SPANS FOR 2-CATEGORIES

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SPAN

$\underline{A}$ CAT W. PB $\rightsquigarrow$ $\underline{\mathrm{SPAN}}(\underline{A})$

$A \xleftarrow{s} S \xrightarrow{s'} A'$

- BICATEGORY
- $\underline{A} \xrightarrow{(\ )_*} \underline{\mathrm{SPAN}}(\underline{A})$ STRONG MORPH
- f_* HAS RT ADJOINT f^*
- BECK CONDITION

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "PB"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
P \arrow[r, "p_*"] \arrow[dr, phantom, "="] & A \\
B \arrow[u, "q^*"] \arrow[r, "g_*"'] & C \arrow[u, "f^*"']
\end{tikzcd}
$$

THEOREM (DPP)

$$
\begin{tikzcd}
\underline{A} \arrow[r, "(\ )_*"] \arrow[dr, "\forall F"'] & \underline{\mathrm{SPAN}}(\underline{A}) \arrow[d, dashed, "\exists!\ \Phi"] \\
& \underline{\underline{B}}
\end{tikzcd}
$$

(PB)

SINISTER

BICAT

OPLAX $\begin{cases} \Phi(ST) \to \Phi(S)\Phi(T) \\ \Phi(I_A) \to 1_{\Phi A} \end{cases}$

NORMAL

PRES f_*S & Tf^*

(PARANORMAL)

- UNIV PROP DOES NOT REFER TO PB
  → DROP IT!
- BUT NEED IT FOR COMPOSITION IN SPAN
  → DROP IT TOO!!
  (Φ DOESN'T PRESERVE IT.)
- NEED SOME TRACE OF COMPOSITION TO EXPRESS OPLAX, PARANORMAL

SEVERAL OBJECT MULTICATEGORIES

(HERMIDA, LEINSTER, LAMBEK)

CELLS:

$$\begin{tikzcd} A \arrow[rrrr] \arrow[dr] & & \arrow[d, phantom, "\alpha \Downarrow"] & & B \\ & A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[ur, dotted, no head] & \end{tikzcd}$$

CAN DEFINE IDENTITIES & COMPOSITES BY UNIVERSAL PROPERTIES

CAN'T EXPRESS ADJOINTS!

$\mathbb{S}\mathrm{PAN}(\underline{A})$

FOR $\underline{A}$ W. PB, THE (WEAK) DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}(\underline{A})$ HAS CELLS

$$\begin{tikzcd} A \arrow[d] & S \arrow[l] \arrow[r] \arrow[d] & A' \arrow[d] \\ B & T \arrow[l] \arrow[r] & B' \end{tikzcd}$$

- BETTER THAN $\underline{\mathrm{SPAN}}(\underline{A})$ BECAUSE IT REMEMBERS THE ARROWS OF $\underline{A}$ (cf SHULMAN)

- IN A DOUBLE CATEGORY THE NOTION OF ADJOINTNESS FACTORS INTO TWO SIMPLER & DUAL NOTIONS:

COMPANION
+
CONJOINT

COMPANIONS

(BROWN - CONNECTIONS, GP)

IN A DOUBLE CAT $\mathbb{D}$ $f : A \to B$ (HORIZ)

AND $g : A \to B$ (VERT) ARE **COMPANIONS**

IF THERE ARE CELLS

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \& \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd}$$

S.T. $\quad \varepsilon\eta = id_f \quad \& \quad \varepsilon \cdot \eta = 1_g$

CONJOINTS (DUAL)

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[d, "g"] \\ B \arrow[r, equal] & B \end{tikzcd} \quad \& \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "g"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\ B \arrow[r, "u"'] & A \end{tikzcd}$$

S.T $\quad \alpha\beta = id_u \quad \& \quad \beta \cdot \alpha = id_g$

ADJOINTS

ADJOINTNESS IS THE USUAL IN THE BICATEGORY $\underline{\underline{\mathrm{Hor}}}(\mathbb{D})$ OF HORIZONTAL ARROWS AND **SPECIAL CELLS**

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

<u>PROP</u>: ① COMPANIONS (CONJOINTS) ARE UNIQUE UP TO SPECIAL ISO

② COMPANIONS (CONJOINTS) COMPOSE

③ f COMPANION TO g & u CONJOINT TO g $\Rightarrow$ f LEFT ADJOINT TO u.

