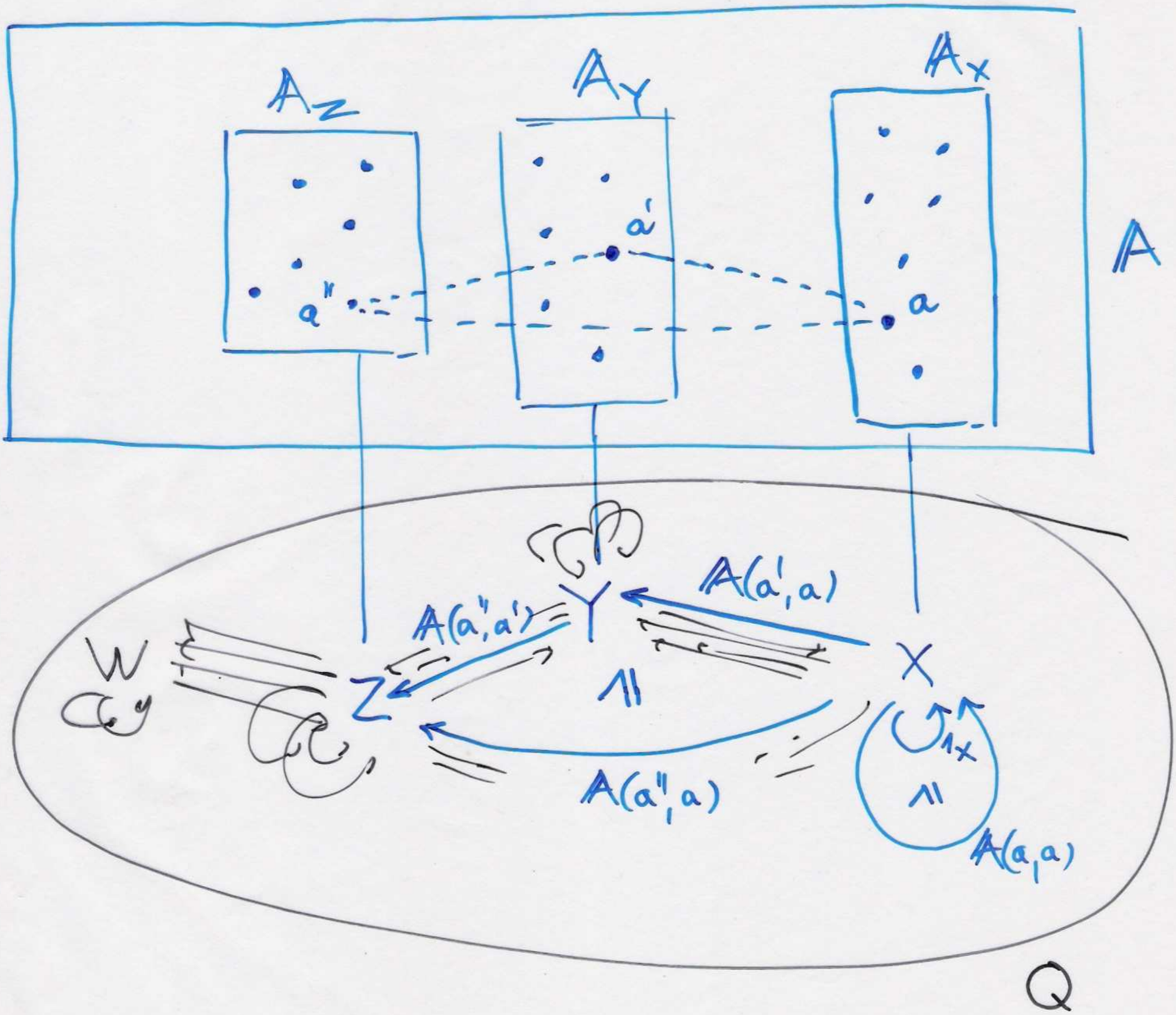
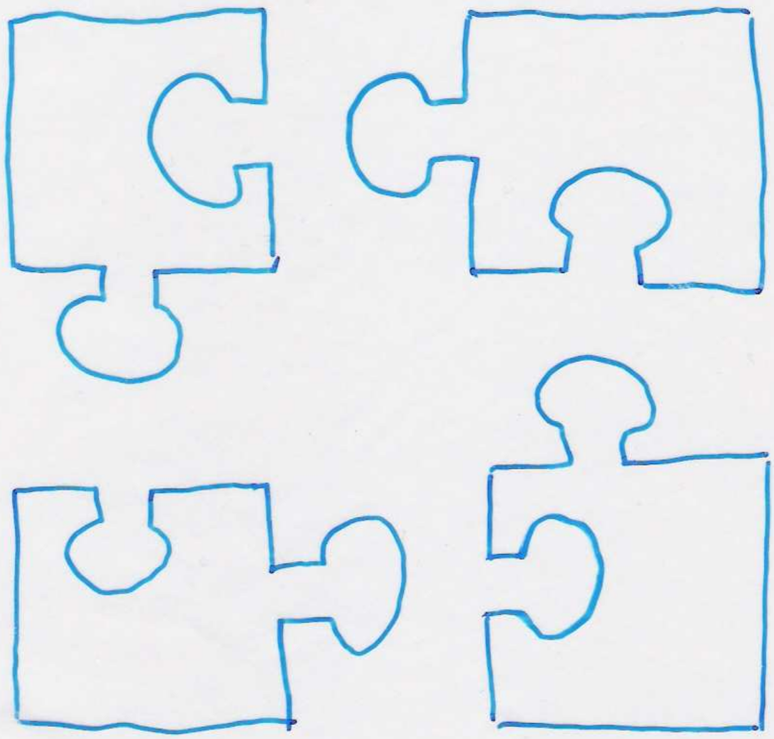
$$\begin{tikzcd} \mathrm{Ord} \arrow[loop above, "\mathcal{P}"] \arrow[r, shift left=2, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathrm{Sup} \arrow[l, shift left=2, "\mathcal{U}"] \end{tikzcd} \simeq \mathrm{Ord}^{\mathcal{P}}$$

