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Lecture Notes 1: The Language of Sentential Logic

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1 Induction and recursion

1.1 Induction

Let X be a set, and let $\mathscr{P}X = \{A \mid A \subseteq X\}$ be the powerset of X. Let $\Phi : \mathscr{P}X \to \mathscr{P}X$ be a monotone operation, i.e., for all $A, B \in \mathscr{P}X$, we have $A \subseteq B$ implies $\Phi(A) \subseteq \Phi(B)$.

Definition. We say that a set $A \subseteq X$ is *closed under* Φ if $\Phi(A) \subseteq A$.

Proposition 1. *(a) Let $(A_i)_{i\in I}$ be a family of sets such that each A_i is closed under Φ. Then the intersection $A = \bigcap_{i\in I} A_i$ is also closed under Φ.*

(b) If $B \subseteq X$ is any set, then there exists a smallest subset $\overline{B} \subseteq X$ such that $B \subseteq \overline{B}$ and $\overline{B}$ is closed under Φ.

Proof. (a) Let $i \in I$ be arbitrary. By hypothesis, we have that A_i is closed under Φ, so $\Phi(A_i) \subseteq A_i$. Since $A \subseteq A_i$ for all i, we have $\Phi(A) \subseteq \Phi(A_i)$ by monotonicity of Φ. Hence $\Phi(A) \subseteq A_i$. Since i was arbitrary, it follows that $\Phi(A) \subseteq \bigcap_{i\in I} A_i$, therefore $\Phi(A) \subseteq A$ and A is closed under Φ.

(b) This is a trivial consequence of (a). Let $\overline{B}$ be the intersection of all sets $A \in \mathscr{P}X$ such that $B \subseteq A$ and A is closed under Φ. By (a), $\overline{B}$ is itself closed under Φ, and it also contains A. Since it is the intersection of all such sets, it is therefore the smallest with those properties. □

Corollary 2 (Induction principle)**.** *Let $\overline{B} \subseteq X$ be defined as in Proposition 1, i.e., $\overline{B}$ is the smallest subset of X that contains B and is closed under Φ. Suppose that P is a property of elements of X. Further suppose that*

(a) for all $x \in B$, the property $P(x)$ holds (base case), and

(b) the set $\{x \mid P(x)\}$ is closed under Φ (induction step).

Then the property $P(x)$ holds for all $x \in \overline{B}$.

Proof. Let $A = \{x \mid P(x)\}$. By the base case, $B \subseteq A$, and by the induction step, A is closed under Φ. Since $\overline{B}$ was the smallest set with these two properties, it follows that $\overline{B} \subseteq A$. □

Example 3. Let X be a set, $0 \in X$ be an element, and $s : X \to X$ a one-to-one function whose image does not include 0. For $A \subseteq X$, let $\Phi(A) = \{s(x) \mid x \in A\}$. Then a set A is closed under Φ if and only if for all $x \in A$, we have $s(x) \in A$. We also say that the set A is *closed under s* in this case.

Let N be the smallest subset of X containing 0 and closed under s. In this case, the induction principle asserts the following: if P is a property such that $P(0)$ holds (base case) and such that for all x, if $P(x)$ holds then $P(s(x))$ holds (induction step), then it follows that $P(x)$ holds for all $x \in N$.

This is just the usual induction principle on the natural numbers; note that N as defined above is isomorphic to the natural numbers. In fact, this is how the natural number are defined from the axiom of infinity (the axiom of infinity in set theory asserts that there exists an infinite set; a set X is by definition infinite if there exists a one-to-one function $s : X \to X$ that is not onto. By taking 0 to be some element of X not in the range of s, one arrives at the above definition of N).

Example 4. Let X be a set, let $B \subseteq X$ be a subset, and let $f : X \to X$ and $g : X \times X \to X$ be functions. Define $\Phi(A) = \{f(x) \mid x \in A\} \cup \{g(x, y) \mid x, y \in A\}$. Then a set A is closed under Φ if and only if for all $x, y \in A$, we have $f(x) \in A$ and $g(x, y) \in A$. We also say that A is *closed under f and g* in this case.

