A Double Approach to Variation and Enrichment for Bicategories

Susan Niefield
(with J.R.B. Cockett and R.J. Wood)

June 2012

CT95 (N) $\mathrm{Moncat}/\mathcal{V} \underset{\mathrm{mod}}{\overset{\mathrm{ev}}{\rightleftarrows}} \mathrm{Mon}\,\mathcal{V}$ *Kelly: works for bicategories*

1997 (NW) Generalize to bicategories?

Relate to $\mathrm{Cat}/B \simeq \mathrm{Fun}(B, \mathcal{S}\mathrm{pan}) \simeq \mathrm{Fun}_{\mathrm{N}}(B, \mathcal{P}\mathrm{rof})$?

2005 (CNW) (1) $\mathrm{Fun}(\mathcal{B}, \mathcal{S}) \simeq \mathrm{Fun}_{\mathrm{N}}(\mathcal{B}, \mathrm{Mod}\,\mathcal{S})$

(2) $\mathrm{LDF}/\mathcal{B} \simeq \mathrm{Fun}(\mathcal{B}^{co}, \mathcal{S}\mathrm{pan}) \underset{(1)}{\simeq} \mathrm{Fun}_{\mathrm{N}}(\mathcal{B}^{co}, \mathcal{P}\mathrm{rof})$

2011 (Paré) $\mathbb{L}\mathrm{ax}(\mathbb{B}, \mathbb{S})$ double category for nice $\mathbb{B}$ and $\mathbb{S}$

$\mathbb{L}\mathrm{ax}((\mathbb{V}B)^{op}, \mathbb{S}\mathrm{pan}) \simeq \mathbb{C}\mathrm{at}/\!/B$

Note: $\mathrm{Fun}(\mathcal{B}^{co}, \mathcal{S}\mathrm{pan}) \simeq \mathrm{H}\mathbb{L}\mathrm{ax}((\mathbb{V}\mathcal{B})^{op}, \mathbb{S}\mathrm{pan})$

Idea: (1) for double categories and vertical structure for (2)

Double Categories

Weak category objects $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \underset{\pi_1}{\overset{\pi_2}{\underset{-\mu\rightarrow}{\rightrightarrows}}} \mathbb{D}_1 \underset{d_1}{\overset{d_0}{\underset{\leftarrow\Delta-}{\rightrightarrows}}} \mathbb{D}_0$ in CAT

Objects: objects of $\mathbb{D}_0$

Horizontal morphisms: morphisms of $\mathbb{D}_0$, $D \longrightarrow D'$

Vertical morphism: objects of $\mathbb{D}_1$, $D \rightarrow\!\!\bullet\!\!\rightarrow \bar{D}$

Cells: morphisms of $\mathbb{D}_1$,

$$\begin{array}{ccc} D & \longrightarrow & D' \\ \bullet\downarrow & \rightarrow & \downarrow\bullet \\ \bar{D} & \longrightarrow & \bar{D}' \end{array}$$

Note: $\mathrm{V}\mathbb{D}$ is a bicategory and $\mathrm{H}\mathbb{D}$ is a 2-category

Examples

$\mathbb{S}\mathrm{pan}$: sets, functions, spans, ...

$\mathbb{C}\mathrm{at}$: categories, functors, profunctors, ...

$\mathbb{V}\mathcal{B}$: vertically $\mathcal{B}$, a bicategory (horizontally discrete)

$\mathbb{L}\mathrm{ax}(\mathbb{B},\mathbb{S})$: lax functors, transformations (horizontal), modules, modulations (CKSW)

$\mathbb{M}\mathrm{od}\,\mathbb{D}$: monads in $\mathrm{V}\mathbb{D}$, homomorphisms, modules, ...

