Differential structure, tangent structure, and SDG

Geoff Cruttwell
(joint work with Robin Cockett)

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Introduction

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- Define tangent structure, give examples and an instance of its theory.
- Show how the "tangent spaces" of the tangent structure form a Cartesian differential category.
- Show how representable tangent structuture gives a model of synthetic differential geometry (SDG).

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- a natural transformation $T \xrightarrow{p} I$;
- for each M, the pullback of n copies of $TM \xrightarrow{p_M} M$ along itself exists (and is preserved by T), call this pullback $T_n M$;
- such that for each $M \in \mathbb{X}$, $TM \xrightarrow{p_M} M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformation $T_2 \xrightarrow{+} T$, $I \xrightarrow{0} T$;

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Definition

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- various other coherence equations for ℓ and c;
- (universality of vertical lift) the map

$$T_2 M \xrightarrow{v := \langle \pi_1 \ell, \pi_2 0_T \rangle T(+)} T^2 M$$

is the equalizer of

$$T^2 M \underset{T(p)p0}{\overset{T(p)}{\rightrightarrows}} TM.$$

Analysis examples

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- Any Cartesian differential category $\mathbb{X}$ has an associated tangent structure:

$$TM := M \times M, Tf := \langle Df, \pi_1 f \rangle$$

with:

  - $p := \pi_1$;
  - $T_n(M) := M \times M \ldots \times M$ ($n + 1$ times);
  - $+\langle x_1, x_2, x_3 \rangle := \langle x_1 + x_2, x_3 \rangle$, $0(x_1) := \langle 0, x_1 \rangle$;
  - $\ell(\langle x_1, x_2 \rangle) := \langle \langle x_1, 0 \rangle, \langle 0, x_2 \rangle \rangle$;
  - $c(\langle \langle x_1, x_2 \rangle, \langle x_3, x_4 \rangle \rangle) := \langle \langle x_1, x_3 \rangle, \langle x_2, x_4 \rangle \rangle$.

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- This is one way to show that the category of finite-dimensional smooth manifolds has tangent structure.
- Similarly, convenient vector spaces have an associated tangent structure, as do manifolds built on convenient vector spaces.

Algebra examples

- The category **cRing** of commutative rings has tangent structure, with:

$$TA := A[\epsilon] = \{a + b\epsilon : a, b \in A, \epsilon^2 = 0\},$$

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(symmetric ring of the Kahler differentials of A).

- More generally, if $(\mathbb{X}, T)$ is tangent structure with T having a left adjoint L, then (X^{op}, L) is also tangent structure.

SDG examples

Recall that a model of SDG consists of a topos with an internal commutative ring R that satisfies the "Kock-Lawvere axiom": if we define

$$D := \{d \in R : d^2 = 0\},$$

then the canonical map

$$\phi : R \times R \longrightarrow R^D,$$

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- When restricted to the "microlinear" objects, any model of SDG gives an instance of tangent structure, with

$$TM := M^D.$$

Lie bracket

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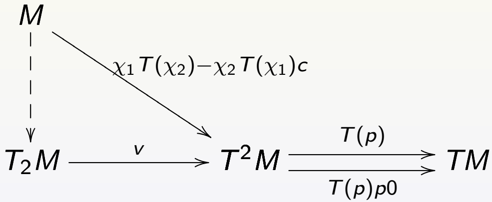
- Rosicky showed how to use the universal property of vertical lift to define the Lie bracket of vector fields $M \xrightarrow{\chi_1, \chi_2} TM$:



following this by $T_2M \xrightarrow{\pi_1} TM$ gives a unique map

$$M \xrightarrow{[\chi_1, \chi_2]} TM$$

which has the abstract properties of a bracket operation.

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Definition

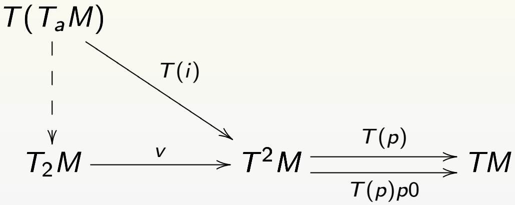
For a point $1 \xrightarrow{a} M$ of an object of tangent structure, say that **the tangent space at** a **exists** if the pullback of a along p_M exists:

$$\begin{array}{ccc} T_a(M) & \xrightarrow{i} & TM \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle p_M} \\ 1 & \xrightarrow[a]{} & M \end{array}$$

and this pullback is preserved by T.

Tangent bundle of a tangent space

- The tangent bundle of a tangent space has a particularly simple form:



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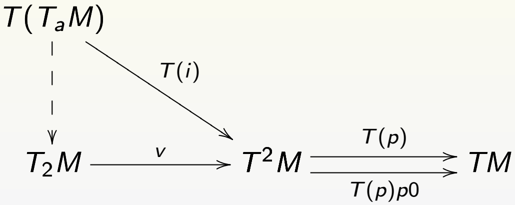

- The existence of the unique map to T_2M gives $T(T_aM) \cong T_aM \times T_aM$, and $p \cong \pi_1$.