Let $\overline{B}$ be the smallest subset of X containing B and closed under f and g. In this case, the induction principle asserts that if some property P is true of all the elements of B (base case), and moreover, if for all elements x, y satisfying P, it is also true that $f(x)$ and $g(x, y)$ satisfy P (induction step), then it follows that P is true for all elements of $\overline{B}$.

If some property of subsets A of X can be expressed in the form "A is closed under Φ", for some monotone operation $\Phi : \mathscr{P}X \to \mathscr{P}X$, then we say that it is a *closure property*.

1.2 Recursion

Let us consider again the situation from Example 4, i.e.:

- X is a set,

- $f : X \to X$ and $g : X \times X \to X$ are functions.

For a given subset $B \subseteq X$, we know that there exists a smallest set $\overline{B}$ that contains B and is closed under f and g.

We are now interested in the question of how one can define functions on the set $\overline{B}$. In particular, we would like to define such functions by recursion. We have the following recursion principle:

Theorem 5 (Recursion principle)**.** *Assume that $\overline{B}$ is defined as in the preceding paragraph. Moreover, assume that the functions $f : X \to X$ and $g : X \times X \to X$ are one-to-one and have disjoint ranges, and that their ranges are disjoint from B. Let S be some set and let there be given three functions $\varphi : B \to S$, $\psi_f : S \to S$, and $\psi_g : S \times S \to S$. Then there exists a unique function $h : \overline{B} \to S$ such that:*

(a) for all $x \in B$, $h(x) = \varphi(x)$;

(b) for all $x \in \overline{B}$, $h(f(x)) = \psi_f(h(x))$; and

(c) for all $x, y \in \overline{B}$, $h(g(x, y)) = \psi_g(h(x), h(y))$.

Proof. Uniqueness is easy, because we can prove it by induction: suppose that h and h' are two such functions. Then use induction to prove that $h(x) = h'(x)$ for all $x \in \overline{B}$. Both the base case and the induction step are trivial.

The difficult part of this proof is existence of a function h having the stated properties. We prove this by first considering an analogous property for *partial* functions. We say that a partial function $k : \overline{B} \rightharpoonup S$ is *consistent* if it satisfies the following partial version of properties (a)–(c) above:

(a') for all $x \in B$, if $k(x)$ is defined, then $k(x) = \varphi(x)$;

(b') for all $x \in \overline{B}$, if $k(f(x))$ is defined, then $k(x)$ is defined, and $k(f(x)) = \psi_f(k(x))$; and

(c') for all $x, y \in \overline{B}$, if $k(g(x, y))$ is defined, then $k(x)$ and $k(y)$ are defined, and $k(g(x, y)) = \psi_g(k(x), k(y))$.

We then prove the following claims (the details were done in class):

(1) There exists a consistent partial function k. (Proof: the empty partial function $k = \emptyset$ will do.)

(2) If k and k' are consistent partial functions, then for all $x \in \overline{B}$, if $k(x)$ and $k'(x)$ are both defined, then $k(x) = k'(x)$. (Proof: by induction).

(3) If $(k_i)_{i \in I}$ is any family of consistent partial functions, then $\bigcup_{i \in I} k_i$ is a consistent partial function. (Proof: the fact that it's a partial function follows from (2). The proof of its consistency is trivial).

(4) Every $x \in \overline{B}$ is in the domain of some consistent partial function k. (Proof: by induction. Base case: For $x \in B$, we can choose $k : \overline{B} \rightharpoonup S$ defined by $k(x) = \varphi(x)$ if $x \in B$, and $k(x) = \mathrm{undefined}$ otherwise. Clearly this satisfies (a'); the fact that it also satisfies (b') and (c') follows from the assumption that the ranges of ψ_f and ψ_g are disjoint from B. For the first induction step, assume that x is in the domain of k; we want show that $f(x)$ is in the domain of some consistent k'. We define $k'(y) = k(y)$ if $y \neq f(x)$, and $k'(f(x)) = \psi_f(k(x))$ otherwise. The fact that this satisfies (a')–(c') hinges on the fact that the ranges of f, g are disjoint from each other and from B, and that f is one-to-one. The details are left to the reader).