$\mathbb{D}/\!/B$: $(\mathbb{D}/\!/B)_0 = \mathbb{D}_0/B$, $(\mathbb{D}/\!/B)_1 = \mathbb{D}_1/\mathrm{id}^{\bullet}_B$

$\mathbb{S}\mathrm{pan}\backslash\!\backslash 1 = \mathbb{S}\mathrm{pan}_*$, pointed sets

The Double Category $\mathbb{L}\mathrm{ax}(\mathbb{B}, \mathbb{S})$ (Paré)

$F\colon \mathbb{B} \longrightarrow \mathbb{S}$
lax functor

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle b}\downarrow^{\bullet} & \beta & \downarrow^{\bullet}{\scriptstyle b'} \\ \bar{B} & \xrightarrow[\bar{f}]{} & \bar{B}' \end{array} \quad \mapsto \quad \begin{array}{ccc} FB & \xrightarrow{Ff} & FB' \\ {\scriptstyle Fb}\downarrow^{\bullet} & F\beta & \downarrow^{\bullet}{\scriptstyle Fb'} \\ F\bar{B} & \xrightarrow[F\bar{f}]{} & F\bar{B}' \end{array}$$

horiz functorial, vert lax, . . .

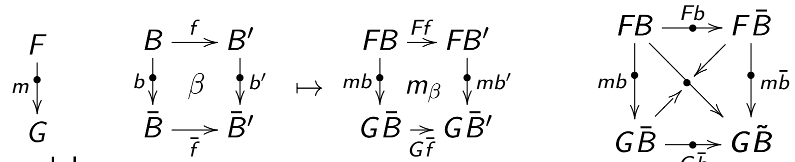
$$F^{\circ}_{B}\colon \mathrm{id}^{\bullet}_{FB} \longrightarrow F\mathrm{id}^{\bullet}_{B}, \quad \widetilde{F}_{b,\bar{b}}\colon F\bar{b}\cdot Fb \longrightarrow F(\bar{b}\cdot b)$$

$t\colon F \longrightarrow F'$
transformation

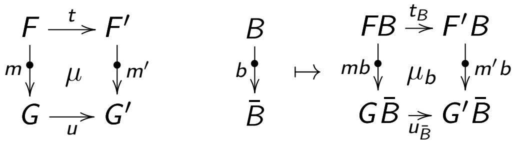
$$\begin{array}{c} B \\ {\scriptstyle b}\downarrow^{\bullet} \\ \bar{B} \end{array} \quad \mapsto \quad \begin{array}{ccc} FB & \xrightarrow{t_B} & F'B \\ {\scriptstyle Fb}\downarrow^{\bullet} & t_b & \downarrow^{\bullet}{\scriptstyle F'b} \\ F\bar{B} & \xrightarrow[t_{\bar{B}}]{} & F'\bar{B} \end{array}$$

horiz natural, vert functorial, . . .

The Double Category $\mathbb{L}\mathrm{ax}(\mathbb{B}, \mathbb{S})$, cont.



module



modulation

Define: $\mathbb{F}\mathrm{un}(\mathcal{B}^{co}, \mathcal{S}\mathrm{pan}) = \mathbb{L}\mathrm{ax}((\mathbb{V}\mathcal{B})^{op}, \mathbb{S}\mathrm{pan})$

The Equivalence $\mathbb{L}\mathrm{ax}(\mathbb{B},\mathbb{S}) \xrightarrow{\mathrm{Mon}} \mathbb{L}\mathrm{ax}_{\mathrm{N}}(\mathbb{B}, \mathbb{M}\mathrm{od}\,\mathbb{S})$

Given $F\colon \mathbb{B} \longrightarrow \mathbb{S}$, define $\mathrm{Mon}\,F\colon \mathbb{B} \longrightarrow \mathbb{M}\mathrm{od}\,\mathbb{S}$ by

$$B \;\mapsto\; (FB \overset{F\mathrm{id}^\bullet_B}{\longrightarrow\!\!\!\!\bullet} FB, \widetilde{F}_{\mathrm{id}^\bullet_B,\mathrm{id}^\bullet_B}, F^\circ_B)$$

Ff homomorphism, Fb module, $F\beta$ equivariant (since F is lax)

Mon: transformations, modules, modulations $\;\mapsto\;$ same

$(\mathrm{Mon})^{-1}$ is composition with $U\colon \mathbb{M}\mathrm{od}\,\mathbb{S} \longrightarrow \mathbb{S}$

$\mathbb{L}\mathrm{ax}(\mathbb{B}, \mathbb{S}\mathrm{pan}) \simeq \mathbb{L}\mathrm{ax}_{\mathrm{N}}(\mathbb{B}, \mathbb{C}\mathrm{at})$, $\;\mathbb{F}\mathrm{un}(\mathcal{B},\mathcal{S}) \simeq \mathbb{F}\mathrm{un}_{\mathrm{N}}(\mathcal{B}, \mathrm{Mod}\,\mathcal{S})$

