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Galois theories of internal groupoids via congruence relations for Maltsev varieties

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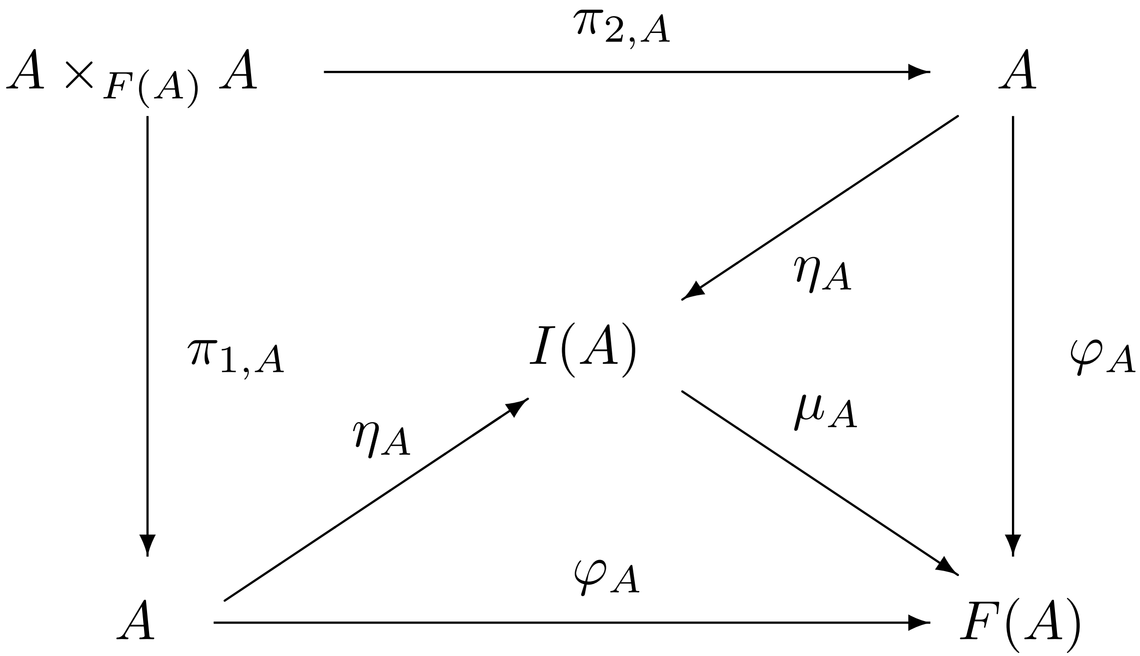
1 Coequalizer of the kernel pair

$\mathbb{C}$ finitely-complete; (F, φ) pointed endofunctor on $\mathbb{C}$, s.t. the kernel pair of $\varphi_A : A \to F(A)$ has a coequalizer for every object A in $\mathbb{C}$.



2 Idempotency of (I, η)

$Fix(I, \eta)$, $Mono(F, \varphi)$ full subcategories of $\mathbb{C}$.

Lemma 2.1

(I, η) well-pointed endofunctor (i.e., $I\eta = \eta I$);
$Fix(I, \eta) = Mono(F, \varphi)$.

Proposition 2.2

μ, $F\eta$ monics $\Rightarrow$ (I, η) idempotent

Remark 2.3

(I, η) idempotent $\Leftrightarrow$ $I\eta = \eta I$ <u>and</u> ηI iso $\Leftrightarrow$ $Fix(I, \eta)$ reflective in $\mathbb{C}$

3 Stabilization and m.-l. factorization

Proposition 3.1

All η_A *pullback stable regular epis* <u>*and*</u> μ *monic* <u>*and*</u> $F\eta$ *iso*

<u>*and*</u> F *preserves*

$$\begin{array}{ccc} C \times_{I(A)} A & \longrightarrow & A \\ \downarrow & & \downarrow \eta_A \\ C & \xrightarrow{\;g\;} & I(A) \end{array}$$

$\Rightarrow$ (I, η) *idempotent with stable units;*

<u>*and*</u> $\forall_{B \in \mathbb{C}} \exists_{p:E \to B \ e.d.m.} \ E \in Mono(F, \varphi)$

$\Rightarrow$ $(\mathcal{E}', \mathcal{M}^*)$ *factorization system (monotone-light).*

4 First example: internal categories

(F, φ) idempotent associated to the localization

$$\mathbf{Cat}(\mathbb{S}) \to \mathbf{LEqRel}(\mathbb{S}) \simeq \mathbb{S}$$

$$C \mapsto \nabla_{C_0}$$

$$\begin{array}{lccccc}
\mathrm{C}= & C_1 \times_{C_0} C_1 & \xrightarrow{\gamma} & C_1 & \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{i} \\ \xrightarrow{d_1} \end{array} & C_0 \\
\varphi_C = & \Big\downarrow {\scriptstyle d_C \times d_C} & & \Big\downarrow {\scriptstyle d_C} & & \Big\downarrow {\scriptstyle 1_{C_0}} \\
\nabla_{C_0} = & C_0 \times C_0 \times C_0 & \longrightarrow & C_0 \times C_0 & \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} & C_0
\end{array}$$

Lemma 4.1 *$\mathbb{S}$ regular $\Rightarrow$ for every $C \in \mathbf{Cat}(\mathbb{S})$ the kernel pair of $\varphi_C = (d_C, 1_{C_0})$ has a coequalizer in $\mathbf{Cat}(\mathbb{S})$.*

$$
\begin{array}{ccccc}
 & & & & \\
p_C \times p_C \downarrow\downarrow q_C \times q_C & & p_C \downarrow\downarrow q_C & & \downarrow 1_{C_0} \\
C_1 \times_{C_0} C_1 & \xrightarrow{\gamma} & C_1 & \overset{d_0,\ d_1}{\underset{i}{\rightleftarrows}} & C_0 \\
\downarrow e_C \times e_C & & \downarrow e_C & & \downarrow 1_{C_0} \\
I(C_1) \times_{C_0} I(C_1) & \xrightarrow{\gamma^I} & I(C_1) & \overset{d_0^I,\ d_1^I}{\underset{e_C i}{\rightleftarrows}} & C_0
\end{array}
$$

