# A convenient 2-category of bicategories

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# CATEGORIES

$\underline{cat}$ : category of categories & functors

lower-case because it's a category not a 2-category

since $\underline{cat}$ is cartesian closed it is enriched over itself, giving

$\underline{Cat}$ : 2-category of categories, functors, & natural transformations

$$\begin{tikzcd} \Delta \arrow[r, hook, "j"'] & \underline{cat} \arrow[r, hook, bend right, "n"'] \arrow[r, phantom, "\perp"] & {[\Delta^{op}, \underline{set}]} \arrow[l, bend right] \end{tikzcd}$$

nerve

$n = \underline{cat}(j, 1)$

$\mathcal{C} \longmapsto \underline{cat}(j-, \mathcal{C})$

$$\underline{n} = \{0 < 1 < \ldots < n\}$$

$$(N\mathcal{C})_0 = \underline{cat}(\underline{0}, \mathcal{C}) = ob\, \mathcal{C}$$

$$(N\mathcal{C})_1 = cat(\underline{1}, \mathcal{C}) = mor\, \mathcal{C}$$

$$(N\mathcal{C})_2 = \underline{cat}(\underline{2}, \mathcal{C}) = \text{composable pairs}$$

etc.

$$\begin{tikzcd}
\cdots\; C_3 \arrow[r, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & C_2 \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & C_1 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, shift left=2] \arrow[r, shift right=2] & C_0 \arrow[l]
\end{tikzcd}$$

# BICATEGORIES

lax morphism $\mathcal{A} \xrightarrow{F} \mathcal{B}$

$$Fg \cdot Ff \xrightarrow{\varphi} F(gf)$$

$$1_{FA} \xrightarrow{\iota} F1_A$$

+ coherence conditions (assoc. + unit)

$\rightsquigarrow$ $\underline{\mathrm{lax}}$

(category of bicats & lax morphisms)

F <u>normal</u>: $\iota$ identity $\underline{\mathrm{nlax}}$

<u>homomorphism</u>: $\varphi, \iota$ invertible $\underline{\mathrm{hom}}$

<u>strict</u>: $\varphi, \iota$ identities $\underline{\mathrm{shom}}$

(also $\underline{\mathrm{nhom}}$)

# MONOIDAL CATEGORIES

monoidal functor $\mathcal{V} \xrightarrow{U} \mathcal{W}$

$$UA \otimes UB \xrightarrow{\varphi} U(A \otimes B)$$

$$I_{\mathcal{W}} \xrightarrow{\iota} UI_{\mathcal{V}}$$

<u>normal</u> : $\iota$ identity

<u>strong</u> : $\varphi, \iota$ invertible

<u>strict</u> : $\varphi, \iota$ identities

- monoidal functors take monoids to monoids:

$$UM \otimes UM \xrightarrow{\varphi} U(M \otimes M) \xrightarrow{U_m} UM$$

- monoidal functors $1 \longrightarrow \mathcal{V}$ are just monoids in $\mathcal{V}$

- Similarly for lax morphisms of bicategories and monads in bicats

- $(\underline{Ab}, \otimes, \mathbb{Z}) \longrightarrow (\underline{Set}, \times, 1)$

- lax morphisms do not in general preserve adjunctions the categories $\underline{lax}$, $\underline{nlax}$, $\underline{hom}$, $\underline{nhom}$

- have products
- have (stable disjoint) coproducts
- few other limits or colimits
- $\underline{nhom}$ (and $\underline{nlax}$?) cartesian closed

$\rightsquigarrow$ 3-dimensional structure
for $\underline{nhom}$

(bicats, normal homomorphisms, "enhanced" pseudonaturals, modifications)

$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ lax morphisms

an oplax natural transformation $F \xrightarrow{\alpha} G$ consists of

- $FA \xrightarrow{\alpha A} GA$ each $A \in \mathcal{A}$
  "components"
- $$\begin{tikzcd} FA \arrow[r, "\alpha A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Rightarrow_{\alpha_f}"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha B"'] & GB \end{tikzcd}$$ each $A \xrightarrow{f} B$
  "pseudonaturality"

+ conditions.