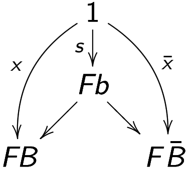
Note: loosely related to $\mathrm{ev} \dashv \mathrm{mod}$ $\quad \dfrac{\mathcal{W} \longrightarrow \mathcal{V}}{\mathbb{1}_{\mathcal{W}} \longrightarrow \mathrm{Mod}\,\mathcal{V}}$ in Moncat / normal in Bicat

Variation for Bicategories

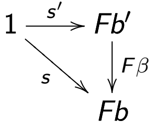
Given $(\mathbb{V}\mathcal{B})^{op} \xrightarrow{F} \mathbb{S}\mathrm{pan}$, consider the projection $\mathcal{B}_F \xrightarrow{P} \mathcal{B}$, where

objects of $\mathcal{B}_F$: $(B, x \in FB)$

morphisms of $\mathcal{B}_F$: $(B, x) \xrightarrow{(b,s)} (\bar{B}, \bar{x})$, with



cells of $\mathcal{B}_F$: $(B, x) \underset{(b',s')}{\overset{(b,s)}{\Downarrow \beta}} (\bar{B}, \bar{x})$, with



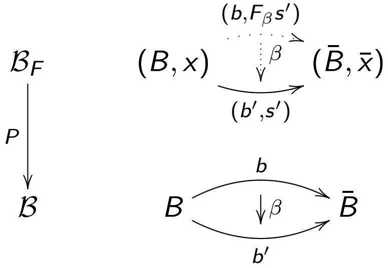
Note: $\mathbb{V}\mathcal{B}_F$ is Paré's "elements of F" $\mathbb{E}\mathrm{l}\, F$, for $F\colon (\mathbb{V}\mathcal{B})^{op} \longrightarrow \mathbb{S}\mathrm{pan}$

Local Discrete Fibrations

A lax functor $P\colon \mathcal{X} \longrightarrow \mathcal{B}$ is a *local discrete fibration* (LDF) if $\mathcal{X}(X,\bar{X}) \longrightarrow \mathcal{B}(PX, P\bar{X})$ is a discrete fibration, for all $X, \bar{X}$

Proposition: $\mathcal{B}_F \xrightarrow{P} \mathcal{B}$ is an LDF strict functor with small fibers

Proof:



Remark: $\begin{array}{ccc} \mathcal{B}_F^{co} & \longrightarrow & \mathcal{S}\mathrm{pan}_* \\ \downarrow & \mathrm{pb} & \downarrow \\ \mathcal{B}^{co} & \longrightarrow & \mathcal{S}\mathrm{pan} \end{array}$ in the category of bicats and lax functors

Local Discrete Fibrations, cont.

A transformation $t\colon F \longrightarrow F'\colon (\mathbb{V}\mathcal{B})^{op} \longrightarrow \mathbb{S}\mathrm{pan}$

$$\begin{array}{c} B \\ b \big\downarrow \bullet \\ \bar{B} \end{array} \quad \mapsto \quad \begin{array}{ccc} FB & \xrightarrow{t_B} & F'B \\ Fb \big\downarrow \bullet & t_b & \bullet \big\downarrow F'b \\ F\bar{B} & \xrightarrow[t_{\bar{B}}]{} & F'\bar{B} \end{array}$$

induces an LDF functor $\mathcal{B}_t\colon \mathcal{B}_F \longrightarrow \mathcal{B}_{F'}$ over $\mathcal{B}$ defined by

$$(B,x) \underset{(b',s')}{\overset{(b,s)}{\rightrightarrows}} (\bar{B},\bar{x}) \ \ {\scriptstyle \downarrow \beta} \quad \mapsto \quad (B,t_B x) \underset{(b',t_{b'}s')}{\overset{(b,t_b s)}{\rightrightarrows}} (\bar{B},t_{\bar{B}}\bar{x}) \ \ {\scriptstyle \downarrow \beta}$$

since the following diagram commutes when the triangle does

by horiz naturality of t

$$\begin{array}{ccccc} 1 & \xrightarrow{s'} & Fb' & \xrightarrow{t_{b'}} & F'b' \\ & {\scriptstyle s} \searrow & \big\downarrow F\beta & & \big\downarrow F'_\beta \\ & & Fb & \xrightarrow[t_b]{} & F'b \end{array}$$

Local Discrete Fibrations, cont.

