

SPANS
FOR
2-CATEGORIES

(JOINT WITH R. DAWSON & D. PRONK)

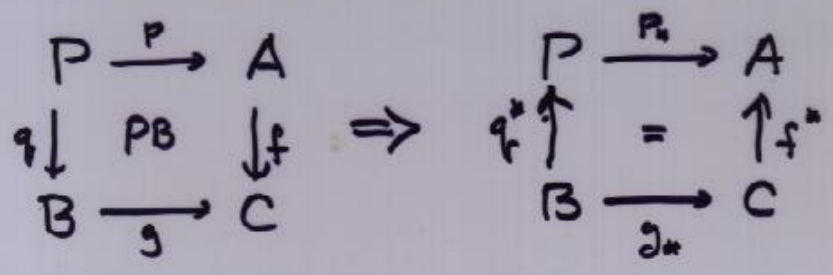
WHITE POINT, NS
JUNE 2006

SPAN

A CAT W. PB \rightsquigarrow SPAN(A)

$A \xrightarrow{S} A'$

- BICATEGORY
- A \xrightarrow{C}^* SPAN(A) STRONG MORPH
- f_* HAS RT ADJOINT f^*
- BECK CONDITION



THEOREM (DPP)

(PB) A \xrightarrow{C}^* SPAN(A)

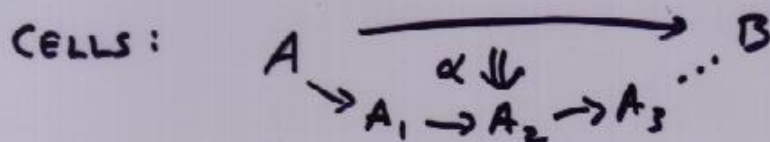


- OPLAX $\Phi(ST) \rightarrow \Phi(S)\Phi(T)$
- $\Phi(\Gamma_A) \rightarrow 1_{\Phi A}$
- NORMAL
- PRES f_*S & Tf^*
- (PARANORMAL)

- UNIV PROP DOES NOT REFER TO PB
→ DROP IT!
- BUT NEED IT FOR COMPOSITION IN SPAN
→ DROP IT TOO!!
(\oplus DOESN'T PRESERVE IT.)
- NEED SOME TRACE OF COMPOSITION
TO EXPRESS OPLAX, PARANORMAL

SEVERAL OBJECT MULTICATEGORIES

(HERMIDA, LEINSTER, LAMBEK)



CAN DEFINE IDENTITIES & COMPOSITES
BY UNIVERSAL PROPERTIES

CAN'T EXPRESS ADJOINTS!

SPAN(A)

FOR A W. PB, THE (WEAK) DOUBLE CATEGORY SPAN(A) HAS CELLS

$$\begin{array}{ccccc} A & \leftarrow & S & \rightarrow & A' \\ \downarrow & & \downarrow & & \downarrow \\ B & \leftarrow & T & \rightarrow & B' \end{array}$$

- BETTER THAN SPAN(A) BECAUSE IT REMEMBERS THE ARROWS OF A (cf SHULMAN)
- IN A DOUBLE CATEGORY THE NOTION OF ADJOINTNESS FACTORS INTO TWO SIMPLER & DUAL NOTIONS:

COMPANION
+
CONJOINT

COMPANIONS

(BROWN - CONNECTIONS, GP)

IN A DOUBLE CAT \mathbb{D} $f: A \rightarrow B$ (HORIZ)

AND $g: A \rightarrow B$ (VERT) ARE **COMPANIONS**

IF THERE ARE CELLS

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \parallel & \eta & \downarrow g \\ A & \xrightarrow{f} & B \end{array} \quad \& \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \varepsilon & \parallel \\ B & \xlongequal{\quad} & B \end{array}$$

S.T. $\varepsilon \eta = id_f$ & $\varepsilon \cdot \eta = id_g$

CONJOINTS (DUAL)

$$\begin{array}{ccc} B & \xrightarrow{\mu} & A \\ \parallel & \beta & \downarrow g \\ B & \xlongequal{\quad} & B \end{array} \quad \& \quad \begin{array}{ccc} A & \xlongequal{\quad} & A \\ g \downarrow & \alpha & \parallel \\ B & \xrightarrow{\mu} & A \end{array}$$

S.T. $\alpha \beta = id_\mu$ & $\beta \cdot \alpha = id_g$

ADJOINTS

ADJOINTNESS IS THE USUAL IN THE
 BICATEGORY HOR(ID) OF HORIZONTAL
 ARROWS AND **SPECIAL CELLS**

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & \alpha & \parallel \\ A & \xrightarrow{k} & B \end{array}$$

PROP: ① COMPANIONS (CONJOINTS)

ARE UNIQUE UP TO SPECIAL ISO

② COMPANIONS (CONJOINTS) COMPOSE

③ f COMPANION TO g & u CONJOINT
 TO $g \Rightarrow f$ LEFT ADJOINT TO u .

