Quantum information-flow, concretely, abstractly

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Abstract

These 'lecture notes' are based on joint work with Samson Abramsky. I will survey and informally discuss the results of [3, 4, 5, 12, 13] in a pedestrian not too technical way. These include:

- *'The logic of entanglement'*, that is, the identification and abstract axiomatization of the *'quantum information-flow'* which enables protocols such as quantum teleportation.[1] To this means we defined *strongly compact closed categories* which abstractly capture the behavioral properties of quantum entanglement.

- '*Postulates for an abstract quantum formalism'* in which *classical information-flow* (e.g. token exchange) is part of the formalism. As an example, we provided a purely formal description of quantum teleportation and *proved correctness in abstract generality*.[2] In this formalism *types reflect kinds*, contra the essentially typeless von Neumann formalism [25]. Hence even concretely this formalism manifestly improves on the usual one.

- *'A high-level approach to quantum informatics'*.[3] Indeed, the above discussed work can be conceived as aiming to solve:

$$\frac{???}{\text{von Neumann quantum formalism}} \simeq \frac{\text{high-level language}}{\text{low-level language}} \, .$$

I also provide a brief discussion on how *classical and quantum uncertainty* can be mixed in the above formalism (cf. density matrices).[4]

1   What? When? Where? Why?

First of all, for us 'quantum' stands for the concepts (both operational and formal) which had to be added to classical physics in order to understand observed phenomena such as

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[1]The identification of quantum information-flow was the content of my QPL I talk.

[2]This abstract quantum formalism will be presented at LiCS'04 in a joint talk by Samson Abramsky and myself, and also in Samson Abramsky's invited lecture.

[3]This will be discussed in Samson Abramsky's invited lecture at LiCS'04.

[4]My talk at QPL II will to a great extent cover this particular feature.

the structure of the spectral lines in atomic spectra, experiments exposing non-local correlations, seemingly 4π symmetries, etc. While the basic part of classical mechanics deals with the (essentially) reversible unitary dynamics of physical systems, quantum required adding the notions of measurement and (possibly non-local) correlations to the discussion. The corresponding mathematical formalism was considered to have reached its maturity in [25]. However!

Where does "it" flow?   Consider quantum teleportation [9],
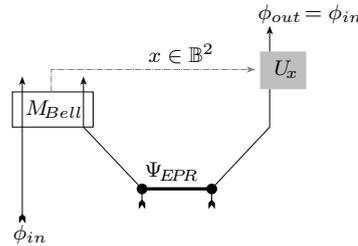
where Ψ_{EPR} denotes an EPR-pair, M_{Bell} a Bell-base measurement, $x \in \mathbb{B}^2$ the transmission of a two-bit token representing the measurement outcome of M_{Bell} and U_x the corresponding unitary correction required for correctly teleporting the state ϕ_{in}. In this process continuous data is transmitted (the state $\phi_{out} = \phi_{in}$) while only using a two-bit classical channel. So where does the 'additional information' flow? The quantum formalism does not tell us in an explicit manner. Clearly it has something to do with the nature of quantum compoundness, but, what exactly? Note that this reasonably simple protocol was only discovered some 60 years after von Neumann's formalism. *Wouldn't it be nice to have a formalism in which inventing quantum teleportation would be an undergraduate exercise?*

Where are the types?   While in the lab measurements are applied to physical systems, application of the corresponding self-adjoint operator $M : \mathcal{H} \to \mathcal{H}$ to the vector $\psi \in \mathcal{H}$ which represents the system's state, hence yielding $M(\psi)$, does not reflect how the state changes during the act of measurement! The actual change is $\psi \mapsto \mathrm{P}_i(\psi)$ for spectral decomposition[5] $M = \sum_i a_i \cdot \mathrm{P}_i$, where a_i is the outcome of the measurement. In addition to this change of state a measurement involves provision of data to 'the observer' cf. teleportation where this data determines the choice of the unitary correction. This contradicts what the corresponding types seem to indicate. The same argument goes for the composite of two self-adjoint operators which in general is not self-adjoint while measurements can be performed sequentially in the lab. *Wouldn't it be nice if types reflect kinds?*

Much worse even, where is the classical information and its flow?   Indeed, the problem regarding types is directly connected to the fact that in von Neumann's formalism there is no place for storage, manipulation and exchange of the classical data obtained from measurements. *We want a quantum formalism which allows to encode classical information and its flow, and hence also one which has enough types to reflect this!*

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[5] Recall that each self-adjoint operator can be written as a weighted sum (or infinite dimensionally, an integral) of mutually orthogonal projectors, i.e. $\mathrm{P}_i \circ \mathrm{P}_j = 0$ for $i \neq j$, projectors themselves being idempotent self-adjoint linear operators (e.g. cf. [12] §A,B).