A module $m\colon F \to\!\!\!\!\bullet\; G\colon (\mathbb{V}\mathcal{B})^{op} \longrightarrow \mathbb{S}\mathrm{pan}$ is given by a lax functor $M\colon (\mathbb{V}(\mathcal{B} \times 2))^{op} \longrightarrow \mathbb{S}\mathrm{pan}$ s.t. $M(-,0) = F$ and $M(-,1) = G$

Thus, m induces an LDF functor $(\mathcal{B} \times 2)_M \longrightarrow \mathcal{B} \times 2$, together with a diagram

$$\begin{array}{ccc}
\mathcal{B}_F & \xrightarrow{P_F} & \mathcal{B} \\
{\scriptstyle \mathrm{LDF}}\downarrow & \mathrm{pb} & \downarrow{\scriptstyle (-,0)} \\
(\mathcal{B} \times 2)_M & \longrightarrow & \mathcal{B} \times 2 \\
{\scriptstyle \mathrm{LDopF}}\uparrow & \mathrm{pb} & \uparrow{\scriptstyle (-,1)} \\
\mathcal{B}_G & \xrightarrow[P_G]{} & \mathcal{B}
\end{array}$$

Local Discrete Fibrations, cont.

A modulation $\begin{array}{ccc} F & \xrightarrow{t} & F' \\ {\scriptstyle m}\downarrow\!\!\bullet & \mu & \bullet\!\!\downarrow{\scriptstyle m'} \\ G & \xrightarrow[u]{} & G' \end{array}$ induces a lax functor

$(\mathcal{B} \times \mathbb{2})_M \longrightarrow (\mathcal{B} \times \mathbb{2})_{M'}$ over $\mathcal{B} \times \mathbb{2}$, and a diagram

$$\begin{array}{ccc} \mathcal{B}_F & \xrightarrow{\mathcal{B}_t} & \mathcal{B}_{F'} \\ \downarrow & \mathrm{pb} & \downarrow \\ (\mathcal{B} \times \mathbb{2})_M & \longrightarrow & (\mathcal{B} \times \mathbb{2})_{M'} \\ \uparrow & \mathrm{pb} & \uparrow \\ \mathcal{B}_G & \xrightarrow[\mathcal{B}_u]{} & \mathcal{B}_{G'} \end{array}$$

The Double Category $\mathbb{LDF}//\mathcal{B}$

objects: $\mathcal{X} \xrightarrow{P} \mathcal{B}$ LDF functors with small fibers

morphisms:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathcal{X}' \\ & {}_{P}\searrow \quad \swarrow_{P'} & \\ & \mathcal{B} & \end{array}$$

horizontal

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{P} & \mathcal{B} \\ {\scriptstyle \mathrm{LDF}}\downarrow & \mathrm{pb} & \downarrow{\scriptstyle (-,0)} \\ \mathcal{M} & \longrightarrow & \mathcal{B} \times \mathbb{2} \\ {\scriptstyle \mathrm{LDopF}}\uparrow & \mathrm{pb} & \uparrow{\scriptstyle (-,1)} \\ \mathcal{Y} & \xrightarrow[Q]{} & \mathcal{B} \end{array}$$

vertical

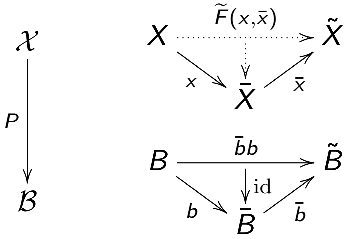
cells:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ \downarrow & \mathrm{pb} & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{M}' \\ \uparrow & \mathrm{pb} & \uparrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}' \end{array} \qquad \text{over} \qquad \begin{array}{c} \mathcal{B} \\ \downarrow{\scriptstyle (-,0)} \\ \mathcal{B} \times \mathbb{2} \\ \downarrow{\scriptstyle (-,1)} \\ \mathcal{B} \end{array}$$