EX: QUINTETS

QA

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h \downarrow & \alpha & \downarrow g \\ C & \xrightarrow{u} & D \end{array}$$

Ex: IN $\text{SPAN}(A)$

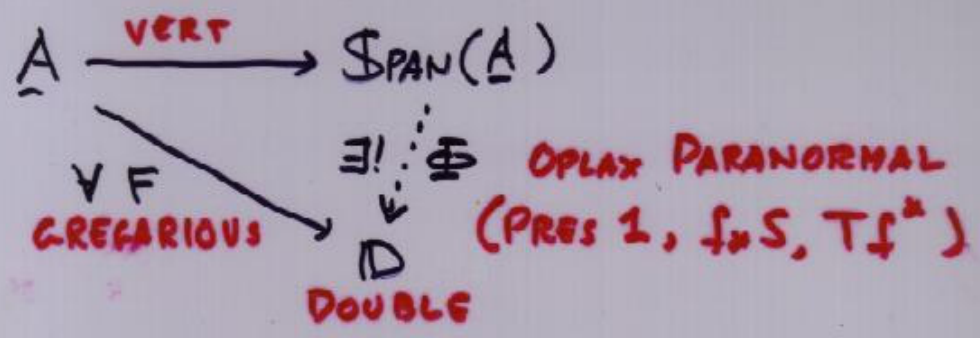
$$\begin{array}{ccc}
 A \xleftarrow{1} A \xrightarrow{1} A & & A \xleftarrow{1} A \xrightarrow{f} B \\
 \parallel & \parallel & \downarrow f \quad \& \quad f \downarrow \quad f \downarrow \quad \parallel \\
 A \xleftarrow{1} A \xrightarrow{f_*} B & & B \xleftarrow{1} B \xrightarrow{1} B
 \end{array}$$

MAKE f_* COMPANION TO f

SIMILARLY f^* CONJOINT TO f

$$\Rightarrow f_* = f^*$$

UNIVERSAL PROPERTY

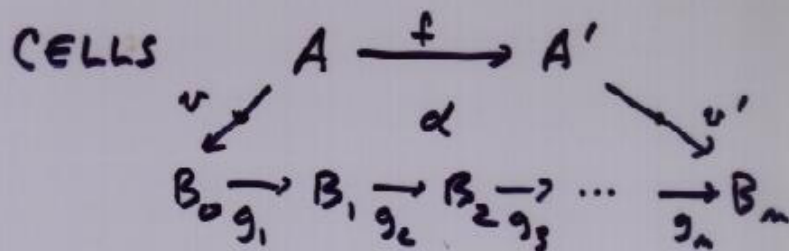


OPLAX DOUBLE CATEGORIES

(LEINSTER'S $\underline{\mathcal{L}}_C$ -MULTICATEGORIES)

WHEN \underline{A} DOESN'T HAVE PB SPANS
DON'T COMPOSE - IT IS NATURAL TO
CONSIDER OPLAX DOUBLE CATEGORIES:

HAVE: OBJECTS, HORIZONTAL ARROWS
VERTICAL ARROWS - FORM A CAT



CELLS COMPOSE VERTICALLY IN THE
OBVIOUS WAY $((\beta_m, \dots, \beta_2, \beta_1) \circ \alpha)$

ASSOCIATIVE + UNITARY

HORIZONTAL COMPOSITES

SAY THAT THE COMPOSITE OF g_1, \dots, g_n
EXISTS (OR IS **STRONGLY REPRESENTABLE**)

IF THERE IS A CELL

$$\begin{array}{ccc}
 B_0 & \xrightarrow{g_n \cdots g_2 g_1} & B_n \\
 \parallel & \wr & \parallel \\
 B_0 & \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} & B_n
 \end{array}$$