Now let k be the union of *all* consistent partial functions $k'\overline{B} \rightharpoonup S$. By (3), k is consistent, and by (4), k is total. It follows that k satisfies (a)–(c), which proves the theorem. □

An analogous theorem of course holds for any number of functions $f_1, \ldots, f_n$ of any arity, instead of f, g.

2 The language of sentential logic

2.1 Well-formed formulas

We define the ***alphabet*** of sentential logic to be the set consisting of the following symbols:

$\neg$	$\wedge$	$\vee$	$\to$	$\leftrightarrow$	$\top$	$\bot$	(connectives)
(	)						(parentheses)
$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	...				(sentence symbols)

Notice that there are infinitely many distinct sentence symbols. Each of them counts as an individual, indivisible symbol. The connectives fall into three classes:

binary connectives ($\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$), unary connectives ($\neg$), and nullary connectives, also known as logical constants ($\top$ and $\bot$). We often write $\Box$ to denote any of the binary connectives.

Let $\mathcal{A}^*$ denote the set of finite strings over the alphabet $\mathcal{A}$. If α and β are strings, then we denote their concatenation by $\alpha\beta$.

Definition 6. The set $\mathcal{W} \subseteq \mathcal{A}^*$ of ***well-formed formulas (wff's)*** of sentential logic is the smallest subset of $\mathcal{A}^*$ such that

1. $\mathbf{A}_n \in \mathcal{W}$, for all n. Also $\top, \bot \in \mathcal{W}$.
2. If $\alpha \in \mathcal{W}$ then $(\neg \alpha) \in \mathcal{W}$.
3. If $\alpha, \beta \in \mathcal{W}$ then $(\alpha \Box \beta) \in \mathcal{W}$, where $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

The formulas $\mathbf{A}_n$, $\top$, and $\bot$ are called ***atomic formulas***. All other well-formed formulas are called ***composite formulas***.

Example 7. Which of the following are well-formed formulas?

1. $(\mathbf{A}_1 \wedge \mathbf{A}_2)$ 2. $((\mathbf{A}_3))$ 3. $(\mathbf{A}_4 \vee \mathbf{A}_7) \rightarrow \mathbf{A}_6$
4. $(((\top \wedge \mathbf{A}_{13}) \vee \mathbf{A}_1) \rightarrow \bot)$ 5. $(\neg \mathbf{A}_3 \rightarrow \mathbf{A}_3)$

Answer: only 1. and 4. are well-formed.

Remark. The existence of a "smallest" set $\mathcal{W}$ satisfying conditions 1–3 is of course guaranteed by the fact that conditions 1–3 are closure properties in the sense of Section 1.1.

2.2 Induction for well-formed formulas

From Section 1.1, we get an induction principle for well-formed formulas. Since we will use it a lot, let's state it explicitly:

Theorem 8 (Induction Principle for well-formed formulas)**.** *To prove that a certain property P holds for all well-formed formulas α, it suffices to show that*

1. *P holds for all atomic formulas α.*
2. *If P holds for a well-formed formula β, then also for $(\neg \beta)$.*
3. *If P holds for well-formed formulas β and γ, then also for $(\beta \Box \gamma)$, for each binary connective $\Box$.* □

We now give some examples of proofs by induction on well-formed formulas.

Example 9. Every well-formed formula α has an equal number of left and right parentheses.

Proof. By induction on the well-formed formula α.

1. *Base case:* Suppose α is atomic. Then it has neither left nor right parentheses, and thus an equal number of each.
2. *Induction step for negation:* Suppose $\alpha = (\neg \beta)$ for some well-formed formula β. By the induction hypothesis, we may assume that β has an equal number of left and right parentheses, say, n of each. Then $\alpha = (\neg \beta)$ has $n + 1$ left and $n + 1$ right parentheses.
3. *Induction step for binary connectives:* Let $\Box$ be a binary connective. Suppose $\alpha = (\beta \Box \gamma)$ for some well-formed formulas β and γ. By the induction hypothesis, we may assume that β and γ each have an equal number of left and right parentheses, say, n and m of each, respectively. Then $\alpha = wbin\beta\gamma$ has $n+m+1$ left parentheses and $n+m+1$ right parentheses.