$$\bigvee D \longleftarrow\!\shortmid D$$

$$\begin{tikzcd} A \arrow[r, "a \mapsto \downarrow a"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathcal{P}A \arrow[l, dashed, bend right=50] \end{tikzcd}$$

"sketch of a Q-category":





Q-orders :

$\mathcal{Q}$ = small quantaloid

$$\mathrm{Ord}(\mathcal{Q}) = \mathrm{Cat}_{cc}(\mathcal{Q}_{si})$$

split idempts

enriched cats

take the Cauchy complete ones

$$\simeq \mathrm{Map}\big(\mathrm{TRSDist}(\mathcal{Q})\big)$$

easy elem descript.

ex. $\mathrm{Ord}(\underline{2}) = \mathrm{Ord}$

$\mathrm{Ord}(\Omega) = \mathrm{Ord}(\mathrm{Sh}(\Omega))$

$\mathrm{Ord}(\bar{\mathbb{R}}^{+}) = \mathrm{CCMet}$

With involution τ on $\mathcal{Q}$, define "τ-symmetric $\mathcal{Q}$-orders" : Mulvey's quantal sets.

Q - modules :

$$\mathrm{Mod}(Q) = \mathrm{QUANT}(Q^{op}, \mathrm{Sup})$$

$$= \mathrm{Sup\text{-}Cat}(Q^{op}, \mathrm{Sup})$$

Because idempotents split in Sup :

$$\mathrm{Mod}(Q) \simeq \mathrm{Mod}(Q_{si})$$

Cocomplete Q-categories :

Cocont (Q) = cocompl. Q-cats + cocont. functors

$\updownarrow$

admit all weighted colimits

$\updownarrow$

tensored, cotensored, "order-cocomplete"

$\updownarrow$

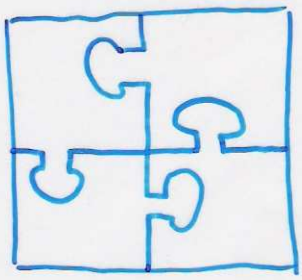
$$\begin{tikzcd} A \arrow[r, "Y"', ""{name=Y}] \arrow[r, phantom, bend left=30, "\perp"] & \mathcal{P}A \arrow[l, bend right=40, dashed, "S"'] \end{tikzcd}$$

In fact,

Cocont (Q) $\simeq$ Mod (Q).

KZ-doctrines :

$$
\begin{tikzcd}
\mathrm{Cat}(Q) \arrow[loop above, "\mathcal{P}"] \arrow[rr, bend left, "\mathcal{P}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Cocont}(Q) \simeq \mathrm{Cat}(Q)^{\mathcal{P}} \simeq \mathrm{Cat}_{cc}(Q)^{\mathcal{P}} \arrow[ll, "U"] \arrow[dl, "V"'] \\
& \mathrm{Cat}_{cc}(Q) \arrow[ul, "\text{loc. equl. } W"] \arrow[ur, bend right, "\mathcal{P}"'] \arrow[ur, phantom, "\perp"] \arrow[loop below, "\mathcal{P}"] &
\end{tikzcd}
$$



"Q-modules are Q-suplattices":

$$
\begin{tikzcd}
\mathrm{Mod}(Q) \arrow[d, phantom, "\cong" rotate=90] & & \\
\mathrm{Mod}(Q_{si}) \arrow[d, phantom, "\cong" rotate=90] & & \\
\mathrm{Cocont}(Q_{si}) \arrow[rr, "\bigvee"] \arrow[rr, phantom, shift right=2, "\top"] \arrow[d, phantom, "\cong" rotate=90] & & \mathrm{Cat}_{cc}(Q_{si}) \arrow[ll, bend left, "\mathcal{P}"] \arrow[loop above, "\mathcal{P}"] \\
\mathrm{Cat}_{cc}(Q_{si})^{\mathcal{P}} \arrow[d, phantom, "="] & & \\
\mathrm{Ord}(Q)^{\mathcal{P}} & &
\end{tikzcd}
$$

<u>Comment</u> :

$$\begin{tikzcd}
Q \simeq Q' \arrow[d, Rightarrow] \\
\mathrm{Ord}(Q) \simeq \mathrm{Ord}(Q') \arrow[d, Rightarrow] \\
\mathrm{Mod}(Q) \simeq \mathrm{Mod}(Q') \arrow[d, Rightarrow] \arrow[u, bend right, "?"'] \\
Z(Q) \simeq Z(Q') \arrow[uuu, bend right=60]
\end{tikzcd}$$

false in general,
but true for
comm. quantales
(in part. locales)

$$Z(Q) = \mathrm{QUANT}(Q,Q)(\mathrm{Id}_Q, \mathrm{Id}_Q)$$

$$= \text{centre of } Q$$