Differential objects

For objects A with $TA \cong A \times A, p_A \cong \pi_1$, the tangent bundle functor gives a differential: for $f : A \longrightarrow B$,

$$D(f) := A \times A \xrightarrow{T(f)} B \times B \xrightarrow{\pi_0} B,$$

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and the axioms for T give the Cartesian differential axioms, eg.:

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- naturality of ℓ gives CD6: $\langle \langle a, 0 \rangle, \langle b, c \rangle \rangle D^2(f) = \langle a, c \rangle D(f)$;
- naturality of c gives CD7: $\langle \langle a, b \rangle \langle c, d \rangle \rangle D^2(f) = \langle \langle a, c \rangle, \langle b, d \rangle \rangle D^2(f)$.

Differentials and tangent functors

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- This exhibits the category of small Cartesian differential categories as a coreflective subcategory of small tangent structures (with appropriate morphisms):

$$\mathbf{cartDiffCats} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{tangCats}$$

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- For the tangent structure on smooth finite-dimensional manifolds, the tangent spaces are the Cartesian spaces.
- For the tangent structure on convenient manifolds, the tangent spaces are the convenient vector spaces.
- What are the tangent spaces in models of SDG?
  - In the Dubuc topos, by a result of Kock and Reyes, the tangent spaces contain convenient vector spaces (do they contain more?).

Representable tangent structure

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Macroscopic level (Functorial properties)	Microscopic level (Infinitesimal object operations)
$p : T \longrightarrow I$ projection	$\wp : 1 \longrightarrow D$ zero
$\ell : T \longrightarrow T^2$ vertical lift	$\odot : D \times D \longrightarrow D$ multiplication
$+ : T_2 \longrightarrow T$ bundle addition	$\delta : D \longrightarrow D(2)$ comultiplication.
$0 : I \longrightarrow T$ bundle zero	$! : D \longrightarrow 1$ final map
$c : T^2 \longrightarrow T^2$ canonical flip	$c_\times : D \times D \longrightarrow D \times D$ symmetry

Infinitesimal objects

Definition

A Cartesian category $\mathbb{X}$ has an **infinitesimal object** D in case:

[Infsml.1] D is a commutative semigroup with multiplication $\_ \odot \_ : D \times D \longrightarrow D$ and a zero $\wp : 1 \longrightarrow D$;

[Infsml.2] $D(n)$ is the pushout of n copies of $\wp : 1 \longrightarrow D$;

[Infsml.3] there is a map $\delta : D \longrightarrow D \star D$ which makes the object $\wp$, in the pointed category $1/\mathbb{X}$, a commutative comonoid with respect to the coproduct;

[Infsml.4] certain coherence equations;

[Infsml.5] the following is a coequalizer:

$$D \underset{\langle \wp, 1 \rangle}{\overset{\langle \wp, \wp \rangle}{\rightrightarrows}} D \times D \xrightarrow{(\delta \times 1)\langle \odot |_{\star} \pi_0 \rangle} D \star D$$

[Infsml.6] The objects D^n and D_n are exponent objects.

Infinitesimal objects continued...

Theorem

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Also: for an infinitesimal object D, every element has square zero, in the sense that the diagram

$$\begin{array}{ccc} D & & \\ {\scriptstyle \Delta}\downarrow & \overset{\wp}{\searrow} & \\ D \times D & \xrightarrow[\odot]{} & D \end{array}$$

commutes.

The associated ring

Given an infinitesimal object, we can construct a ring R_0 as certain structure preserving endomorphisms of D. Then Rosicky showed R_0 satisfies the Kock-Lawvere axiom:

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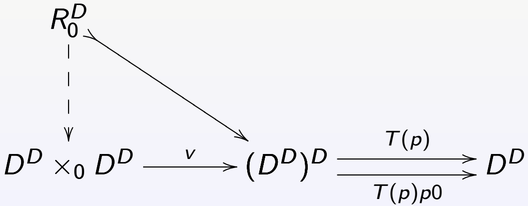
Again, the key is the universality of vertical lift, in this case at $M = D$.



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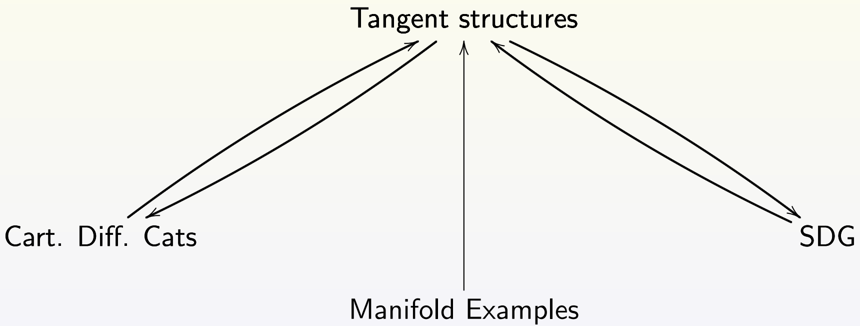
- For example, the tangent structure on $\mathbf{cRing}^{\mathrm{op}}$ is representable, with $D = \mathbb{Z}[\epsilon]$, $R = R_0 = \mathbb{Z}[x]$.
- Any instance of representable tangent structure gives a model of SDG (If the ambient category is not a topos, can embed in a category which is).

Tangent structure subsumes both notions



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