What is the true essence of quantum? John von Neumann himself was the first to look for this, teaming up with the 'king of lattices' Garrett Birkhoff [10]. It is fair to say that as an attempt to understand 'the whole of quantum mechanics' this particular 'quantum logic' program has failed. While it provided a better understanding of quantum superposition and the superselection rules, it failed at teaching us anything about quantum entanglement, and definitely didn't teach us anything on how quantum and classical information interact. So lattices don't seem to be capable of doing the job. *Which mathematical setting provides an abstract quantum formalism, and its corresponding logic?*

2 The logic of entanglement

A mathematics exercise. The 'Where does "it" flow?' question was addressed and solved in [12, 13]. But the result challenges quantum mechanics' faithfulness to vector spaces! We start by playing a quiz testing the reader's knowledge on the Hilbert space tensor product. Consider the situation depicted below where all boxes represent bipartite projectors on one-dimensional subspaces of Hilbert spaces $\mathcal{H}_i \otimes \mathcal{H}_j$, that is, linear maps

$$\mathrm{P}_\Xi : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_i \otimes \mathcal{H}_j :: \Phi \mapsto \langle \Psi_\Xi \mid \Phi \rangle \cdot \Psi_\Xi$$

with $\Psi_\Xi \in \mathcal{H}_i \otimes \mathcal{H}_j$ and $|\Psi_\Xi| = 1$ so $\mathrm{P}_\Xi(\Psi_\Xi) = \Psi_\Xi$, $\phi_{in} \in \mathcal{H}_1$, $\phi_{out} \in \mathcal{H}_5$, $\Phi_{in} \in \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5$ and hence $\Psi_{\mathbf{in}}, \Psi_{\mathbf{out}} \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5$,
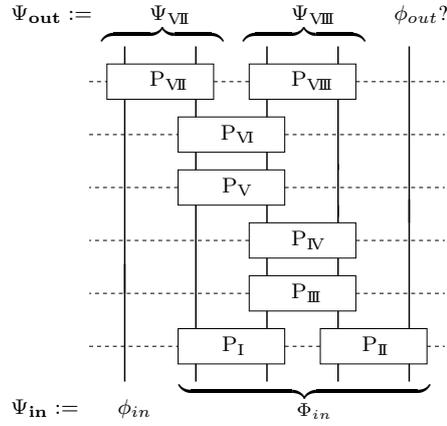
What is ϕ_{out}? (up to a scalar multiple is ok)

In algebraic terms this means solving

$$k \cdot \zeta \left(\phi_{in} \otimes \Phi_{in} \right) = \Psi_{\mathrm{VII}} \otimes \Psi_{\mathrm{VIII}} \otimes \phi_{out}$$

in the unknown ϕ_{out} for $k \in \mathbb{C}$ and

$$\begin{aligned}
\zeta := \ & (\mathrm{P}_{\mathrm{VII}} \otimes \mathrm{P}_{\mathrm{VIII}} \otimes 1_5) \circ (1_1 \otimes \mathrm{P}_{\mathrm{VI}} \otimes 1_{4,5}) \circ (1_1 \otimes \mathrm{P}_{\mathrm{V}} \otimes 1_{4,5}) \circ \\
& (1_{1,2} \otimes \mathrm{P}_{\mathrm{IV}} \otimes 1_5) \circ (1_{1,2} \otimes \mathrm{P}_{\mathrm{III}} \otimes 1_5) \circ (1_1 \otimes \mathrm{P}_{\mathrm{I}} \otimes \mathrm{P}_{\mathrm{II}})
\end{aligned}$$

where 1_i is the identity on $\mathcal{H}_i$ and 1_{ij} is the identity on $\mathcal{H}_i \otimes \mathcal{H}_j$.

At first sight this seems a randomly chosen nasty problem without conceptual significance. But it is not! Observe that bipartite vectors $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ are in bijective correspondence with linear maps $f : \mathcal{H}_1 \to \mathcal{H}_2$ through matrix representation in bases $\{e_i^{(1)}\}_i$ and $\{e_j^{(2)}\}_j$ of $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$\Psi = \sum_{ij} m_{ij} \cdot e_i^{(1)} \otimes e_j^{(2)} \;\overset{\simeq}{\longleftrightarrow}\; \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kn} \end{pmatrix} \;\overset{\simeq}{\longleftrightarrow}\; f :: e_i^{(1)} \mapsto \sum_j m_{ij} \cdot e_j^{(2)},$$

or in bra-ket/qu-nit notation,

$$\sum_{ij} m_{ij} \mid i\,j \rangle = \sum_{ij} m_{ij} \mid i \rangle \otimes \mid j \rangle \;\overset{\simeq}{\longleftrightarrow}\; \sum_{ij} m_{ij} \langle i \mid - \rangle \cdot \mid j \rangle \,.$$

This correspondence lifts to an isomorphism of vector spaces. As an example, the (non-normalized) EPR-state corresponds to the identity

$$\mid 00 \rangle + \mid 11 \rangle \;\overset{\simeq}{\longleftrightarrow}\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\overset{\simeq}{\longleftrightarrow}\; \mathbf{1} = \langle 0 \mid - \rangle \cdot \mid 0 \rangle + \langle 1 \mid - \rangle \cdot \mid 1 \rangle \,.$$

In fact, the correspondence between $\mathcal{H}_1 \otimes \mathcal{H}_2$ and anti-linear maps is a more natural one, since it is independent on the choice of a base for $\mathcal{H}_1$,

$$\sum_{ij} m_{ij} \mid i\,j \rangle = \sum_{ij} m_{ij} \mid i \rangle \otimes \mid j \rangle \;\overset{\simeq}{\longleftrightarrow}\; \sum_{ij} m_{ij} \langle - \mid i \rangle \cdot \mid j \rangle \,,$$

or equivalently, the correspondence between $\mathcal{H}_1^* \otimes \mathcal{H}_2$ and linear maps, where $\mathcal{H}_1^*$ is the vector space of linear functionals $\varphi : \mathcal{H}_1 \to \mathbb{C}$ which arises by setting $\varphi := \langle \psi \mid - \rangle$ for each $\psi \in \mathcal{H}_1$. We will ignore this for now (see [12] for a detailed discussion) and come back to this issue later.