<u>pseudonatural</u> if each $\alpha f$ invertible

<u>enhanced pseudonatural</u> if also have chosen $FA \to GB$ isomorphic to $\alpha B . Ff$

... also modifications

.. the story so far:

tricategories <u>HOM</u>, <u>NHOM</u>

(not <u>LAX</u> etc!)

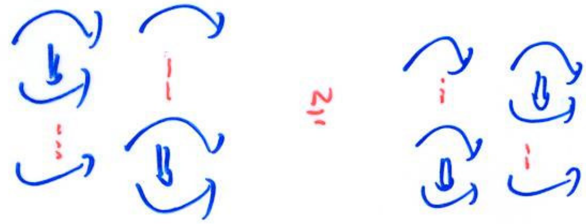
how about a 2-category?
can't just throw away 3-cells:

- composition of pseudonaturals is not strictly associative

$$\begin{tikzcd}
A \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=50, "H"'{name=H}] & B
\arrow[from=F, to=G, phantom, "\Downarrow\alpha"]
\arrow[from=G, to=H, phantom, "\Downarrow\beta"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FA \arrow[d, "\alpha A"] \\
GA \arrow[d, "\beta A"] \\
HA
\end{tikzcd}$$

- "middle-four" fails

$$\begin{tikzcd}
A \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "G"'{name=G}] & B \arrow[r, bend left=50, "F'"{name=F2}] \arrow[r, bend right=50, "G'"'{name=G2}] & C
\arrow[from=F, to=G, phantom, "\Downarrow\alpha"]
\arrow[from=F2, to=G2, phantom, "\Downarrow\beta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\beta F"'] \arrow[dr, phantom, "\cong"] & F'G \arrow[d, "\beta G"] \\
G'F \arrow[r, "G'\alpha"'] & G'G
\end{tikzcd}$$



$\cong$

$\mathcal{V}, \mathcal{W}$ monoidal categories

$\mathcal{V} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{W}$ monoidal functors

$$\begin{tikzcd} \Sigma\mathcal{V} \arrow[r, bend left=50, "\Sigma F"{name=U}] \arrow[r, bend right=50, "\Sigma G"'{name=D}] & \Sigma\mathcal{W} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$ oplax natural

- object $W \in \mathcal{W}$
- $W \otimes FX \xrightarrow{\alpha_X} GX \otimes W$

  each $X \in \mathcal{V}$

etc.

more general than monoidal natural transformations.

Consider

Identity-Component Oplax Natural transformations (icons):

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{B} \end{tikzcd}$$ lax morphisms

- $FA = GA$
- $$\begin{tikzcd} FA \arrow[r, squiggly, "Ff"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \alpha f"] & FB \arrow[d, equal] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$ + conditions

2-category $\underline{Lax}$ of bicategories, lax morphisms, and icons.

Similarly $\underline{NLax}$, $\underline{Hom}$, $\underline{NHom}$

(only 1-cells change)

# Plan for rest of talk

an (honest) advertisement for these 2-categories

with occasional breaks for mathematical content.

☺ the usual 2-categories of monoidal categories embed fully

☹ cartesian closed structure of $\underline{\text{nhom}}$ doesn't extend to $\underline{\text{NHom}}$.

☺ every object equivalent to a strict one (i.e. a 2-category)

i.e. $\forall \mathcal{B}\ \exists \mathcal{A}$ with

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B} \qquad \begin{matrix} GF \cong 1 \\ FG \cong 1 \end{matrix} \quad \text{in } \underline{\text{Hom}}$$

and $\mathcal{A}$ strict.

not every biequivalence of bicats is an equivalence in $\underline{\mathrm{Hom}}$

$\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a biequivalence

(i) each $\mathcal{A}(A,B) \longrightarrow \mathcal{B}(FA,FB)$ is an equivalence of categories

(ii) $\forall B \in \mathcal{B}$, $\exists A \in \mathcal{A}$ with $FA \simeq B$ in $\mathcal{B}$

but for equivalence in $\underline{\mathrm{Hom}}$ need (i) and

(ii') $F$ bijective on objects

$\underline{\mathrm{Hom}}$ and $\underline{\mathrm{NHom}}$ have good properties: they are bicategorically complete & cocomplete

[interrupt advertisement for mathematical content]

2-dimensional theory of monads
(Blackwell - Kelly - Power)