IF EVERY CELL

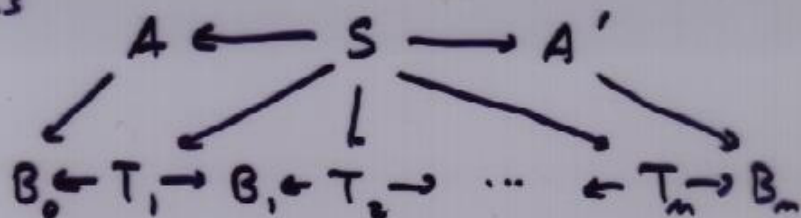
$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & A' \\
 \swarrow & & \searrow \\
 X_0 \rightarrow \cdots \rightarrow X_1 \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \rightarrow B_n \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_2
 \end{array}$$

FACTORS UNIQUELY AS

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & A' \\
 \swarrow & \wr & \searrow \\
 X_0 \rightarrow \cdots \rightarrow X_1 \rightarrow B_0 \rightarrow B_n \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_2 \\
 \parallel & \parallel \text{id} \parallel & \parallel \text{id} \parallel \\
 X_0 \rightarrow \cdots \rightarrow X_1 \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \rightarrow B_n \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_2
 \end{array}$$

SPAN(A)

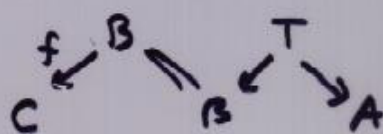
CELLS



- THE COMPOSITE OF $T_1 \dots T_n$ EXISTS IFF THE GENERALIZED PB (= LIM) EXISTS.

- IDENTITIES EXIST $A \overset{!}{\leftarrow} A \overset{!}{\rightarrow} A$

- ALSO $f \circ S$ AND $T \circ f^*$ EXIST



PARANORMAL

AN OPLAX DOUBLE CATEGORY IS **NORMAL**
IF ALL IDENTITIES EXIST

COMPANIONS & CONJOINTS MAKE
SENSE — NORMAL MORPHISMS
PRESERVE THEM

IT IS **PARANORMAL** IF

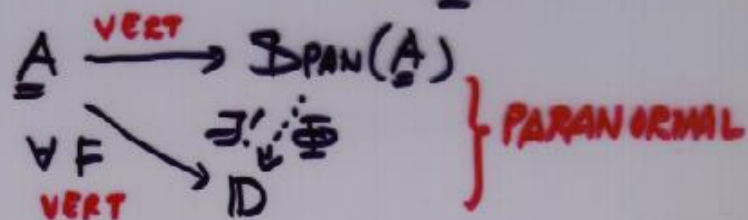
- (1) IT IS NORMAL
- (2) EVERY VERTICAL v HAS A COMPANION v_*
AND A CONJOINT v^*
- (3) ALL COMPOSITES $v_* h$ & $k v^*$ EXIST.

THEOREM $\text{SPAN} : \underline{\text{CAT}} \longrightarrow \underline{\text{PARA}}$
IS 2-LAJ TO $\text{VERT} : \underline{\text{PARA}} \longrightarrow \underline{\text{CAT}}$.

2-CATEGORIES

A 2-CATEGORY

WANT THE FREE PARANORMAL DOUBLE
CATEGORY GENERATED BY A VERTICALLY



THE UNDERLYING 2-CATEGORY OF A PDC

VERT(ID) HAS SAME OBJECTS AS ID,

THE ARROWS ARE THE VERTICAL ARROWS OF ID,

THE 2-CELLS ARE SPECIAL CELLS

$$\begin{array}{ccc}
 A & \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} & B \\
 \hline
 A & = & A \\
 \downarrow v & \alpha & \downarrow u \\
 B & = & B
 \end{array}$$