Since we can now 'represent' vectors $\Psi_\Xi \in \mathcal{H}_i \otimes \mathcal{H}_j$ by linear functions of type $\mathcal{H}_i \to \mathcal{H}_j$, and hence also the projectors P_Ξ which appear in the above picture, we can redraw that picture as
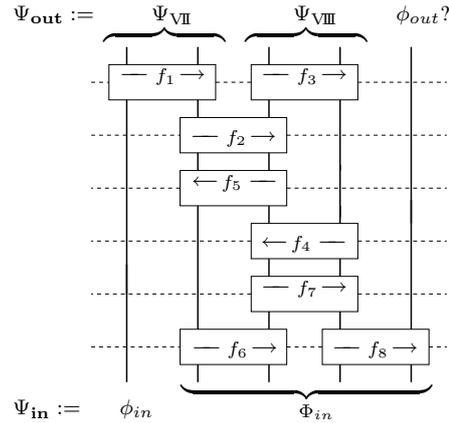

where now $\Psi_{\mathrm{VII}} \overset{\simeq}{\longleftrightarrow} f_1$ and $\Psi_{\mathrm{VIII}} \overset{\simeq}{\longleftrightarrow} f_3$, and the arrows $— f_i \to$ specify the domain and the codomain of the functions f_i, and, I should mention that the new (seemingly somewhat random) numerical labels of the functions and the direction of the arrows are well-chosen

(since, of course, I know the answer to the quiz question). We claim that, provided $k \neq 0$ (see [12]),

$$\phi_{out} = (f_8 \circ f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(\phi_{in})$$

(up to a scalar multiple), and we also claim that this is due to the fact that we can draw a 'line' of which the allowed passages through a projector are restricted to
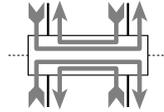

that is, if the line enters at an input (resp. output) of a bipartite box then it has to leave by the other input (resp. output) of that box (note the deterministic nature of the path),

$$\phi_{out} = (f_8 \circ \ldots \circ f_1)(\phi_{in})$$
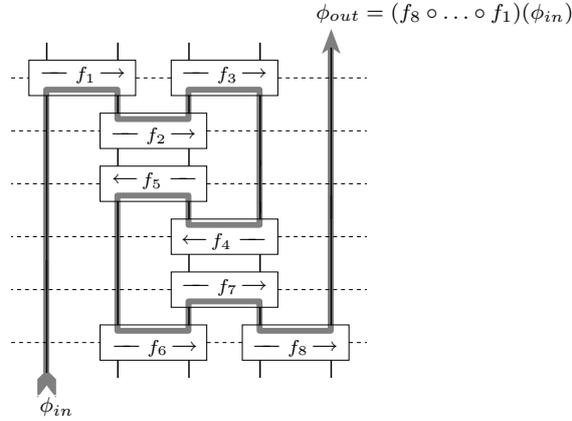
When we follow this line, we first pass through the box labeled f_1, then the one labeled f_2 and so on until f_8. Hence it seems **"as if"** *the information flows from ϕ_{in} to ϕ_{out} following that line and that the functions f_i labeling the boxes act on this information.* Also, $\phi_{out} = (f_8 \circ \ldots \circ f_1)(\phi_{in})$ does not depend on the input of the projectors at $\mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5$ and, more importantly, the order in which we apply the projectors does not reflect the order in which $f_1, \ldots, f_8$ are applied to ϕ_{in} in the expression $(f_8 \circ \ldots \circ f_1)(\phi_{in})$. Doesn't this have a somewhat 'acausal' flavor to it?

The logic of quantum entanglement. We claim that the above purely mathematical observation exposes a *quantum information-flow*. It suffices to conceive the projectors P_Ξ as appearing in the spectral decompositions of self-adjoint operators $M_\Xi := \sum_i a_{\Xi,i} \cdot P_{\Xi,i}$ representing quantum measurements, that is, for some i we have $P_\Xi = P_{\Xi,i}$ (hence the outcome of the measurement represented by M_Ξ is $a_{\Xi,i}$). As an example, consider

$$\phi_{out} = (1 \circ 1)(\phi_{in}) = \phi_{in}$$
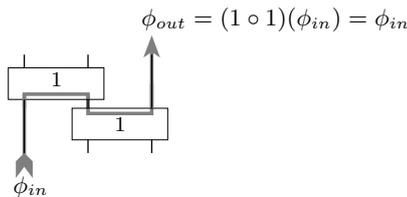
where, since all labeling functions are identities, both projectors project on the EPR-state. Since the first projector corresponds to 'preparing an EPR-state', this picture seems to provide us with a teleportation protocol,

$$\phi_{out} = \phi_{in}$$

$$P_{EPR}$$

$$\Psi_{EPR}$$

$$\phi_{in}$$
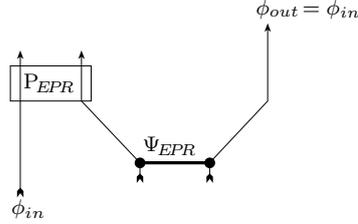
However, physically we cannot implement P_{EPR} on its own 'with certainty'. But P_{EPR} is part of Bell-base measurement together with three other projectors. We denote the corresponding labeling functions by $\beta_2, \beta_3, \beta_4$. The grey boxes below denote unitary transformations. We have

