Mackey functors and Green functors

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Main References

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The Compact closed category $\mathbf{Spn}(\mathscr{E})$

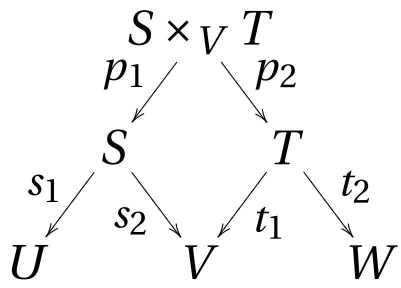
- Let $\mathscr{E}$ be a finitely complete category.
- Objects of $\mathbf{Spn}(\mathscr{E})$ are the objects of $\mathscr{E}$.
- Morphisms $U \longrightarrow V$ are the isomorphisms class of *spans* from U to V.
- A span from U to V is a diagram,

$$(s_1, S, s_2): \qquad \begin{array}{ccc} & S & \\ {}^{s_1}\swarrow & & \searrow^{s_2} \\ U & & V \end{array}$$

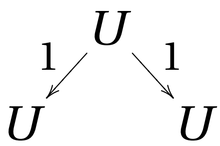
- An isomorphism of two spans $(s_1, S, s_2): U \longrightarrow V$ and $(s'_1, S', s'_2): U \longrightarrow V$ is an invertible arrow $h: S \longrightarrow S'$ such that following diagram commutes.

$$\begin{array}{ccccc} & {}^{s_1} & S & {}^{s_2} & \\ & \swarrow & & \searrow & \\ U & & \cong \downarrow h & & V \\ & \nwarrow & & \nearrow & \\ & {}_{s'_1} & S' & {}_{s'_2} & \end{array}$$

- The composite of two spans $(s_1, S, s_2) : U \longrightarrow V$ and $(t_1, T, t_2) : V \longrightarrow W$ is $(s_1 \circ p_1, S \times_V T, t_2 \circ p_2)$



- The identity span $(1, U, 1) : U \longrightarrow U$ is



- This defines the category $\mathbf{Spn}(\mathscr{E})$.

- We write $\mathbf{Spn}(\mathscr{E})(U, V) \cong [\mathscr{E} / (U \times V)]$.

- The category $\mathbf{Spn}(\mathscr{E})$ is monoidal. Tensor product

$$\mathbf{Spn}(\mathscr{E}) \times \mathbf{Spn}(\mathscr{E}) \xrightarrow{\times} \mathbf{Spn}(\mathscr{E})$$

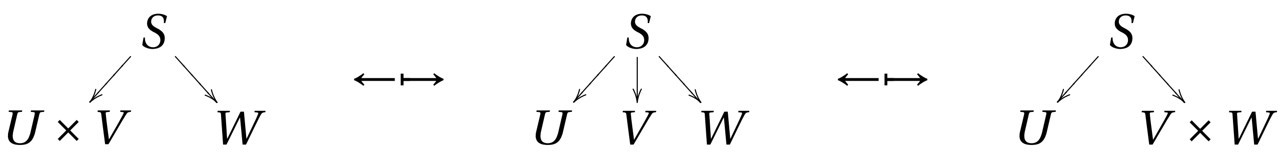
  is defined by

$$(U, V) \longmapsto U \times V$$

$$[U \xrightarrow{S} U', V \xrightarrow{T} V'] \longmapsto [U \times V \xrightarrow{S \times T} U' \times V'].$$

- It is also compact closed.
  In fact, we have the following isomorphisms:
  - ⋆ $\mathbf{Spn}(\mathscr{E})(U, V) \cong \mathbf{Spn}(\mathscr{E})(V, U)$
  - ⋆ $\mathbf{Spn}(\mathscr{E})(U \times V, W) \cong \mathbf{Spn}(\mathscr{E})(U, V \times W)$

    The second isomorphism can be shown by the following diagram



Direct sums in $\mathbf{Spn}(\mathscr{E})$

- Let $\mathscr{E}$ be a *lextensive* category.