<u>EX</u>: QUINTETS $\mathbb{Q}\underline{\underline{A}}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Ex: In $\mathbb{S}\mathrm{PAN}(\underline{A})$

$$
\begin{tikzcd}
A & A \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, equal] & A \arrow[d, "f"] \\
A \arrow[u, equal] & A \arrow[l, "1"] \arrow[r, "f"'] & B
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\
B & B \arrow[l, "1"] \arrow[r, "1"'] & B
\end{tikzcd}
$$

MAKE f_* COMPANION TO f

SIMILARLY f^* CONJOINT TO f

$\Rightarrow f_* \dashv f^*$

UNIVERSAL PROPERTY

$$
\begin{tikzcd}
\underline{A} \arrow[r, "\text{VERT}"] \arrow[dr, "\forall F"', "\text{GREGARIOUS}"'] & \mathbb{S}\mathrm{PAN}(\underline{A}) \arrow[d, dashed, "\exists!\ \Phi"] \\
& \mathbb{D}
\end{tikzcd}
$$

OPLAX PARANORMAL (PRES 1, $f_* S$, $T f^*$)

DOUBLE

OPLAX DOUBLE CATEGORIES

(LEINSTER'S fc-MULTICATEGORIES)

WHEN $\underline{A}$ DOESN'T HAVE PB SPANS DON'T COMPOSE — IT IS NATURAL TO CONSIDER OPLAX DOUBLE CATEGORIES:

HAVE: OBJECTS, HORIZONTAL ARROWS

VERTICAL ARROWS - FORM A CAT

CELLS

$$\begin{tikzcd}
 & A \arrow[rrr, "f"] \arrow[dl, "v"'] & & \alpha & A' \arrow[dr, "v'"] & \\
B_0 \arrow[r, "g_1"'] & B_1 \arrow[r, "g_2"'] & B_2 \arrow[r, "g_3"'] & \cdots \arrow[rr, "g_n"'] & & B_n
\end{tikzcd}$$

CELLS COMPOSE VERTICALLY IN THE OBVIOUS WAY ($(\beta_n, \dots, \beta_2, \beta_1) \cdot \alpha$)

ASSOCIATIVE + UNITARY

HORIZONTAL COMPOSITES

SAY THAT THE COMPOSITE OF $g_1, \ldots, g_m$ **EXISTS** (OR IS **STRONGLY REPRESENTABLE**)

IF THERE IS A CELL

$$\begin{tikzcd}
B_0 \arrow[rrrr, "g_m \cdots g_2 g_1"] \arrow[d, equal] \arrow[drrrr, phantom, "\iota"] & & & & B_m \arrow[d, equal] \\
B_0 \arrow[r, "g_1"'] & B_1 \arrow[r, "g_2"'] & B_2 \arrow[r, "g_3"'] & \cdots \arrow[r, "g_m"'] & B_m
\end{tikzcd}$$

IF EVERY CELL

$$\begin{tikzcd}
& & & A \arrow[rr] \arrow[dlll] \arrow[drr, phantom, "\alpha"] & & A' \arrow[drrrr] & & & & \\
X_k \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_m \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_l
\end{tikzcd}$$

FACTORS UNIQUELY AS

$$\begin{tikzcd}
& & & A \arrow[rrr] \arrow[dlll] \arrow[drrr, phantom, "\bar{\alpha}"] & & & A' \arrow[drrr] & & & \\
X_k \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "id"] & B_0 \arrow[rrr] \arrow[d, equal] \arrow[drrr, phantom, "\iota"] & & & B_m \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "id"] & Y_1 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & Y_l \arrow[d, equal] \\
X_k \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_m \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_l
\end{tikzcd}$$

SPAN($\underline{A}$)

CELLS

$$
\begin{tikzcd}
 & A \arrow[dl] & & S \arrow[ll] \arrow[rr] \arrow[dll] \arrow[d] \arrow[drr] & & A' \arrow[dr] & \\
B_0 & T_1 \arrow[l] \arrow[r] & B_1 & T_2 \arrow[l] \arrow[r] & \cdots & T_m \arrow[l] \arrow[r] & B_m
\end{tikzcd}
$$

- THE COMPOSITE OF $T_1 \cdots T_m$ EXISTS IFF THE GENERALIZED PB (= LIM) EXISTS.