By induction, this proves the claim. □

As you can see in this example, a proof by induction on well-formed formulas looks very much like a case distinction: Case 1: α is atomic, cases 2 and 3: α is composite. The only difference to a case distinction is the presence of an induction hypothesis: if α is composite, then we may already assume that the induction hypothesis holds for its immediate subformulas.

To show that a case distinction is really a special case of a proof by induction, consider the next example, in which the induction hypothesis is actually never used!

Example 10. A well-formed formula α does never start with the symbol $\neg$.

Proof. By induction on the well-formed formula α.

1. *Base case:* If α is atomic, then it is either $\mathbf{A}_n$ or $\top$ or $\bot$. In neither case does it start with $\neg$.

2. *Induction step for negation:* Suppose $\alpha = (\neg\beta)$ for some well-formed formula β. Then α starts with (and not with $\neg$.

3. *Induction step for binary connectives:* Let $\Box$ be a binary connective. Suppose $\alpha = (\beta \Box \gamma)$ for some well-formed formulas β and γ. As in the previous case, α starts with (and not with $\neg$. $\Box$

We give a third example of an induction proof, to establish a property of well-formed formulas that we will need later. Here, a string $\beta \in \mathcal{A}^*$ is called an ***initial segment*** of a string $\alpha \in \mathcal{A}^*$ if there exists some $\gamma \in \mathcal{A}^*$ such that $\alpha = \beta\gamma$. We say that β is a ***proper initial segment*** of α if it is an initial segment, and if β is neither equal to α nor to the empty string. We say that a string contains an excess of left parentheses if it contains strictly more left than right parentheses.

Example 11. Every proper initial segment of a well-formed formula α contains an excess of left parentheses.

Proof. By induction on the well-formed formula α.

1. *Base case:* If α is atomic, then it has length 1 and thus it has no proper initial segments. Thus, there is nothing to show.

2. *Induction step for negation:* Suppose $\alpha = (\neg\beta)$ for some well-formed formula β. Then the proper initial segments of α are:

   $($,
   $(\neg$,
   $(\neg\beta'$ where β' is a proper initial segment of β, and
   $(\neg\beta$.

   In the first two cases, there is one left parenthesis and zero right ones. Thus, there is an excess of left parentheses. In the third case, β' contains an excess of left parentheses by induction hypothesis. Adding one more left parenthesis certainly leaves the left in the majority. In the last case, β contains an equal number of left and right parentheses by Example 9. Adding one more left parenthesis creates, again, a majority of left parentheses.

3. *Induction step for binary connectives:* This is very similar to the previous case. $\Box$

Corollary 12. *No proper initial segment of a well-formed formula is a well-formed formula.*

Proof. This is an easy consequence of Examples 9 and 11: Every proper initial segment of a well-formed formula contains an excess of left parentheses, and thus cannot be a well-formed formula. $\Box$

2.3 An alternative definition of well-formed formulas

Instead of defining the set of well-formed formulas "from above", as the smallest set satisfying a certain closure property, we could have alternatively defined it "from below", by starting from the atoms and iteratively defining more and more formulas. We will now give this alternative definition and prove that they are equivalent.

Definition. Let $\mathcal{W}_0 = \{\top, \bot, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots\}$ be the set of atomic formulas. For every $n = 0, 1, 2, \ldots$, define

$$\begin{aligned}\mathcal{W}_{n+1} \quad := \quad & \mathcal{W}_n \cup \{(\neg\alpha) \mid \alpha \in \mathcal{W}_n\} \\ & \cup \{(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta) \mid \alpha, \beta \in \mathcal{W}_n\}.\end{aligned}$$

We define the set $\mathcal{W}_* = \bigcup_{n=0}^{\infty} \mathcal{W}_n$.