- A category $\mathscr{E}$ is called lextensive when it has finite limits and finite coproducts such that the functor

$$\mathscr{E}/A \times \mathscr{E}/B \longrightarrow \mathscr{E}/A+B \ ; \quad \begin{array}{c} X \\ \downarrow f \\ A \end{array} \ , \ \begin{array}{c} Y \\ \downarrow g \\ B \end{array} \longmapsto \begin{array}{c} X+Y \\ \downarrow f+g \\ A+B \end{array}$$

  is an equivalance of categories for all objects A and B.

- In a lextensive category, coproducts are disjoint and universal and 0 is strictly initial. Also we have that the canonical morphism

$$(A \times B) + (A \times C) \longrightarrow A \times (B + C)$$

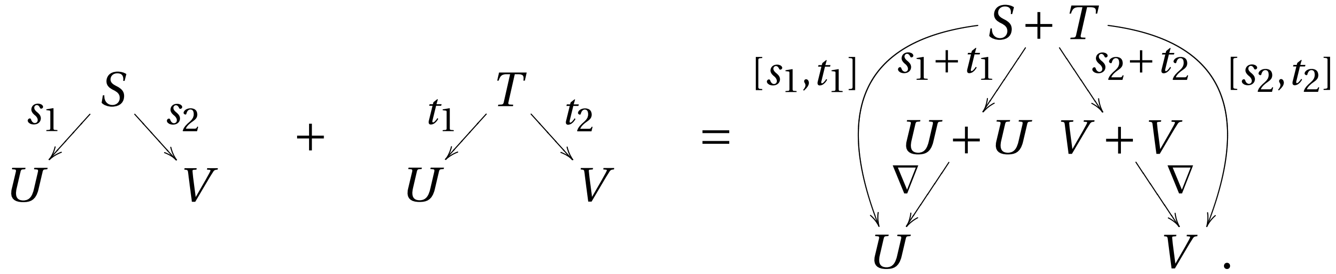
  is invertible.

- In $\mathbf{Spn}(\mathscr{E})$ the object $U+V$ is the direct sum of U and V. This can be shown as follows:

$$\begin{aligned}\mathbf{Spn}(\mathscr{E})(U+V,W) &\cong [\mathscr{E}/((U+V)\times W)]\\ &\cong [\mathscr{E}/((U\times W)+(V\times W))]\\ &\simeq [\mathscr{E}/(U\times W)]\times[\mathscr{E}/(V\times W)]\\ &\cong \mathbf{Spn}(\mathscr{E})(U,W)\times \mathbf{Spn}(\mathscr{E})(V,W);\end{aligned}$$

  and so $\mathbf{Spn}(\mathscr{E})(W,U+V)\cong \mathbf{Spn}(\mathscr{E})(W,U)\times \mathbf{Spn}(\mathscr{E})(W,V)$.

- The addition of two spans $(s_1,S,s_2):U\longrightarrow V$ and $(t_1,T,t_2):U\longrightarrow V$ is given by



- $\mathbf{Spn}(\mathscr{E})$ is a monoidal commutative-monoid-enriched category.

Mackey functors on $\mathscr{E}$

- A Mackey functor

$$M : \mathscr{E} \longrightarrow \mathbf{Mod}_k$$

consists of two functors $M^* : (\mathscr{E})^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k$, $M_* : \mathscr{E} \longrightarrow \mathbf{Mod}_k$ such that

⋆ $M_*(U) = M^*(U) \quad (= M(U))$ for all U in $\mathscr{E}$.

⋆ For all pullbacks

$$\begin{array}{ccc} P & \xrightarrow{q} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ U & \xrightarrow[r]{} & W \end{array} ,$$

in $\mathscr{E}$, the square(Mackey square)

$$\begin{array}{ccc} M(P) & \xrightarrow{M_*(q)} & M(V) \\ {\scriptstyle M^*(p)}\uparrow & & \uparrow{\scriptstyle M^*(s)} \\ M(U) & \xrightarrow[M_*(r)]{} & M(W) \end{array}$$

commutes.