- IDENTITIES EXIST $\quad A \xleftarrow{1} A \xrightarrow{1} A$
- ALSO $\quad f_* S \quad$ AND $\quad T f^* \quad$ EXIST

$$
\begin{tikzcd}[column sep=small]
 & S \arrow[dl] \arrow[dr] & & B \arrow[dl, equal] \arrow[dr, "f"] & \\
A & & B & & C
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & B \arrow[dl, "f"'] \arrow[dr, equal] & & T \arrow[dl] \arrow[dr] & \\
C & & B & & A
\end{tikzcd}
$$

PARANORMAL

AN OPLAX DOUBLE CATEGORY IS **NORMAL** IF ALL IDENTITIES EXIST

COMPANIONS & CONJOINTS MAKE SENSE — NORMAL MORPHISMS PRESERVE THEM

IT IS **PARANORMAL** IF

(1) IT IS NORMAL

(2) EVERY VERTICAL v HAS A COMPANION v_* AND A CONJOINT v^*

(3) ALL COMPOSITES $v_* h$ & $k v^*$ EXIST.

THEOREM $\mathcal{S}$PAN : $\underline{\underline{\text{CAT}}} \longrightarrow \underline{\underline{\text{PARA}}}$ IS 2-LAJ TO VERT : $\underline{\underline{\text{PARA}}} \longrightarrow \underline{\underline{\text{CAT}}}$.

2-CATEGORIES

$\underline{\underline{A}}$ 2-CATEGORY

WANT THE FREE PARANORMAL DOUBLE CATEGORY GENERATED BY $\underline{\underline{A}}$ VERTICALLY

$$\left.\begin{tikzcd} \underline{\underline{A}} \arrow[r, "\text{VERT}"] \arrow[dr, "\forall F"', "\text{VERT}"'] & \mathbb{S}\mathrm{PAN}(\underline{\underline{A}}) \arrow[d, dashed, "\exists!\,\Phi"] \\ & \mathbb{D} \end{tikzcd}\right\}\text{PARANORMAL}$$

THE UNDERLYING 2-CATEGORY OF A PDC

$\underline{\mathrm{VERT}}(\mathbb{D})$ HAS SAME OBJECTS AS $\mathbb{D}$,

THE ARROWS ARE THE VERTICAL ARROWS OF $\mathbb{D}$,

THE 2-CELLS ARE SPECIAL CELLS

$$\frac{\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}}{\begin{tikzcd} A \arrow[r, equal] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "u"] \\ B \arrow[r, equal] & B \end{tikzcd}}$$

PRECELLS

CONSTRUCT $\mathbb{S}\mathrm{pan}'(\underline{\underline{A}})$

OBJECTS - THOSE OF $\underline{\underline{A}}$

VERTICAL ARROWS - ARROWS OF $\underline{\underline{A}}$

HORIZONTAL ARROWS - SPANS IN $\underline{\underline{A}}$

PRECELLS

$$
\begin{tikzcd}
 & A \arrow[dl, "x"'] \arrow[d, phantom, "\Leftarrow\alpha_0"] & & S \arrow[ll, "s"'] \arrow[rr, "s'"] \arrow[dll, "x_1"'] \arrow[dl, phantom, "\Leftarrow\alpha_1"] \arrow[d, "x_2"] \arrow[d, phantom, "\cdots"] \arrow[drr, "x_n"] \arrow[dr, phantom, "\Leftarrow\alpha_n"] & & A' \arrow[dr, "x'"] & \\
B_0 & T_1 \arrow[l, "t_1"] \arrow[r, "t_1'"'] & B_1 & T_2 \arrow[l, "t_2"] \arrow[r, "t_2'"'] & \cdots & T_n \arrow[l, "t_n"] \arrow[r, "t_n'"'] & B_n
\end{tikzcd}
$$

WITH OBVIOUS COMPOSITION OF PRECELLS

$\mathbb{S}\mathrm{pan}'(\underline{\underline{A}})$ IS AN OPLAX DOUB CAT.

