in cm^3).

$\boxed{\text{Pass on A}}$

Solution: Let x be the length of each side. Then the increase in surface area is $6(x+1)^2 - 6x^2 = 6(2x+1)$. This is 66 when $x = 5$. The corresponding increase in volume is $\mathsf{A} = 6^3 - 5^3 = 91$

□

P-B. You will receive A.

The average of two numbers is A/7. If one of the numbers is doubled and the other is tripled then their average becomes 30.

Let B be the smaller of the numbers. $\boxed{\text{Pass on B}}$

Solution: Let the numbers be x and y and let $n = \mathsf{A}/7$. Then $x + y = 2n$ and $2x + 3y = 60$. The second equation less twice first gives $y = 60 - 4n$, whence $x = 6n - 60$. With $\mathsf{A} = 91$ get $n = 13$, $y = 8$ and $x = 18$. So $\mathsf{B} = 8$.

□

P-C. You will receive B.

A closed rectangular box is twice as long as it is wide, and twice as tall as it is long. Its volume is B times its surface area.

Let C be the length of the shortest side of the box. $\boxed{\text{Pass on C}}$

Solution: Let the sides be $\mathsf{C}, 2\mathsf{C}$ and $4\mathsf{C}$, so volume is $8\mathsf{C}^3$ and surface area is $2\mathsf{C}^2(2+4+8) = 28\mathsf{C}^2$. Then $8\mathsf{C}^3 = BB(28\mathsf{C}^2) \implies \mathsf{C} = 7\mathsf{B}/2$. With $\mathsf{B} = 8$ get $\mathsf{C} = 28$.

□

P-D. You will receive C.

A ship sails 34 km north, followed by C km east, then C km south, and finally 36 km west.

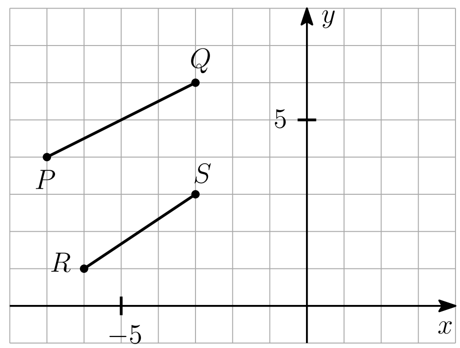
Let D be the distance between the ship and its starting point.

$\boxed{\text{Done!}}$

Solution: The distance is $\sqrt{(34-\mathsf{C})^2 + (36-\mathsf{C})^2}$. With $\mathsf{C} = 28$ get $\sqrt{6^2 + 8^2} = 10$. □

Individual Relay

I-A. If segments $\overline{PQ}$ and $\overline{RS}$ are extended, the lines intersect at the point (x, y). Let $\mathsf{A} = x$.



Pass on A

Solution: The lines are $y - 6 = \frac{1}{2}(x+3)$ and $y - 3 = \frac{2}{3}(x+3)$. Subtract to get $-3 = -\frac{1}{6}(x+3)$, so $x + 3 = 18 \implies x = 15$. Thus $\mathsf{A} = 15$.

□

I-B. You will receive A.

Find B such that

$$\frac{81^{\mathsf{B}}}{27^{\mathsf{A}}} = \frac{1}{3}.$$

Pass on B

Solution: Simplify to $3^{4\mathsf{B}-3\mathsf{A}} = 3^{-1}$, so $\mathsf{B} = \frac{1}{4}(3\mathsf{A} - 1)$. With $\mathsf{A} = 15$ get $\mathsf{B} = 11$.

□

I-C. You will receive B.

Let C be the sum of the first B terms of the series

$$1 - 4 + 7 - 10 + 13 - 16 + \cdots$$

Pass on C

Solution: Pairing terms as $(1-4) + (7-10) + \cdots$ shows that the sum of the first $2n$ terms is $-3n$. The $(2n+1)$-st term is $1 + 2n \cdot 3 = 1 + 6n$, so the sum of the first $2n+1$ terms is $-3n + (1+6n) = 1 + 3n$. Thus the sum of the first $\mathsf{B} = 11$ terms is $\mathsf{C} = 1 + 3(5) = 16$.

□

I-D. You will receive C.

Compute

$$\mathsf{D} = \frac{(1+\sqrt{\mathsf{C}})^4 - (1-\sqrt{\mathsf{C}})^4}{1+\mathsf{C}}.$$

$\boxed{\text{Done!}}$

Solution: Simplifying using the difference of squares factorization $x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$ yields $\mathsf{D} = 8\sqrt{\mathsf{C}}$. With $\mathsf{C} = 16$ get $\mathsf{D} = 32$.

$\square$