⋆ For all coproduct diagrams

$$U \xrightarrow{i} U+V \xleftarrow{j} V$$

in $\mathscr{E}$, the diagram

$$M(U) \underset{M_* i}{\overset{M^* i}{\rightleftarrows}} M(U+V) \underset{M_* j}{\overset{M^* j}{\rightleftarrows}} M(V)$$

is a direct sum situation in $\mathbf{Mod}_k$.
(This implies $M(U+V) \cong M(U) \oplus M(V)$.)

- A morphism $\theta : M \longrightarrow N$ of Mackey functors is a family $\theta_U : M(U) \longrightarrow N(U)$ of morphisms for U in $\mathscr{E}$. This gives natural transformations $\theta_* : M_* \longrightarrow N_*$ and $\theta^* : M^* \longrightarrow N^*$.

- **Proposition:** (Due to Lindner)

  The category $\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k)$ *of Mackey functors is equivalent to* $[\mathbf{Spn}(\mathscr{E}), \mathbf{Mod}_k]_+$ *of the category of coproduct-preserving functors. That is:*

$$\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k) \simeq [\mathbf{Spn}(\mathscr{E}), \mathbf{Mod}_k]_+$$

- **Proof**

  Let $M:\mathscr{E}\longrightarrow\mathbf{Mod}_k$ be a Mackey functor.

  We Define a morphism $M:\mathbf{Spn}(\mathscr{E})\longrightarrow\mathbf{Mod}_k$ by $M(U)=M_*(U)=M^*(U)$ and

$$M\left(\begin{array}{ccc} & S & \\ {}^{s_1}\swarrow & & \searrow^{s_2} \\ U & & V \end{array}\right) \quad = \quad \left(M(U)\xrightarrow{M^*(s_1)}M(S)\xrightarrow{M_*(s_2)}M(V)\right).$$

  Conversely, let $M:\mathbf{Spn}(\mathscr{E})\longrightarrow\mathbf{Mod}_k$ be a functor.

  Then we can define two functors M_* and M^*,

$$\begin{array}{ccccc} \mathscr{E} & \xrightarrow{(-)_*} & \mathbf{Spn}(\mathscr{E}) & \xrightarrow{M} & \mathbf{Mod}_k \\ & \nearrow_{(-)^*} & & & \\ \mathscr{E}^{\mathrm{op}} & & & & \end{array},$$

  by putting $M_*=M\circ(-)_*$ and $M^*=M\circ(-)^*$.

- Denote $\mathbf{Mky}=\mathbf{Mky}(\mathscr{E},\mathbf{Mod}_k)\simeq[\mathbf{Spn}(\mathscr{E}),\mathbf{Mod}_k]_+$

Tensor products in $\mathbf{Mky}$

- Let $\mathscr{T}$ be general compact closed, commutative-monoid-enriched category. (The main example is $\mathbf{Spn}(\mathscr{E})$).

- The tensor product of Mackey functors can be defined by convolution in $[\mathscr{T}, \mathbf{Mod}_k]_+$ since $\mathscr{T}$ is a monoidal category.

- The tensor product is:

$$\begin{aligned}(M * N)(Z) &= \int^{X,Y} \mathscr{T}(X \otimes Y, Z) \otimes M(X) \otimes_k N(Y) \\ &\cong \int^{X,Y} \mathscr{T}(Y, X^* \otimes Z) \otimes M(X) \otimes_k N(Y) \\ &\cong \int^{X} M(X) \otimes_k N(X^* \otimes Z) \\ &\cong \int^{Y} M(Z \otimes Y^*) \otimes_k N(Y).\end{aligned}$$

Hom functor and Burnside functor

- Let $\mathscr{T} = \mathbf{Spn}(\mathscr{E})$ where $\mathscr{E}$ the category of finite G-sets for the finite group G.