$$\phi_{out} = \phi_{in} \qquad \phi_{out} = \phi_{in} \qquad \phi_{out} = \phi_{in} \qquad \phi_{out} = \phi_{in}$$
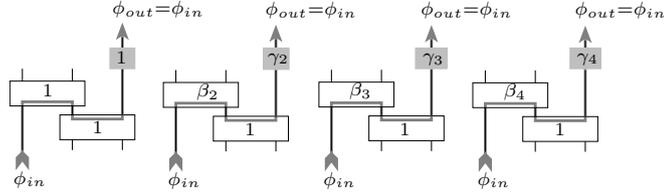
where $\gamma_i \circ \beta_i$ has to be the identity so $\gamma_i = \beta_i^{-1}$. These four pictures together yield the full teleportation protocol! The classical communication is encoded in the fact that in each picture the unitary correction γ_i depends on β_i, that is, the measurement outcome. Hence the classical communication does not contribute to the transmission of the data, it only *distributes the knowledge* about 'which of the four pictures is actually taking place'.

To conclude this paragraph we stress that the functional labels are not actual physical operations but only arise in the above discussed mathematical isomorphism. Further, in the generic example

$$\phi_{out} = (f_2 \circ f_1)(\phi_{in})$$

$$f_1$$

$$f_2$$

$$\phi_{in}$$
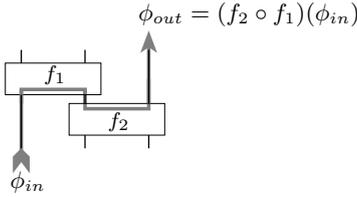
the order of the physical operations is opposite to the order in which their labels apply to the input state in the expression $(f_2 \circ f_1)(\phi_{in})$. Algebraically,[6]

$$k \cdot \zeta(\phi_{in} \otimes \Phi_{in}) = \Psi_{f_1} \otimes (f_{\underline{2}} \circ f_{\underline{1}})(\phi_{in}) \quad \text{for} \quad \zeta = (P_{f_{\underline{1}}} \otimes 1) \circ (1 \otimes P_{f_{\underline{2}}})$$

with $\Psi_f \xleftrightarrow{\simeq} f$ and $P_f(\Psi_f) = \Psi_f$ as a new notation. Slightly simpler,

$$(P_{f_{\underline{1}}} \otimes 1)(\phi_{in} \otimes \Psi_{f_{\underline{2}}}) = \Psi_{f_1} \otimes (f_{\underline{2}} \circ f_{\underline{1}})(\phi_{in}),$$

by conceiving the first projector as a state. Furthermore, the above discussed $*$ in $\mathcal{H}_1^* \otimes \mathcal{H}_2$ which is necessary to have a base-independent correspondence with linear functions '*is*

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[6]The pictures really look much better than the formulas, don't they?

not a bug but a feature', it actually witnesses (by means of a phase conjugation) the fact that the line changes its temporal direction every time it passes a projector box (see [12]).

Using the same line of thought it is also easy to reconstruct other protocols such as *logic-gate teleportation* [15] and *entanglement swapping* [27], and, the quantum information-flow interpretation also extends to multipartite projectors. We refer the reader to [12, 13] for details on this.

The question Samson and I then asked was: "Are these information-flow features specifically related to the Hilbert space structure? Or to ..."

Sets, relations and the cartesian product. Doesn't sound very 'quantum' you say? Let's see. We make the following substitutions in the above:

$$
\begin{aligned}
\text{Hilbert space } \mathcal{H} &\rightsquigarrow \textbf{set } X \\
\text{linear function } f &\rightsquigarrow \textbf{relation } R \\
\text{tensor product } \otimes &\rightsquigarrow \textbf{cartesian product } \times
\end{aligned}
$$

Can we also translate projectors to this world of relations? Observe that for projectors on one-dimensional subspaces, which take the general form $P_\psi = \langle \psi \mid - \rangle \cdot \mid \psi \rangle : \mathcal{H} \to \mathcal{H}$, we have $\mid \psi \rangle \otimes \mid \psi \rangle \overset{\simeq}{\longleftrightarrow} \langle \psi \mid - \rangle \cdot \mid \psi \rangle$,[7] that is, projectors correspond with symmetric pure tensors. By analogy we define a projector of type $X \to X$ as $A \times A \subseteq X \times X$ in the world of relations.[8] Hence $R \times R \subseteq (X \times Y) \times (X \times Y)$ with $R \subseteq (X \times Y)$ is a bipartite projector in the world of relations which we denote by P_R in analogy with P_f. Since for the identity relation $1 \subseteq X \times X$ we have $x_1 1 x_2 \Leftrightarrow x_1 = x_2$ and since

$$
P_R := R \times R = \left\{ \big((x_1, y_1), (x_2, y_2)\big) \in (X \times Y) \times (X \times Y) \,\Big|\, x_1 R y_1, x_2 R y_2 \right\},
$$

for $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$ we have

$$
(x_1, y_1, z_1)(1_X \otimes P_{R_2})(x_2, y_2, z_2) \;\Leftrightarrow\; y_1 R_2 z_1 \text{ and } \underline{y_2 R_2 z_2}, \text{ and, } \underline{x_1 = x_2}\,,
$$

$$
(x_2, y_2, z_2)(P_{R_1} \otimes 1_Z)(x_3, y_3, z_3) \;\Leftrightarrow\; \underline{x_2 R_1 y_2} \text{ and } x_3 R_1 y_3, \text{ and, } \underline{z_2 = z_3}\,.
$$

Setting $s_{in} := x_1$, $s_{out} := z_3$ and using the underlined expressions,

$$
(s_{in}, y_1, z_1)\big((P_{R_{\underline{1}}} \otimes 1) \circ (1 \otimes P_{R_{\underline{2}}})\big)(x_3, y_3, s_{out})
$$

entails $s_{in}(R_{\underline{2}} \circ R_{\underline{1}}) s_{out}$.[9] And this is not an accident!