PRECELLS

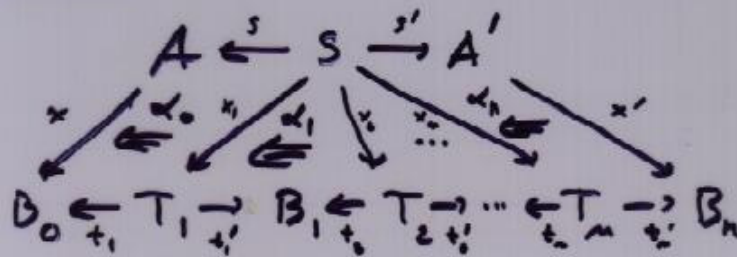
CONSTRUCT $\text{SPAN}'(\underline{A})$

OBJECTS - THOSE OF \underline{A}

VERTICAL ARROWS - ARROWS OF \underline{A}

HORIZONTAL ARROWS - SPANS IN \underline{A}

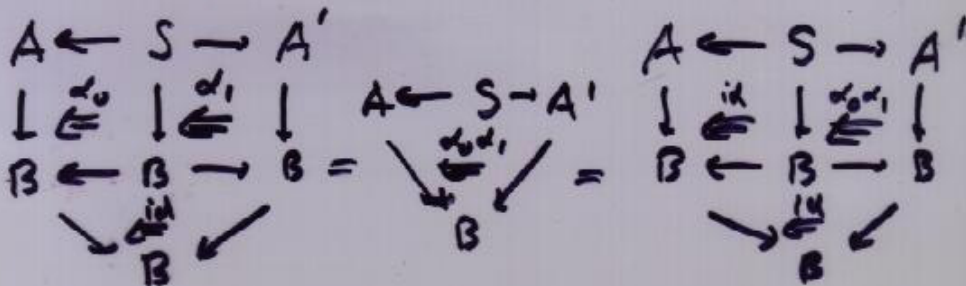
PRECELLS



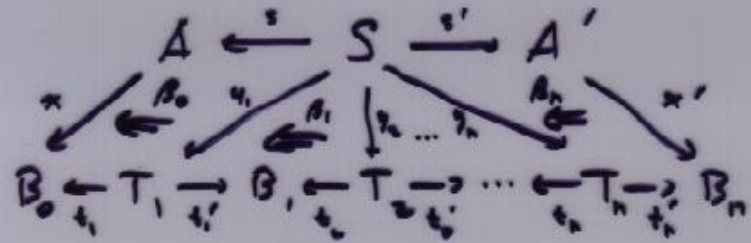
WITH OBVIOUS COMPOSITION OF PRECELLS

$\text{SPAN}'(\underline{A})$ IS AN OPLAX DOUBLE CAT.

BUT NOT NORMAL !



EQUIVALENCE RELATION

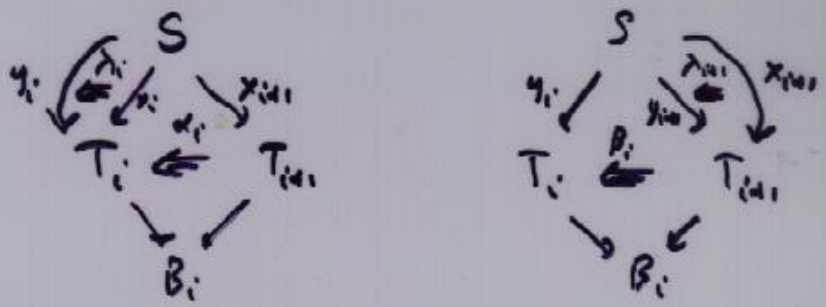


ANOTHER PRECELL WITH SAME BOUNDARY

IDENTIFY IT WITH PREVIOUS ONE IF

THERE ARE $\lambda_i : x_i \rightarrow y_i \quad i = 1, \dots, n$

S.T.



(END PTS FIXED - $\lambda_0 = id, \lambda_{n+1} = id$)

TAKE EQUIVALENCE RELATION GENERATED

DENOTE EQUIV CLASS BY $\alpha_0 \otimes \alpha_1 \otimes \dots \otimes \alpha_n$

SPAN(A)

THEOREM TAKING CELLS TO BE EQUIVALENCE CLASSES OF DIAGRAMS AS ABOVE GIVES US AN OPLAX DOUBLE CATEGORY $\mathbb{S}PAN(\underline{A})$

- $\mathbb{S}PAN(\underline{A})$ IS PARANORMAL
- $\mathbb{S}PAN : \underline{2-CAT} \longrightarrow \underline{PARA}$ IS LEFT 2-ADJOINT TO $\underline{VERT} : \underline{PARA} \longrightarrow \underline{2-CAT}$

- $\mathbb{Q}(\underline{A}) \xrightarrow{(\)_*} \mathbb{S}PAN(\underline{A})$

$$\begin{array}{ccc}
 A \xrightarrow{f} B & & A \xleftarrow{i_a} A \xrightarrow{i_b} B \\
 g \downarrow \swarrow \alpha \downarrow h & \xrightarrow{1} & s \downarrow = s \downarrow \swarrow \alpha \downarrow h \\
 C \xrightarrow{k} D & & C \xrightarrow{i_c} C \xrightarrow{k} D
 \end{array}$$