- The Hom Mackey functor is

$$\mathrm{Hom}(M,N)(V) = \mathbf{Mky}(M(V\times -),N),$$

  functorially in V.

$$\frac{\dfrac{\dfrac{\dfrac{(L*M)(U) \longrightarrow N(U)}{L(V)\otimes_k M(V\times U)\longrightarrow N(U)}}{L(V)\longrightarrow \mathrm{Hom}_k(M(V\times U),N(U))}}{L(V)\longrightarrow \displaystyle\int_U \mathrm{Hom}_k(M(V\times U),N(U))}}{L(V)\longrightarrow \mathbf{Mky}(M(V\times -),N)}$$

- The Burnside functor $J:\mathscr{E}\longrightarrow \mathbf{Mod}_k$ has value at U equal to the free k-module on $\mathbf{Spn}(\mathscr{E})(1,U) = [\mathscr{E}/U]$.

Green functors on $\mathscr{E}$

- A Green functor $A : \mathscr{E} \longrightarrow \mathbf{Mod}_k$ is
  - ⋆ A Mackey functor (that is, a coproduct preserving functor $A : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$) with
  - ⋆ A monoidal structure made up of a natural transformation

    $$\mu : A(U) \otimes_k A(V) \longrightarrow A(U \times V),$$

    for which we use the notation $\mu(a \otimes b) = a.b$ for $a \in A(U)$, $b \in A(V)$, and
  - ⋆ a morphism $\eta : k \longrightarrow A(1)$ such that $\eta(1) = 1$.

- Green functors are the monoids in $\mathbf{Mky}$.

- The Burnside functor J and $\mathrm{Hom}(A, A)$ are monoids in $\mathbf{Mky}$ and therefore are Green functors.

Finite dimensional Mackey functors

- Let $\mathbf{Mky}_{\mathsf{fin}}$ be the category of finite-dimensional-valued Mackey functors. Define $\mathbf{Mky}_{\mathsf{fin}} = [\mathscr{T}, \mathbf{Vect}_{\mathsf{fin}}]_+$.

- Let $\mathscr{C}$ be the full sub-category of $\mathscr{T}$ consisting of the connected G-sets. The functor $F : \mathscr{C} \to \mathscr{T}$ is a fully faithful functor. The category $\mathscr{C}$ has finitely many objects. Each $X \in \mathscr{T}$ can be written as
$$X \cong \bigoplus_{i=1}^{n} F(U_i).$$

- We can show that
$$M(X) \cong \int^{C} \mathscr{T}(C, X) \otimes M(C).$$

- **Lemma** *If S is a commutative monoid generated by a finite set of elements $s_1, \ldots, s_m$ and V is a vector space with basis $v_1, \ldots, v_n$ then $S \otimes V$ is a finite dimensional vector space.*

- The tensor product $M, N \in \mathbf{Mky}_{\mathsf{fin}}$ is finite dimensional.

$$(M * N)(Z) = \int^{X,Y} \mathscr{T}(X \times Y, Z) \otimes M(X) \otimes_k N(Y)$$

$$\cong \int^{X,Y,C,D} \mathscr{T}(X \times Y, Z) \otimes \mathscr{T}(C, X) \otimes \mathscr{T}(D, Y) \otimes M(C) \otimes_k N(D)$$

$$\cong \int^{C,D} \mathscr{T}(C \times D, Z) \otimes M(C) \otimes_k N(D).$$

  Here $\mathscr{T}(C \times D, Z)$ is finitely generated as a commutative monoid and $M(C)$ and $N(D)$ are finite dimensional.