BUT NOT NORMAL !

$$
\begin{tikzcd}
A \arrow[d] \arrow[dr, phantom, "\Leftarrow\alpha_0"] & S \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Leftarrow\alpha_1"] & A' \arrow[d] \\
B \arrow[dr] & B \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow \mathrm{id}"] & B \arrow[dl] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[dr] & S \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\alpha_0\alpha_1"] & A' \arrow[dl] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d] \arrow[dr, phantom, "\Leftarrow\mathrm{id}"] & S \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Leftarrow\alpha_0\alpha_1"] & A' \arrow[d] \\
B \arrow[dr] & B \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow \mathrm{id}"] & B \arrow[dl] \\
 & B &
\end{tikzcd}
$$

EQUIVALENCE RELATION

$$
\begin{tikzcd}
 & A \arrow[dl, "\pi"'] & & S \arrow[ll, "s"'] \arrow[rr, "s'"] \arrow[dl, "y_1"'] \arrow[d, "y_2 \dots y_n"] \arrow[drrr] & & A' \arrow[dr, "\pi'"] & \\
B_0 \arrow[rr, phantom, "\Leftarrow \beta_0"] & T_1 \arrow[l, "t_1"] \arrow[r, "t_1'"'] & B_1 \arrow[r, phantom, "\Leftarrow \beta_1"] & T_2 \arrow[l, "t_2"] \arrow[r, "t_2'"'] & \cdots & T_n \arrow[l, "t_n"] \arrow[r, "t_n'"'] \arrow[u, phantom, "\Leftarrow \beta_n"] & B_n
\end{tikzcd}
$$

ANOTHER PRECELL WITH SAME BOUNDARY

IDENTIFY IT WITH PREVIOUS ONE IF

THERE ARE $\lambda_i : x_i \to y_i \qquad i = 1, \dots, n$

S.T.

$$
\begin{tikzcd}
 & S \arrow[dl, bend right, "y_i"'] \arrow[dl, "x_i"] \arrow[dr, "x_{i+1}"] & \\
T_i \arrow[dr] \arrow[rr, phantom, "\Leftarrow \alpha_i"] & & T_{i+1} \arrow[dl] \\
 & B_i &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S \arrow[dl, "y_i"'] \arrow[dr, "y_{i+1}"'] \arrow[dr, bend left, "x_{i+1}"] & \\
T_i \arrow[dr] \arrow[rr, phantom, "\Leftarrow \beta_i"] & & T_{i+1} \arrow[dl] \\
 & B_i &
\end{tikzcd}
$$

(END PTS FIXED — $\lambda_0 = id$, $\lambda_{n+1} = id$)

TAKE EQUIVALENCE RELATION GENERATED

DENOTE EQUIV CLASS BY $\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n$

SPAN($\underline{\underline{A}}$)

THEOREM TAKING CELLS TO BE EQUIVALENCE CLASSES OF DIAGRAMS AS ABOVE GIVES US AN OPLAX DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}(\underline{\underline{A}})$

- $\mathbb{S}\mathrm{PAN}(\underline{\underline{A}})$ IS PARANORMAL
- $\mathbb{S}\mathrm{PAN} : \underline{\underline{2\text{-}\mathrm{CAT}}} \longrightarrow \underline{\underline{\mathrm{PARA}}}$ IS LEFT 2-ADJOINT TO $\mathrm{VERT} : \underline{\underline{\mathrm{PARA}}} \longrightarrow \underline{\underline{2\text{-}\mathrm{CAT}}}$

- $\mathbb{Q}(\underline{\underline{A}}) \xrightarrow{(\ )_*} \mathbb{S}\mathrm{PAN}(\underline{\underline{A}})$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "h"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A \arrow[d, "g"'] \arrow[dr, phantom, "="] & A \arrow[l, "1_A"'] \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[d, "h"] \\
C \arrow[r, "1_C"'] & C \arrow[r, "k"'] & D
\end{tikzcd}
$$

IS LOCALLY FULL & FAITHFUL ∎

PROPERTIES

- $$\begin{tikzcd} A \arrow[d, "x"'] & S \arrow[l, "s"'] \arrow[r, "s'"] \arrow[d, "xs"] \arrow[l, phantom, shift right=4, "\Leftarrow \alpha_0"] \arrow[r, phantom, shift right=4, "\Leftarrow \alpha_1"] & A' \arrow[d, "x'"] \\ B & B \arrow[l, "1"] \arrow[r, "f"'] & B' \end{tikzcd} = \begin{tikzcd} A \arrow[d, "x"'] & S \arrow[l, "s"'] \arrow[r, "s'"] \arrow[d, "xs"] \arrow[l, phantom, shift right=4, "="] \arrow[r, phantom, shift right=4, "\Leftarrow \alpha_0\alpha_1"] & A' \arrow[d, "x'"] \\ B & B \arrow[l, "1"] \arrow[r, "f"'] & B' \end{tikzcd}$$