3 The abstract algebra of entanglement

Categories for physical systems. Which abstract structure do Hilbert spaces and relations share? First of all, the above construction would not work if instead of relations we had taken functions. The importance of considering appropriate maps indicates that we will have to consider *categories*. As theoretical computer scientists know, categories are

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[7]Again we ignore un-naturality, that is, the slight base-dependency.
[8]Recall that a relation of type $X \to Y$ is a subset of $X \times Y$ (cf. its 'graph').
[9]We invite the reader to make a picture of this.

not just a language, nor metamathematics, nor hyper abstraction. They are mathematical objects in their own right which arise very naturally in 'real situations'. E.g. one takes the state spaces of the systems under consideration to be the *objects*, and (physical) operations on these systems to be *morphisms* (including a *skip* operation), the axioms of a category are then satisfied by the mere fact that operations can be *composed*. We denote by **Rel** the category of sets and relations, by **Set** the category of sets and functions, by **FdHilb** finite dimensional (complex) Hilbert spaces and linear maps, and more generally, by **FdVec**$_\mathbb{K}$ finite dimensional vector spaces over a field $\mathbb{K}$.

If instead of the cartesian product we would have considered disjoint union on sets, again things wouldn't have worked out. Also in the quantum case the use of the tensor product is crucial. All this indicates that we want some specific *bifunctor* $\boxtimes$ to live on our category, $\times$ on **Rel** and $\otimes$ on **FdVec**$_\mathbb{K}$. Intuitively, we think of a bifunctor as an operation which allows to combine systems, and also the operations thereon, and, the *bifunctoriality* property has a clear physical interpretation: if S_1 and S_2 are distinct physical entities, when performing operation O_1 on S_1 and O_2 on S_2, the order in which we perform O_1 and O_2 doesn't matter. One typically thinks of *local operations* on spatially separated systems.

In categories, *elements* of an object A can be thought of as morphisms $q : I \to A$ where I is a unit for the bifunctor, i.e. $A \boxtimes I \simeq I \boxtimes A \simeq A$. In (**FdHilb**, $\otimes$) we have $I := \mathbb{C}$, and indeed, maps $q : \mathbb{C} \to \mathcal{H}$ are in bijective correspondence with $\mathcal{H}$ itself, by considering $q(1) \in \mathcal{H}$. In (**Set**, $\times$) and (**Rel**, $\times$) we have $I := \{*\}$, i.e., a singleton. In (**Set**, $\times$) maps $q : \{*\} \to X$ are in bijective correspondence with elements of X by considering $q(*) \in X$. But not in (**Rel**, $\times$)! Morphisms $q \subseteq \{*\} \times X$ now correspond to all subsets of X, which can be thought of as *superpositions* of the individual elements.[10]

We want not only a unit I for $\boxtimes$, but a full *symmetric monoidal* structure, that is, we want the following *natural isomorphisms* [11]

$$\lambda_A : A \simeq I \boxtimes A \qquad \rho_A : A \simeq A \boxtimes I \qquad \sigma_{A,B} : A \boxtimes B \simeq B \boxtimes A$$

$$\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \simeq (A \boxtimes B) \boxtimes C .$$

Note here that we do not require $\boxtimes$-*projections* $p_{A,B} : A \boxtimes B \to A$ nor $\boxtimes$-*diagonals* $\Delta_A : A \to A \boxtimes A$ to exist. More precisely, we don't want them to exist, and this will be guaranteed by a piece of structure we shall introduce. In physical terms this non-existence means *no-cloning* [26] and *no-deleting* [23]. In categorical terms it means that $\boxtimes$ is *not a categorical product*.[12] In logical terms this means that we are doing *linear logic* [14, 20, 21, 24] as opposed to classical logic. In linear logic we are not allowed to copy and delete assumptions, that is, $A \wedge B \Rightarrow A$ and $A \Rightarrow A \wedge A$ are not valid.

Compact closure and information-flow.   Crucial in the analysis of the quantum information-flow was $\mathcal{H}_1^* \otimes \mathcal{H}_2 \simeq \mathcal{H}_1 \to \mathcal{H}_2$. In categorical terms, making sense of $\mathcal{H}_1 \to$

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[10]Compare this to 'superposition' in lattice theoretic terms: an atomic lattice has superposition states if the join of two atoms has additional atoms below it (e.g. cf. [11]).

[11]A categorical isomorphism is a morphism $f : A \to B$ with an inverse $f^{-1} : B \to A$, that is, $f \circ f^{-1} = 1_A$ and $f^{-1} \circ f = 1_B$. A natural isomorphism is a strong notion of categorical isomorphism. For vector spaces it essentially boils down to 'base independent', e.g. there exists a natural isomorphism of type $(\mathcal{H}_1^* \otimes \mathcal{H}_2) \longrightarrow (\mathcal{H}_1 \to \mathcal{H}_2)$ but not one of type $(\mathcal{H}_1 \otimes \mathcal{H}_2) \longrightarrow (\mathcal{H}_1 \to \mathcal{H}_2)$, where we treat $\mathcal{H}_1 \to \mathcal{H}_2$ as a Hilbert space.