- The promonoidal structure on $\mathbf{Mky}_{\mathsf{fin}}$ for the Mackey functors M, N, and L is

$$\begin{aligned} P(M, N; L) &= \mathsf{Nat}_{X,Y,Z}(\mathscr{T}(X \times Y, Z) \otimes M(X) \otimes_k N(Y), L(Z)) \\ &\cong \mathsf{Nat}_{X,Y}(M(X) \otimes_k N(Y), L(X \times Y)) \\ &\cong \mathsf{Nat}_{X,Z}(M(X) \otimes_k N(X^* \times W), L(Z)) \\ &\cong \mathsf{Nat}_{Y,Z}(M(Z \times Y^*) \otimes_k N(Y), L(Z)). \end{aligned}$$

  Therefore the category $\mathbf{Mky}_{\mathsf{fin}}$ is monoidal for the promonoidal structure; that is,

$$P(M, N; L) \cong \mathbf{Mky}_{\mathsf{fin}}(M * N, L).$$

- A monoidal category $\mathcal{V}$ is $*$-*autonomous* when it is equipped with an equivalence $S : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}$ of categories and

$$\mathcal{V}(A \otimes B, SC) \cong \mathcal{V}(B \otimes C, S^{-1}A).$$

In the category $\mathbf{Mky}_{\mathrm{fin}}$ we can write $(SA)X = A(X^*)^*$.

- **Theorem** The category $\mathbf{Mky}_{\mathrm{fin}}$ is $*$-autonomous.

- **Proof** The promonoidal structure $P(M, N; SL)$ for the category $\mathbf{Mky}_{\mathrm{fin}}$ can be written as:

$$\begin{aligned} P(M, N; SL) &= \mathrm{Nat}_{X,Y}(M(X) \otimes_k N(Y), L(X^* \times Y^*)^*) \\ &\cong \mathrm{Nat}_{X,Y}(N(Y) \otimes L(X^* \times Y^*), (MX)^*) \\ &\cong \mathrm{Nat}_{X,Y}(N(Y) \otimes L(X \times Y^*), M^*(X)) \\ &\cong P(N, L; M^*). \end{aligned}$$

- There is a possibility that for a class of finite G (including the cyclic ones) that $\mathbf{Mky}_{\mathrm{fin}}$ could be compact (autonomous).

Modules over a Green functor

- A *module* M over A, or A-*module* means A acts on M via the convolution.

- The monoidal action $\alpha^M : A * M \longrightarrow M$ is defined by a family of morphisms

$$\bar{\alpha}^M_{U,V} : A(U) \otimes_k M(V) \longrightarrow M(U \times V),$$

where we put $\bar{\alpha}^M_{U,V}(a \otimes m) = a.m$ for $a \in A(U)$, $m \in M(V)$.

- If M is an A-module, then M is of course a Mackey functor.

- Let $\mathbf{Mod}(A)$ denote the category of left A-modules. Objects are A-modules and morphisms are A-module morphisms.

Morita equivalence of Green functors

- For any good monoidal category $\mathscr{W}$ we have the monoidal bicategory $\mathbf{Mod}(\mathscr{W})$. We spell this out in the case $\mathscr{W} = \mathbf{Mky}$:
  - ⋆ Objects are monoids A in $\mathscr{W}$ (i.e. $A : \mathscr{E} \longrightarrow \mathbf{Mod}_k$ are Green functors)
  - ⋆ morphisms are modules $M : A \longrightarrow\!\!\!\!\!\!+\;\, B$ with a two-sided action $\alpha^M : A * M * B \longrightarrow M$, that is

$$\alpha^M_{U,V,W} : A(U) \otimes_k M(V) \otimes_k B(W) \longrightarrow M(U \times V \times W)$$

  - ⋆ Composition of morphisms $M : A \longrightarrow\!\!\!\!\!\!+\;\, B$ and $N : B \longrightarrow\!\!\!\!\!\!+\;\, C$ is $M *_B N$ and it is defined via the coequalizer

$$M * B * N \underset{1_M * \alpha^N}{\overset{\alpha^M * 1_N}{\rightrightarrows}} M * N \longrightarrow M *_B N = N \circ M$$

that is,

$$(M *_B N)(U) = \sum_{X,Y} \mathbf{Spn}(\mathscr{E})(X \times Y, U) \otimes M(X) \otimes_k N(Y) / \sim_B .$$

  - ⋆ The identity morphism is given by $A : A \longrightarrow\!\!\!\!\!\!+\;\, A$.