Proposition 13. *The two definitions of well-formed formulas coincides, i.e.,* $\mathcal{W} = \mathcal{W}_*$.

Proof. First, note that $\mathcal{W}_n \subseteq \mathcal{W}_{n+1}$, and by a simple induction on the natural numbers, $\mathcal{W}_n \subseteq \mathcal{W}_m$ whenever $n \leqslant m$.

To prove the proposition, we first show that $\mathcal{W} \subseteq \mathcal{W}_*$. Since $\mathcal{W}$ is the smallest inductive set, it will suffices to prove that $\mathcal{W}_*$ is inductive. *1.:* If α is atomic, then $\alpha \in \mathcal{W}_0$, hence $\alpha \in \mathcal{W}_*$. *2.:* Suppose $\beta \in \mathcal{W}_*$. Then $\beta \in \mathcal{W}_n$, for some n. It follows from the definition of $\mathcal{W}_{n+1}$ that $(\neg\beta) \in \mathcal{W}_{n+1}$, and thus $(\neg\beta) \in \mathcal{W}_*$. *3.:* Suppose $\beta, \gamma \in \mathcal{W}_*$. Then $\beta \in \mathcal{W}_n$ and $\gamma \in \mathcal{W}_m$ for some n, m. Assume without loss of generality that $n \leqslant m$. In this case, $\beta, \gamma \in \mathcal{W}_m$, and thus $(\beta \Box \gamma) \in \mathcal{W}_{m+1}$. It follows that $(\beta \Box \gamma) \in \mathcal{W}_*$. This proves that $\mathcal{W}_*$ is inductive, and thus that $\mathcal{W} \subseteq \mathcal{W}_*$.

Conversely, we will show that $\mathcal{W}_* \subseteq \mathcal{W}$. It suffices to show that $\mathcal{W}_n \subseteq \mathcal{W}$, for all $n = 0, 1, 2, \ldots$. This is easy to show by induction on the natural number n. $\Box$

Remark 14. We define the ***rank*** of a well-formed formula α to be the least n such that $\alpha \in \mathcal{W}_n$. Thus, the rank of a formula is the number of nesting levels of its unary and binary connectives.

2.4 Unique readability

We can think of the set of well-formed formulas as an algebra with one unary and four binary operations. On the set $\mathcal{W}$, consider the following five operations:

$$\begin{array}{llllll} F_\neg: & \mathcal{W} & \to \mathcal{W} & : & \alpha & \mapsto & (\neg\alpha) \\ F_\wedge: & \mathcal{W}\times\mathcal{W} & \to \mathcal{W} & : & \langle\alpha,\beta\rangle & \mapsto & (\alpha\wedge\beta) \\ F_\vee: & \mathcal{W}\times\mathcal{W} & \to \mathcal{W} & : & \langle\alpha,\beta\rangle & \mapsto & (\alpha\vee\beta) \\ F_\to: & \mathcal{W}\times\mathcal{W} & \to \mathcal{W} & : & \langle\alpha,\beta\rangle & \mapsto & (\alpha\to\beta) \\ F_\leftrightarrow: & \mathcal{W}\times\mathcal{W} & \to \mathcal{W} & : & \langle\alpha,\beta\rangle & \mapsto & (\alpha\leftrightarrow\beta) \end{array}$$

The following theorem ensures that every well-formed formula can be read in a unique way. In practical terms, this means that we have put enough parentheses into our definition of well-formed formulas to avoid any ambiguities.