[12]See below where we discuss biproducts.

$\mathcal{H}_2$ requires the category to be *closed*.[13] To give sense to the $*$ we require it to be $*$-*autonomous* [8],[14] and finally, requiring $\mathcal{H}_1^* \otimes \mathcal{H}_2 \simeq \mathcal{H}_1 \to \mathcal{H}_2$ implies that the category is *compact closed* [18]. In logical terms this means that we have the multiplicative fragment of linear logic, with negation, and where *conjunction is self-dual*, that is, it coincides with disjunction — indeed, you read this correct, $A \wedge B \simeq A \vee B$.

But we will follow a different path which enables us to use less categorical jargon. This path is known in category theory circles as *Australian* or *Max Kelly* style category theory. Although this style is usually conceived (even by category theoreticians) as of an abstract$^\infty$ nature, in our particular case, it's bull's-eye for understanding the quantum information-flow.[15]

In [19] a category $\mathbf{C}$ is defined to be compact closed iff for each object A three additional pieces of data are specified, an object denoted A^*, a morphism $\eta_A : \mathrm{I} \to A^* \boxtimes A$ called *unit* and a morphism $\epsilon_A : A \boxtimes A^* \to \mathrm{I}$ called *counit*, which are such that the diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\quad\simeq\quad} & A \boxtimes \mathrm{I} & \xrightarrow{1_A \boxtimes \eta_A} & A \boxtimes (A^* \boxtimes A) \\
\Big\downarrow{\scriptstyle 1_A} & & & & \Big\downarrow{\scriptstyle \simeq} \\
A & \xleftarrow{\quad\simeq\quad} & \mathrm{I} \boxtimes A & \xleftarrow{\epsilon_A \boxtimes 1_A} & (A \boxtimes A^*) \boxtimes A
\end{array}
$$

and the same diagram for A^* both commute. Although at first sight this diagram seems quite intangible, we shall see that this diagram perfectly matches the teleportation protocol. Both $(\mathbf{Rel}, \times)$ and $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$ are compact closed, respectively for $X^* := X$, $\eta_X = \{(*, (x,x)) \mid x \in X\}$ and $\epsilon_X = \{((x,x), *) \mid x \in X\}$, and, for V^* the dual vector space of linear functionals, for $\{\bar{e}_i\}_{i=1}^{i=n}$ being the base of V^* satisfying $\bar{e}_i(e_j) = \delta_{ij}$,

$$
\eta_V :: 1 \mapsto \sum_{i=1}^{i=n} \bar{e}_i \otimes e_i \qquad \text{and} \qquad \epsilon_V :: e_i \otimes \bar{e}_j \mapsto \delta_{ij} \,.
$$

(if V has an inner-product, $\bar{e}_i := \langle e_i \mid - \rangle$) Note that $\eta_V(1)$ can be thought of as an abstract generalization of the notion of an EPR-state.

Given the *name* and *coname* of a morphism $f : A \to B$, respectively

$$
\ulcorner f \urcorner := (1 \boxtimes f) \circ \eta_A : \mathrm{I} \to A^* \boxtimes B \ \text{ and } \ \llcorner f \lrcorner := \epsilon_A \circ (f \boxtimes 1) : A \boxtimes B^* \to \mathrm{I},
$$

one can prove the **Compositionality Lemma** ([4] §3.3), diagrammatically,

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[13]For a monoidal category to be closed indeed means that we can 'internalize' morphism sets $A \to B$ as objects, also referred to as the category having exponentials. Typically, one thinks of $\boxtimes$ as conjunction and of this internalization as implication.

[14]$*$-autonomy means that there exists an operation $*$ on the monoidal category from which the internalization of morphism sets follows as $(A \boxtimes B^*)^*$, cf. classical logic where we have $A \Rightarrow B = \neg A \vee B = \neg(A \wedge \neg B)$ by the De Morgan rule.

[15]When we spell out this alternative definition of compact closure it indeed avoids much of the categorical jargon. But it also has a very elegant abstract formulation in terms of *bicategories*: a compact closed category is a symmetric monoidal category in which, when viewed as a one-object bicategory, every one-cell A has a left adjoint A^*.

$$A \longrightarrow_{\simeq} A \boxtimes \mathrm{I} \xrightarrow{1_A \boxtimes \ulcorner f_{\underline{2}} \urcorner} A \boxtimes (B^* \boxtimes C)$$

with left vertical arrow $f_{\underline{2}} \circ f_{\underline{1}}$ from A to C, right vertical arrow $\simeq$, bottom row

$$C \xleftarrow{\simeq} \mathrm{I} \boxtimes C \xleftarrow{\llcorner f_{\underline{1}} \lrcorner \boxtimes 1_A} (A \boxtimes B^*) \boxtimes C$$

for $f_1 : A \to B$ and $f_2 : B \to C$. This lemma generalizes the defining diagram of compact closedness since $\eta_A = \ulcorner 1_A \urcorner$ and $\epsilon_A = \llcorner 1_A \lrcorner$ (cf. EPR-state $\xleftrightarrow{\simeq} 1$). The careful reader will have understood the picture by now,

hence it seems as if there is an information flow through names and conames,
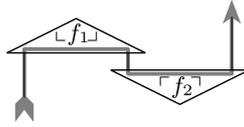
Are we really there yet? We actually have two things, names and conames, and names act as 'the output of a bipartite projector' while conames act as 'the input of a bipartite projector'. The obvious thing to do is to glue a coname and a name together in order to produce a bipartite projector.