The Equivalence

Theorem: $\mathcal{B}_{-}\colon \mathbb{L}\mathrm{ax}((\mathbb{V}\mathcal{B})^{op}, \mathbb{S}\mathrm{pan}) \longrightarrow \mathbb{L}\mathrm{DF}//\mathcal{B}$ is an equivalence

Proof (sketch): Given $P\colon \mathcal{X} \longrightarrow \mathcal{B}$, define $F\colon (\mathbb{V}\mathcal{B})^{op} \longrightarrow \mathbb{S}\mathrm{pan}$ by $FB = \{X \mid PX = B\}$ and $F(B \xrightarrow{b} \bar{B}) = \{X \xrightarrow{x} \bar{X} \mid Px = b\}$ with projections $FB \xleftarrow{d_0} Fb \xrightarrow{d_1} F\bar{B}$, and constraints $FB \xrightarrow{F^\circ} F\mathrm{id}^\bullet_B$ given by $X \mapsto \mathrm{id}^\bullet_X$, and $F\bar{b} \times_{F\bar{B}} Fb \xrightarrow{\widetilde{F}} F(\bar{b}b)$ by



Horizontal and vertical morphisms of $\mathbb{L}\mathrm{DF}//\mathcal{B}$ give rise to transformations and modules, and cells induce modulations.

A Double Approach to Enrichment for Bicategories

2005 (CNW) Showed $\mathrm{LDF}/\mathcal{B} \simeq \hat{\mathcal{B}}\text{-}\mathrm{Cat}$, where $\hat{\mathcal{B}}$ is the bicategory with $|\hat{\mathcal{B}}| = |\mathcal{B}|$ and $\hat{\mathcal{B}}(B, \bar{B}) = \mathrm{Sets}^{\mathcal{B}(B,\bar{B})^{op}}$

For $F \colon \mathcal{B}(B, \bar{B})^{op} \longrightarrow \mathrm{Sets}$ and $\bar{F} \colon \mathcal{B}(\bar{B}, \tilde{B})^{op} \longrightarrow \mathrm{Sets}$,

$\bar{F} \cdot F \colon \mathcal{B}(B, \tilde{B})^{op} \longrightarrow \mathrm{Sets}$ is given for $c \colon B \longrightarrow \tilde{B}$ by

$$(\bar{F} \cdot F)(c) = \int^{b} \int^{\bar{b}} Fb \times \bar{F}\bar{b} \times \mathcal{B}(c, \bar{b}b)$$

and the identity on B is $(-, \mathrm{id}_B) \colon \mathcal{B}(B, B)^{op} \longrightarrow \mathrm{Sets}$

The Double Category $\hat{\mathcal{B}}$-$\mathbb{C}\mathrm{at}$: Objects

$\hat{\mathcal{B}}$-categories $\mathcal{X}$, i.e., a set $|\mathcal{X}|$ together with a function

$P\colon |\mathcal{X}| \longrightarrow |\mathcal{B}|$, $\hat{\mathcal{B}}$-morphisms $\mathcal{X}[X,\bar{X}]\colon PX \longrightarrow P\bar{X}$, and cells

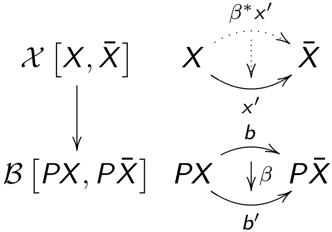
$\mathcal{X}[\bar{X},\tilde{X}] \cdot \mathcal{X}[X,\bar{X}] \longrightarrow \mathcal{X}[X,\tilde{X}]$, and $\mathrm{id}_{PX} \longrightarrow \mathcal{X}[X,X]$ s.t. ...