Theorem 15 (Unique Readability)**.**

1. *Each of the functions $F_\neg$, $F_\wedge$, $F_\vee$, $F_\to$, and $F_\leftrightarrow$ is one-to-one.*
2. *The ranges of these five functions are pairwise disjoint.*
3. *The ranges of these five functions are all disjoint from $\mathcal{W}_0$, the set of atomic formulas.*

Proof. 1. We show, for instance, that the function $F_\wedge$ is one-to-one. All the other cases are similar. So assume that α, β, γ, and δ are well-formed formulas, and that $(\alpha\wedge\beta) = (\gamma\wedge\delta)$. Then, by deleting the first symbol of the strings on the left-hand-side and right-hand-side, it follows that

$$\alpha\wedge\beta) = \gamma\wedge\delta).$$

Then either $\alpha = \gamma$, or α is a proper initial segment of γ, or γ is a proper initial segment of α. Because α and γ are well-formed formulas, it follows from Corollary 12 that the last two cases are impossible. Hence $\alpha = \gamma$. By deleting α from the beginning of each string in our equation, we get

$$\wedge\beta) = \ \wedge\delta),$$

and finally, by deleting the first and last symbol,

$$\beta = \delta.$$

It follows that $F_\wedge$ is one-to-one.

2. We first show that the ranges of $F_\wedge$ and $F_\vee$ are disjoint. The argument is the same for the other pairs of binary connectives. So suppose that α, β, γ, and δ are well-formed formulas, and that $(\alpha\wedge\beta) = (\gamma\vee\delta)$. By deleting the first symbol, we get

$$\alpha\wedge\beta) = \gamma\vee\delta),$$

and as before, we can use Corollary 12 to conclude that $\alpha = \gamma$. Hence, it follows that

$$\wedge\beta) = \ \vee\delta).$$

However, this is clearly a contradiction, since these two strings start with a different symbol.

We now show that the ranges of $F_\neg$ and $F_\Box$ are disjoint, where $\Box$ is a binary connective. So suppose that α, β, and γ are well-formed formulas, and that $(\neg\alpha) = (\beta\,\Box\,\gamma)$. As before, we delete the first symbol to get

$$\neg\alpha) = \beta\,\Box\,\gamma).$$

Thus, the well-formed formula β begins with the symbol "$\neg$", contradicting Example 10.

3. We want to show that the images of the five functions are disjoint from $\mathcal{W}_0$. From the definition of the five functions, it is clear that if α is in the image of any of these functions, then it starts with the symbol "(". On the other hand, none of the elements of $\mathcal{W}_0 = \{\top, \bot, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots\}$ start with the symbol "(". Thus, the claim follows. $\Box$

The Unique Readability Theorem states that a well-formed formula cannot be atomic and composite at the same time. It also states that if a formula is composite, then in a unique way. This is an important property of our syntax: every well-formed formula can be *parsed* in a unique way. Notice that this would not have been true if we had omitted parentheses. For instance, the formula $\neg\alpha\wedge\beta$, which is not well-formed according to our definition, is ambiguous: it could be parsed either as $((\neg\alpha)\wedge\beta)$ or as $(\neg(\alpha\wedge\beta))$. If we had chosen to allow this more liberal syntax, then we would have had to introduce ***precedence rules***, i.e., rules that determine which connectives bind stronger than others, which ones associate to the left and which ones to the right, and so on. This would have made our *formal* treatment much more complicated.

2.5 Recursion for well-formed formulas

The unique readability theorem precisely ensures that the functions $F_\neg, \ldots, F_\leftrightarrow$ and the set $\mathcal{W}_0$ satisfy the hypothesis of Theorem 16. We therefore have the following recursion principle for defining functions on well-formed formulas:

Theorem 16 (Recursion Principle)**.** *Recall that $\mathcal{W}_0$ is the set of atomic formulas. Suppose we are given a set V, together with functions*

$$\begin{array}{ll} H_@: & \mathcal{W}_0 \to V, \\ H_\neg: & V \to V, \quad \textit{and} \\ H_\Box: & V \times V \to V \quad \textit{for each binary connective } \Box. \end{array}$$

Then there exists a unique function $f : \mathcal{W} \to V$ such that for all $\alpha, \beta \in \mathcal{W}$ and for all binary connectives $\Box$,

$$\begin{array}{lcl} f(\alpha) & = & H_@(\alpha), \quad \textit{if } \alpha \textit{ is atomic,} \\ f((\neg\alpha)) & = & H_\neg(f(\alpha)), \\ f((\alpha \Box \beta)) & = & H_\Box(f(\alpha), f(\beta)). \end{array}$$

We give some examples of definitions by recursion.