However, we have a type-mismatch.

$$P_f :\stackrel{?}{=} \ulcorner f \urcorner \circ \llcorner f \lrcorner : A \boxtimes B^* \to A^* \boxtimes B$$

To solve this problem we need a tiny bit of extra structure. This bit of extra structure will capture the idea of *complex conjugation*. When conceiving elements as *Dirac-kets*, it will provide us with a notion of *Dirac-bra*.

Strong compact closure, inner-products and projectors. The assignment $A \mapsto A^*$ which arises as part of the definition of compact closure actually extends to one on morphisms,

$$B^* \longrightarrow_{\simeq} \mathrm{I} \boxtimes B^* \xrightarrow{\eta_A \boxtimes 1_{B^*}} (A^* \boxtimes A) \boxtimes B^*$$

with left vertical arrow f^* from B^* to A^*, right vertical arrow $1_{A^*} \boxtimes f \boxtimes 1_{B^*}$, bottom row

$$A^* \xleftarrow{\simeq} A^* \boxtimes \mathrm{I} \xleftarrow{1_{A^*} \boxtimes \epsilon_B} A^* \boxtimes (B \boxtimes B^*)$$

and again this looks much nicer in a picture,
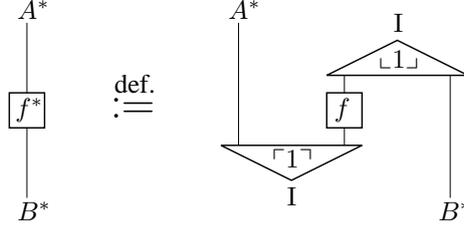
It is as if the information flows backward through f,
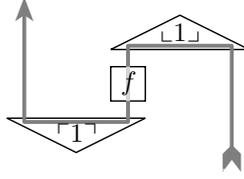
For vector spaces the matrix of f^* is the *transposed* of the matrix of f when taking $\{\bar{e}_i\}_{i=1}^{i=n}$ as base for V^* given base $\{e_i\}_{i=1}^{i=n}$ of V. For relations R^* is the *relational converse* of R. One verifies that $(\ )^* : \mathbf{C} \to \mathbf{C}$ is a *contravariant functor*, that is $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$, and that there exists a natural isomorphism $A^{**} \simeq A$.

Definition. A *strongly compact closed category* [4] is a compact closed category for which $A = A^{**}$ and for which the assignment $A \mapsto A^*$ has also an *involutive covariant functorial extension*, which *commutes with* $(\ )^*$.

We set $f \mapsto f_*$ for this functorial extension. For each morphism $f : A \to B$ we define its *adjoint* and a *bipartite projector* as

$$f^\dagger := (f_*)^* = (f^*)_* : B \to A \ \text{ and } \ \mathrm{P}_f := \ulcorner f \urcorner \circ \llcorner f_* \lrcorner : A^* \boxtimes B \to A^* \boxtimes B,$$

and we call an isomorphism $U : A \to B$ *unitary* iff $U^{-1} = U^\dagger$. An abstract notion of *inner-product* also emerges. Given elements $\psi, \phi : \mathrm{I} \to A$ we set $\langle \psi \mid \phi \rangle := \psi^\dagger \circ \phi \in \mathbf{C}(\mathrm{I}, \mathrm{I})$ where $\mathbf{C}(\mathrm{I}, \mathrm{I})$ are the morphisms of type $\mathrm{I} \to \mathrm{I}$ — we discuss these *scalars* in more detail below. We can now prove the usual defining properties of adjoints and unitarity in abstract generality,

$$\langle f^\dagger \circ \psi \mid \phi \rangle_B = (f^\dagger \circ \psi)^\dagger \circ \phi = \psi^\dagger \circ f \circ \phi = \langle \psi \mid f \circ \phi \rangle_A \,,$$

$$\langle U \circ \psi \mid U \circ \varphi \rangle_B = \langle U^\dagger \circ U \circ \psi \mid \varphi \rangle_A = \langle \psi \mid \varphi \rangle_A \,.$$

When calling $\psi : \mathrm{I} \to A$ a *ket*, then $\psi^\dagger : \mathrm{I} \to A$ is the corresponding *bra* and the scalar $\phi^\dagger \circ \psi : \mathrm{I} \to \mathrm{I}$ is a *bra-ket*. Hence strong compact closure provides a nice and juicy lump of Hilbert space — see [4] §7 and [5] §2 for details.