IS LOCALLY FULL & FAITHFUL ■

PROPERTIES

$$\bullet \begin{array}{ccc} A \xleftarrow{s} S \xrightarrow{s'} A' \\ \downarrow \cong \downarrow \cong \downarrow \cong \\ B \xleftarrow{f} B \xrightarrow{f} B' \end{array} = \begin{array}{ccc} A \xleftarrow{s} S \xrightarrow{s'} A' \\ \downarrow \cong \downarrow \cong \downarrow \cong \\ B \xleftarrow{f} B \xrightarrow{f} B' \end{array}$$

$$\bullet \text{LEMMA} \quad \begin{array}{ccc} A \leftarrow S & S \rightarrow & A' \\ \parallel & \downarrow \cong \uparrow & \parallel \\ A \leftarrow T & T \rightarrow & A' \end{array}$$

$$\Rightarrow (A \leftarrow S \rightarrow A') \cong (A \leftarrow T \rightarrow A') \quad \blacksquare$$

$$\bullet \text{COROLLARY} \quad f \dashv u$$

$$\Rightarrow (A \xleftarrow{f} S \xrightarrow{g} B) \cong (A \xleftarrow{1_A} A \xrightarrow{gu} B) \quad \blacksquare$$

COMMA OBJECTS

THEOREM IF \underline{A} HAS COMMA OBJECTS THEN COMPOSITES EXIST IN $\text{SPAN}(\underline{A})$.
 $\text{SPAN}(\underline{A})$ SATISFIES THE BECK CONDITION:

$$\begin{array}{ccc}
 K \xrightarrow{f} B & & K \xrightarrow{f_2} B \\
 p \downarrow \lrcorner \downarrow & \text{COMMA} \Rightarrow & p^* \uparrow \cong \uparrow g^* \\
 A \xrightarrow{f} C & & A \xrightarrow{f_1} C
 \end{array}$$

REMARK 1: COMPOSING SPANS WITH COMMA OBJECTS IS ASSOCIATIVE WITHOUT EQUIVALENCE RELATION BUT NOT UNITARY

REMARK 2: COMPOSITES IN $\text{SPAN}(\underline{A})$
 \Rightarrow COMMA OBJECTS IN \underline{A}

PROPOSITION IF $B \leftarrow T \rightarrow B'$ IS A BIFIBRATION, EVERY CELL

$$\begin{array}{ccc} A \leftarrow S \rightarrow A' \\ \downarrow \alpha_0 \quad \downarrow \alpha_1 \\ B \leftarrow T \rightarrow B' \end{array}$$

IS EQUAL TO ONE (NOT UNIQUE) IN WHICH α_0, α_1 ARE IDENTITIES. ■

PROPOSITION IF \underline{A} HAS COMMA OBJECTS

(1) EVERY SPAN IS ISOMORPHIC TO A BIFIBRATION

$$(2) (A \xleftarrow{d_1} A^2 \xrightarrow{d_0} A) \cong (A \xleftarrow{1} A \xrightarrow{1} A)$$

"PROOF" (1) $S \cong (\underline{I}_B S) \underline{I}_A$

(2) $\underline{I}_A \underline{I}_A \cong \underline{I}_A$ ■

PROPOSITION CONSIDER SPANS

$$A \xleftarrow{s} S \xrightarrow{s'} B \quad B \xleftarrow{t} T \xrightarrow{t'} C$$

AND ASSUME THE 2-PULLBACK

$$\begin{array}{ccc} P & \xrightarrow{r'} & T \\ p \downarrow & & \downarrow t \\ S & \xrightarrow{s'} & B \end{array}$$

EXISTS. IF t IS AN OPFIBRATION OR s' A FIBRATION, THEN THE COMPOSITE OF S & T EXISTS AND IS ISOMORPHIC TO $A \xleftarrow{sp} P \xrightarrow{t'r'} C$ ■

SO IN PRESENCE OF COMMA OBJECTS AND 2-PULLBACKS WE CAN RESTRICT TO BIFIBRATIONS, COMMUTING DIAGRAMS AS CELLS & PB AS COMPOSITION. **BUT THE EQUIVALENCE REL NOT TRIVIAL!**