Example 17. The ***rank*** of a well-formed formula α, in symbols $r(\alpha)$, is defined recursively as:

$$\begin{array}{lcl} r(\alpha) & = & 0, \quad \text{if } \alpha \text{ is atomic,} \\ r((\neg\alpha)) & = & r(\alpha) + 1 \\ r((\alpha \Box \beta)) & = & \max\{r(\alpha), r(\beta)\} + 1. \end{array}$$

Verify that this definition fits the schema of the general Recursion Principle. What are the functions $H_@$, $H_\neg$, and $H_\Box$? What is V?

Notice that the notion of rank defined in the previous example coincides with the notion of rank defined in Remark 14. It is the "nesting depth" of a formula.

Example 18. The function $\ell : \mathcal{W} \to \mathbb{N}$, is defined recursively as:

$$\begin{array}{lcl} \ell(\alpha) & = & 1, \quad \text{if } \alpha \text{ is atomic,} \\ \ell((\neg\alpha)) & = & \ell(\alpha) + 1 \\ \ell((\alpha \Box \beta)) & = & \ell(\alpha) + \ell(\beta) + 1. \end{array}$$

What does the function ℓ represent?

Example 19. We define a function sub : $\mathcal{W} \to \mathscr{P}\mathcal{W}$ to compute the set of subformulas of a well-formed formula α. It is defined recursively as

$$\begin{array}{lcl} \text{sub}(\alpha) & = & \{\alpha\}, \quad \text{if } \alpha \text{ is atomic,} \\ \text{sub}((\neg\alpha)) & = & \text{sub}(\alpha) \cup \{(\neg\alpha)\} \\ \text{sub}((\alpha \Box \beta)) & = & \text{sub}(\alpha) \cup \text{sub}(\beta) \cup \{(\alpha \Box \beta)\}. \end{array}$$

We say that β is a ***subformula*** of α if $\beta \in \text{sub}(\alpha)$.

Example 20. The set of ***free sentence symbols*** of a well-formed formula α, denoted FS(α), is defined recursively as follows:

$$\begin{array}{lcl} \text{FS}(\mathbf{A}_n) & = & \{\mathbf{A}_n\}, \\ \text{FS}(\top) = \text{FS}(\bot) & = & \emptyset, \\ \text{FS}((\neg\alpha)) & = & \text{FS}(\alpha) \\ \text{FS}((\alpha \Box \beta)) & = & \text{FS}(\alpha) \cup \text{FS}(\beta). \end{array}$$

2.6 Informal precedence rules

Having established unique readability and the recursion principle for well-formed formulas, we now know that the formal language of sentential logic is unambiguous. Secure in this knowledge, we will now relax the syntactic rules as far as our *informal* treatment is concerned. This means that, when we write formulas from now on, we will take more liberties with parentheses.

Convention (Informal precedence rules)**.** From now on, when we write formulas, we will sometimes omit certain parentheses. It is understood that the formulas that we write are only shorthands, and that they denote well-formed formulas in the formal sense. The following rules determine how missing parentheses are to be filled in:

1. Negation takes precedence over binary connectives. Thus, $\neg\alpha \wedge \beta$ means $((\neg\alpha) \wedge \beta)$.
2. "$\wedge$" and "$\vee$" take precedence over "$\to$" and "$\leftrightarrow$". Thus, $\alpha \wedge \beta \to \gamma \wedge \delta$ means $((\alpha \wedge \beta) \to (\gamma \wedge \delta))$.
3. If we mix "$\wedge$" with "$\vee$", or "$\to$" with "$\leftrightarrow$", then we will still write parentheses.
4. All binary connectives associate to the right, so that $\alpha \wedge \beta \wedge \gamma$ means $(\alpha \wedge (\beta \wedge \gamma))$, and $\alpha \to \beta \to \gamma$ means $(\alpha \to (\beta \to \gamma))$.