- **LEMMA**

$$\begin{tikzcd} A \arrow[d, equal] \arrow[r, phantom, shift right=4, "="] & S \arrow[l] \arrow[d, "\ell"] \arrow[r, phantom, "\dashv"] & S \arrow[r] \arrow[r, phantom, shift right=4, "="] & A' \arrow[d, equal] \\ A & T \arrow[l] & T \arrow[u, "r"'] \arrow[r] & A' \end{tikzcd}$$

$$\Rightarrow (A \leftarrow S \rightarrow A') \cong (A \leftarrow T \rightarrow A') \blacksquare$$

- **COROLLARY** $f \dashv u$

$$\Rightarrow (A \xleftarrow{f} S \xrightarrow{g} B) \cong (A \xleftarrow{1_A} A \xrightarrow{gu} B) \blacksquare$$

COMMA OBJECTS

THEOREM IF $\underline{\underline{A}}$ HAS COMMA OBJECTS THEN COMPOSITES EXIST IN $\mathrm{Span}(\underline{\underline{A}})$. $\mathrm{Span}(\underline{\underline{A}})$ SATISFIES THE BECK CONDITION:

$$\begin{tikzcd} K \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\swarrow"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \text{ COMMA } \Rightarrow \begin{tikzcd} K \arrow[r, "q_*"] \arrow[d, phantom, ""] & B \\ A \arrow[u, "p^*"] \arrow[r, "f_*"'] \arrow[ur, phantom, "\cong"] & C \arrow[u, "g^*"'] \end{tikzcd} \quad \blacksquare$$

REMARK 1: COMPOSING SPANS WITH COMMA OBJECTS IS ASSOCIATIVE WITHOUT EQUIVALENCE RELATION **BUT** NOT UNITARY

REMARK 2: COMPOSITES IN $\mathrm{Span}(\underline{\underline{A}})$ $\not\Rightarrow$ COMMA OBJECTS IN $\underline{\underline{A}}$

PROPOSITION IF $B \leftarrow T \rightarrow B'$ IS A BIFIBRATION, EVERY CELL

$$
\begin{tikzcd}
A \arrow[d] \arrow[r, phantom, "\Leftarrow\alpha_0"', shift right=4] & S \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, "\Leftarrow\alpha_1"', shift right=4] & A' \arrow[d] \\
B & T \arrow[l] \arrow[r] & B'
\end{tikzcd}
$$

IS EQUAL TO ONE (NOT UNIQUE) IN WHICH α_0, α_1 ARE IDENTITIES. ∎

PROPOSITION IF $\underline{\underline{A}}$ HAS COMMA OBJECTS

(1) EVERY SPAN IS ISOMORPHIC TO A BIFIBRATION

(2) $(A \xleftarrow{d_1} A^2 \xrightarrow{d_0} A) \cong (A \xleftarrow{1} A \xrightarrow{1} A)$

"PROOF" (1) $S \cong (I_B S) I_A$

(2) $I_A I_A \cong I_A$ ∎

PROPOSITION CONSIDER SPANS

$$A \xleftarrow{s} S \xrightarrow{s'} B \quad B \xleftarrow{t} T \xrightarrow{t'} C$$

AND ASSUME THE 2-PULLBACK

$$\begin{array}{ccc} P & \xrightarrow{p'} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t} \\ S & \xrightarrow[s']{} & B \end{array}$$

EXISTS. IF t IS AN OPFIBRATION OR s' A FIBRATION, THEN THE COMPOSITE OF S & T EXISTS AND IS ISOMORPHIC TO $A \xleftarrow{sp} P \xrightarrow{t'p'} C$ ∎

SO IN PRESENCE OF COMMA OBJECTS AND 2-PULLBACKS WE CAN RESTRICT TO BIFIBRATIONS, COMMUTING DIAGRAMS AS CELLS & PB AS COMPOSITION. BUT THE EQUIVALENCE REL NOT TRIVIAL!