Conclusion 4.2 $\mathbb{S}$ *regular:*

$\mathbf{Cat}(\mathbb{S}) \to \mathbf{Preord}(\mathbb{S})$ *reflection with stable units;*

$\mathbf{Grpd}(\mathbb{S}) \to \mathbf{EqRel}(\mathbb{S})$ *reflection with stable units and monotone-light factorization,*

$(\sigma, d_1) : Eq(d_0) \to G$, *with* $\sigma = \gamma(1_{G_1} \times s)$,

$$
\begin{array}{ccccc}
G_1 \times_{G_0} G_1 \times_{G_0} G_1 & \xrightarrow{p_1 \times p_2} & G_1 \times_{G_0} G_1 & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{<1,1>} \\ \xrightarrow{p_1} \end{array} & G_1 \\
\downarrow{\scriptstyle \sigma \times \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle d_1} \\
G_1 \times_{G_0} G_1 & \xrightarrow{\gamma} & G_1 \circlearrowleft s & \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{i} \\ \xrightarrow{d_1} \end{array} & G_0
\end{array} .
$$

$\sigma < 1_{G_1}, id_0 >= 1_{G_1}$ *and* $d_1 i = 1_{G_0}$.

e.g. $\mathbb{S} = \mathbf{Set}$: $\mathbf{Cat} \to \mathbf{Preord}$,
$(\mathcal{E}', \mathcal{M}^*) = (\mathbf{Full\ and\ Bijective\ on\ Objects}, \mathbf{Faithful})$.

$\mathbb{S}$ Maltsev category: $\mathbf{EqRel}(\mathbb{S}) = \mathbf{RRel}(\mathbb{S})(\Rightarrow \mathbf{Cat}(\mathbb{S}) = \mathbf{Grpd}(\mathbb{S}))$.

$\mathbb{S}$ regular Maltsev category: $\mathbf{Grpd}(\mathbb{S}) \to \mathbf{EqRel}(\mathbb{S}) = \mathbf{RRel}(\mathbb{S})$
reflection with stable units and monotone-light-factorization.

A variety of universal algebras is Maltsev iff its theory has a
Maltsev operator $p : X \times X \times X \to X$, $p(x, y, y) = x = p(y, y, x)$.

e.g. $\mathbf{Grp}$:

$$p(x, y, z) = xy^{-1}z;$$

$$\mathbf{Cat}(\mathbf{Grp}) = \mathbf{Grpd}(\mathbf{Grp}) \simeq \mathbf{CrossMod}.$$

5 Geometric morphisms

Corollary 5.1

$\mathbb{C}$ *admits a (regular epi, mono)-factorization* <u>*and*</u> (F, φ) *idempotent*

$\Rightarrow (I, \eta)$ *idempotent.*

Corollary 5.2

$\mathbb{C}$ *regular* <u>*and*</u> (F, φ) *idempotent* $\Rightarrow (I, \eta)$ *idempotent;*

<u>*and*</u> F *left exact* $\Rightarrow$ *stable units;*

<u>*and*</u> $\forall_{B \in \mathbb{C}} \exists_{p: E \to B \ e.d.m.} \ E \in Mono(F, \varphi) \Rightarrow$ *m.-l. factorization.*

Proposition 5.3 *Let $F : \mathcal{E} \to \mathcal{F}$ be a geometric morphism between regular categories, $F^* \dashv F_* : \mathcal{E} \to \mathcal{F}$, which is an embedding.*

Then, the reflection $I : \mathcal{F} \to Mono(F^*)$, obtained from the localization $F^* : \mathcal{F} \to \mathcal{E}$ through the coequalizer of the kernel pair process, does have stable units. Moreover, there is a monotone-light factorization associated to the reflection $I : \mathcal{F} \to Mono(F^*)$ provided the following four conditions also hold:

1. *the category $\mathcal{F}$ is cocomplete;*
2. *the full subcategory $Mono(F^*)$ is dense in $\mathcal{F}$, i.e., every object of $\mathcal{F}$ is a colimit of objects of $Mono(F^*)$.*
3. *in $\mathcal{F}$ the coproduct of monomorphisms is a monomorphism;*
4. *regular epis are effective descent morphisms in $\mathcal{F}$.*

6 Second example: simplicial sets

$K : \mathcal{B} \to \mathcal{A}$ fully faithful, $\mathcal{S}$ regular and complete

$$\mathcal{S}^K : \mathcal{S}^{\mathcal{A}} \to \mathcal{S}^{\mathcal{B}}$$

$\Delta_n^{op} \subset \Delta^{op}, n \geq 0$, $\mathcal{S} = \mathbf{Set}$

$$\mathbf{Smp} \to \mathbf{Smp}_n$$

$$\mathbf{Smp} \to Mono(F_n)$$

$$(F_n, \varphi^n) \mapsto (I_n, \eta^n)$$

Lemma 6.1 *Every unit morphism of any representable functor*

$$\varphi^n_{\Delta(-,[p])} : \Delta(-,[p]) \to F_n(\Delta(-,[p])),\ \ p \geq 0,$$

is a monomorphism in $\mathbf{Smp} = \mathbf{Set}^{\Delta^{op}}$.