⋆ The 2-cells are natural transformations $\theta : M \longrightarrow M'$ which respect the actions

$$\begin{array}{ccc} A(U)\otimes_k M(V)\otimes_k B(W) & \xrightarrow{\bar{\alpha}^{M}_{U,V,W}} & M(U\times V\times W) \\ {\scriptstyle 1\otimes_k\theta_V\otimes_k 1}\downarrow & & \downarrow{\scriptstyle \theta_{U\times V\times W}} \\ A(U)\otimes_k M'(V)\otimes_k B(W) & \xrightarrow[\bar{\alpha}^{M'}_{U,V,W}]{} & M'(U\times V\times W)\ . \end{array}$$

⋆ The tensor product on $\mathbf{Mod}(\mathscr{W})$ is the convolution $*$. The tensor product of the modules $M : A \longrightarrow\!\!\!\!\!\!+\ B$ and $N : C \longrightarrow\!\!\!\!\!\!+\ D$ is $M * N : A * C \longrightarrow\!\!\!\!\!\!+\ B * D$.

- **Definition:** Green functors A and B are said to be *Morita equivalent* when they are equivalent in $\mathbf{Mod}(\mathscr{W})$.

- **Proposition:** *If A and B are equivalent in $\mathbf{Mod}(\mathscr{W})$ then $\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)$ as categories.*

- **Proof** $\mathbf{Mod}(\mathscr{W})(-, J) : \mathbf{Mod}(\mathscr{W})^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ is a pseudo functor and so takes equivalences to equivalences.

- Now we enriched $\mathbf{Mod}(A)$ to a $\mathscr{W}$-category $\mathscr{P}A$.

- The $\mathscr{W}$-category $\mathscr{P}A$ has underlying category $\mathbf{Mod}(\mathscr{W})(J, A)$. The objects are modules $M : J \relbar\joinrel\mapstochar\joinrel\rightarrow A$ and homs are defined by the following equalizer.

$$\mathbf{Mod}(A)(M,N) \longrightarrow \mathrm{Hom}(M,N) \xrightarrow{\mathrm{Hom}(\alpha^M,1)} \mathrm{Hom}(A * M, N)$$

$$\mathrm{Hom}(M,N) \xrightarrow{(A*-)} \mathrm{Hom}(A * M, A * N) \xrightarrow{\mathrm{Hom}(1,\alpha^N)} \mathrm{Hom}(A * M, N)$$

- The Cauchy completion $\mathscr{Q}A$ of A is the full sub-$\mathscr{W}$-category of $\mathscr{P}A$ consisting of the modules $M : J \relbar\joinrel\mapstochar\joinrel\rightarrow A$ with right adjoints $N : A \relbar\joinrel\mapstochar\joinrel\rightarrow J$.

- Recall the classical result from enriched category theory:

- **Theorem:** *Green functors A and B are Morita equivalent if and only if $\mathscr{Q}A \simeq \mathscr{Q}B$ as $\mathscr{W}$-categories.*

- In our case this theorem can be applied via our characterization of the Cauchy completion.

- **Theorem:** *The Cauchy completion* $\mathcal{Q}A$ *of the monoid* A *in* $\mathbf{Mky}$ *consists of all the retracts of modules of the form*

$$\bigoplus_{i=1}^{k} A(Y_i \times -)$$

for some $Y_i \in \mathbf{Spn}(\mathscr{E})$.

Some applications of Mackey functors

- Let $\mathbf{Rep}(G)$ be the category of k-linear representations of the finite group G.
  The category $\mathbf{Mky}(G)$ provides an extension of ordinary representation theory.
  For example, $\mathbf{Rep}(G)$ can be regarded as a full reflective monoidal sub-category of $\mathbf{Mky}(G)$.

- Mackey functors provide relations between λ- and μ-invariants in Iwasawa theory and between Mordell-Weil groups, Shafarevich-Tate groups, Selmer group and zeta functions of elliptic curves
  (W. Bley and R. Boltje, *Cohomological Mackey functors in number theory*, J. Number Theory **105** (2004), 1–37).