$\mathcal{K}$ good 2-category (say locally finitely presentable)

$T = (T, m, i)$ 2-monad (strict) on $\mathcal{K}$

$\mathcal{K} \xrightarrow{T} \mathcal{K}$ preserves filtered colimits (finitary)

$\underline{\text{T-Alg}}$ 2-category of

- strict T-algebras
- pseudo T-morphisms
- T-transformations

- $\underline{\text{T-Alg}}$ has 2-categorical limits called products, inserters, and equifiers; and so all bicategorical limits. These are formed as in $\mathcal{K}$.
- $\underline{\text{T-Alg}}$ has bicategorical colimits
- and many other good features

[return to advertisement]

☺ $\underline{\mathrm{Hom}}$ is $T\text{-}\underline{\mathrm{Alg}}$ for a finitary 2-monad $T$ on an lfp 2-category $\mathcal{K} = \underline{\mathrm{Cat}}\text{-Gph}$ (can give a presentation for $T$)

Also $\underline{\mathrm{NHom}}$ (but use different $\mathcal{K}$)

Can treat $\underline{\mathrm{Lax}}$, $\underline{\mathrm{NLax}}$ using lax $T$-morphisms.

☹ Some limits in $\underline{\mathrm{Hom}}$ are not what you might have expected.

e.g. $\mathbb{2} \pitchfork \mathcal{B}$ has same objects as $\mathcal{B}$ but $(\mathbb{2} \pitchfork \mathcal{B})(B,B') = \mathcal{B}(B,B')^{\mathbb{2}}$.

As well as every object being equivalent in $\underline{Hom}$ to a strict one, every morphism is isomorphic to a normal one. So in

full sub-2-cats of strict objects

$$
\begin{tikzcd}
N\underline{Ps} \arrow[r, hook] \arrow[d] & N\underline{Hom} \arrow[d] \\
\underline{Ps} \arrow[r, hook] & \underline{Hom}
\end{tikzcd}
$$

all four inclusions are biequivalence

[further interruption]

the inclusion $\Delta \xhookrightarrow{i} \underline{cat}$ induces the functor $\underline{cat} \xrightarrow{n} [\Delta^{op}, \underline{set}]$ sending a category $\mathcal{C}$ to its nerve $\underline{cat}(i-, \mathcal{C})$

$$\cdots \; \mathcal{C}_2 \;\overline{\underline{\Longrightarrow}}\; \mathcal{C}_1 \;\overline{\underline{\leftrightarrows}}\; \mathcal{C}_0$$

and $n$ is fully faithful

similarly $\Delta \xhookrightarrow{j} \underline{nlax}$ induces the nerve functor $n\underline{lax} \xrightarrow{n} [\Delta^{op}, \underline{set}]$

$(N\mathcal{B})_0$ = objects

$(N\mathcal{B})_1$ = morphisms

$(N\mathcal{B})_2 = \{ \cdot \nearrow \Downarrow \searrow \cdot \}$

and this $n$ is fully faithful.

[end interruption]

the inclusion $\Delta \overset{J}{\hookrightarrow} \underline{NHom}$ induces a 2-nerve 2-functor

$$\underline{NHom} \xrightarrow{N} [\Delta^{op}, Cat]$$

$$\mathcal{B} \longmapsto \underline{NHom}(J-, \mathcal{B})$$

$\mathcal{B}_0$ : discrete category of objects of $\mathcal{B}$

$\mathcal{B}_1$ : 1-cells & 2-cells of $\mathcal{B}$

$$\mathcal{B}_1 = \sum_{A,B} \mathcal{B}(A,B)$$

$\mathcal{B}_2$ : category of diagrams

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\cong"] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd}$$

with evident morphisms

and $N$ is fully faithful.

By [BKP] this 2-nerve 2-functor also has a left biadjoint.

Can characterize its image as those $\Delta^{op} \xrightarrow{X} \underline{Cat}$ with

(i) $X_0$ discrete

(ii) $X$ 3-coskeletal

(iii) "Segal maps"

$$X_n \longrightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are equivalences

(iv) $\left.\begin{array}{l} X_2 \longrightarrow Cosk_1(X)_2 \\ X_3 \longrightarrow Cosk_1(X)_3 \end{array}\right\}$ are "discrete isofibrations"