Example: For $P\colon \mathcal{X} \longrightarrow \mathcal{B}$ an LDF, define

$$\mathcal{X}[X,\bar{X}]\colon \mathcal{B}(PX,P\bar{X})^{op} \longrightarrow \mathrm{Sets}$$

$$b \mapsto \mathcal{X}[X,\bar{X}]_b = \{X \xrightarrow{x} \bar{X} \mid Px = b\}$$

$$\beta^*\colon \mathcal{X}[X,\bar{X}]_{b'} \longrightarrow \mathcal{X}[X,\bar{X}]_b$$



The Double Category $\hat{\mathcal{B}}$-$\mathbb{C}\mathrm{at}$: Horizontal Morphisms

$\hat{\mathcal{B}}$-functors $H\colon \mathcal{X} \longrightarrow \mathcal{X}'$

$$\begin{array}{ccc} |\mathcal{X}| & \xrightarrow{H} & |\mathcal{X}'| \\ & {\scriptstyle P}\searrow \quad \swarrow{\scriptstyle P'} & \\ & |\mathcal{B}| & \end{array} \qquad PX \underset{\mathcal{X}'[HX,H\bar{X}]}{\overset{\mathcal{X}[X,\bar{X}]}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}}}} P\bar{X} \quad \text{s.t. } \ldots$$

Example: For $\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathcal{X}' \\ & {\scriptstyle P}\searrow \quad \swarrow{\scriptstyle P'} & \\ & \mathcal{B} & \end{array}$ in $\mathbb{LDF}/\!/\mathcal{B}$, define

$$H_b\colon \mathcal{X}[X,\bar{X}]_b \longrightarrow \mathcal{X}'[HX,H\bar{X}]_b \;\text{ by }\; X \xrightarrow{x} \bar{X} \mapsto HX \xrightarrow{Hx} H\bar{X}$$

The Double Category $\hat{\mathcal{B}}$-$\mathbb{C}\mathrm{at}$: Vertical Morphisms

$\hat{\mathcal{B}}$-modules $M\colon \mathcal{X} \mathrel{-\!\!\bullet\!\!\!\rightarrow} \mathcal{Y}$

$$\begin{array}{ccc} PX & \xrightarrow{\mathcal{X}[X,\bar{X}]} & P\bar{X} \\ {\scriptstyle M[X,Y]}\Big\downarrow & \diagdown \quad \swarrow & \Big\downarrow{\scriptstyle M[\bar{X},\bar{Y}]} \\ & M[X,\bar{Y}] & \\ & \nearrow \quad \searrow & \\ QY & \xrightarrow[\mathcal{Y}[Y,\bar{Y}]]{} & Q\bar{Y} \end{array} \qquad \text{s.t. } \ldots$$

$$\text{Example:} \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{P} & \mathcal{B} \\ {\scriptstyle i}\Big\downarrow & & \Big\downarrow \\ \mathcal{M} & \xrightarrow{R} & \mathcal{B} \times 2 \\ {\scriptstyle j}\Big\uparrow & & \Big\uparrow \\ \mathcal{Y} & \xrightarrow{Q} & \mathcal{B} \end{array} \text{, define } M[X,Y]_b = \{iX \xrightarrow{m} jY \mid Rm = b\}$$

The Double Category $\hat{\mathcal{B}}$-$\mathbb{C}\mathrm{at}$: Cells

$\hat{\mathcal{B}}$-modulations
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathcal{X}' \\ {\scriptstyle M}\big\downarrow\!\bullet & \rightarrow & \bullet\!\big\downarrow{\scriptstyle M'} \\ \mathcal{Y} & \xrightarrow[K]{} & \mathcal{Y}' \end{array} \qquad PX \underset{M'[HX,KY]}{\overset{M[X,Y]}{\rightrightarrows}} P\bar{X} \ (\downarrow) \quad \text{s.t. } \ldots$$

Example:
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathcal{X}' \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i'} \\ \mathcal{M} & \xrightarrow{L} & \mathcal{M}' \\ {\scriptstyle j}\big\uparrow & & \big\uparrow{\scriptstyle j'} \\ \mathcal{Y} & \xrightarrow{K} & \mathcal{Y}' \end{array}$$
, define $M[X, Y]_b \longrightarrow M'[HX, KY]_b$

$$iX \xrightarrow{m} jY \ \mapsto \ i'HX \xrightarrow{Lm} j'KY$$

Theorem: $\mathbb{F}\mathrm{un}(\mathcal{B}^{co}, \mathcal{S}\mathrm{pan}) \simeq \mathbb{L}\mathrm{DF}//\mathcal{B} \simeq \hat{\mathcal{B}}\text{-}\mathbb{C}\mathrm{at}$