The category $(\mathbf{Rel}, \times)$ is trivially strongly compact closed for $R_* := R$, so $R^\dagger = R^*$, that is, adjoints are relational converses. The same goes for any compact closed category where $A^* = A$. For $(\mathbf{FdVec}_\mathbb{K}, \otimes)$ we don't have $V^* = V$, nor does the above defined compact closed structure satisfy $V^{**} = V$, so it cannot be extended to a strong compact closed structure. But for $\mathbb{K} := \mathbb{R}$, finite-dimensional real inner-product spaces are strongly compact closed for $V := V^*$ and $\epsilon_V := \langle - \mid - \rangle$, and for $\mathbb{K} := \mathbb{C}$, our main category $(\mathbf{FdHilb}, \otimes)$ is also strongly compact closed when we take $\mathcal{H}^*$ to be the *conjugate space*, that is, the Hilbert space with the same elements as $\mathcal{H}$ but with $\alpha \bullet_{\mathcal{H}^*} \phi := \bar{\alpha} \bullet_\mathcal{H} \phi$ as scalar multiplication and $\langle \phi \mid \psi \rangle_{\mathcal{H}^*} := \langle \psi \mid \phi \rangle_\mathcal{H}$ as (sesquilinear) inner-product. We can

then set $\epsilon_{\mathcal{H}} : \mathcal{H} \otimes \mathcal{H}^* \to \mathrm{I} :: \psi \otimes \phi \mapsto \langle \phi \mid \psi \rangle$. One verifies that we recover the usual notion of adjoint, that is, the *conjugate transpose*, where $(\ )^*$ provides transposition while $(\ )_*$ provides complex conjugation.

Let us end this paragraph by saying that most things discussed above extend to *infinite dimensional* settings when using ideas from [2].

A note on categorical traces.   This paragraph slightly diverges from our story line, but we do want to mention that much of the inspiration for [12, 13] emerged from [3] where we studied the physical realization of 'abstract traces' [17], which generalize traditional *feedback traces* [1, 7]. It turns out that both on $(\mathbf{Rel}, \times)$ and $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$, due to compact closure, the trace also admits a feedback-loop type interpretation, but a linear 'only-use-once' one. Please consult [5] for more details and some nice pictures.

4   Beyond von Neumann's axiomatics

Biproducts.   Strong compact closure provides a serious lump of Hilbert space, but we need some additional types which enable to encode classical information and its flow in our quantum formalism. They will capture 'gluing pictures together' and 'distributing the knowledge on in which picture we are' (cf. §2). To this means we use *biproducts*, that is, objects $A \boxplus B$ which both are the *product* and the *coproduct* for A and B, and corresponding induced morphisms $f \boxplus g : A \boxplus B \to C \boxplus D$ for $f : A \to C$ and $g : B \to D$. Contrary to $\boxtimes$, biproducts go (by definition) equipped with *projections* $p_j : \boxplus_i A_i \to A_j$, also with *injections* $q_j : A_j \to \boxplus_i A_i$, and with *pairing* and *copairing* operations, $\langle f_i \rangle_i : A \to \boxplus_i A_i$ and $[f_i]_i : \boxplus_i A_i \to A$, for morphisms $f_i : A \to A_i$ and $g_i : A_i \to A$ with coinciding domain and codomain respectively. From these we can construct *diagonals* and *codiagonals*, $\Delta_A := \langle 1_A, 1_A \rangle : A \to A \boxplus A$ and $\nabla_A := [1_A, 1_A] : A \boxplus A \to A$. This 'non-linear' $\boxplus$-structure encodes that there is no difference between looking at two pictures separately, or together — the components of a compound quantum system cannot be considered separately, hence $\boxtimes$ is linear.

We take the projections and injections such that they work nicely together with the strong compact closure by setting $q_i^\dagger = p_i$ (and hence $p_i^\dagger = q_i$). Of crucial importance for us is the *distributivity* of $\boxtimes$ over $\boxplus$,[16] that is, there is a natural isomorphism

$$\mathrm{DIST} : A \boxtimes (B_1 \boxplus B_2) \simeq (A \boxtimes B_1) \boxplus (A \boxtimes B_2) \,.$$

For $(\mathbf{Rel}, \times)$ the *disjoint union* $+$ provides a biproduct structure with inclusion as injections. For $(\mathbf{FdHilb}, \otimes)$ the *direct sum* $\oplus$ provides a biproduct structure with coordinate projections as projections.

Categorical quantum mechanics.   We define a quantum formalism relative to any strongly compact closed category with some biproducts.

i. We take *state spaces* to be objects which do not involve explicit biproducts and use $\boxtimes$ to describe compound systems. The basic *data unit* is a state space Q which is unitary isomorphic to $\mathrm{I} \boxplus \mathrm{I}$, which in the case of $(\mathbf{Rel}, \times, +)$ where $\mathrm{I} \boxplus \mathrm{I} = \{*\} + \{*\}$ yields the

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[16]Which follows by closedness of $\boxtimes$ and $\boxplus$ being a coproduct.

boolean type and in the case of $(\mathbf{FdHilb}, \otimes, \oplus)$ where $I \boxplus I = \mathbb{C} \oplus \mathbb{C}$ yields the *qubit type*.

ii. Explicit biproducts express 'different pictures' due to distinct measurement outcomes, they enable to encode *classical data*. The distributivity isomorphism DIST expresses *exchange of classical data* ! (see below)

iii. We have already defined bipartite projectors. To turn them into a measurement we need to glue a complete family of mutually orthogonal ones to each other. More generally, we define a *spectral decomposition* to be a unitary morphism $U : A \to \boxplus_i A_i$. We define the corresponding *non-destructive measurement* to be the copairing

$$\langle \mathrm{P}_i \rangle_i : A \to \boxplus_i A \quad \text{where} \quad \mathrm{P}_j = \pi_j^\dagger \circ \pi_j : A \to A \quad \text{for} \quad \pi_j = p_j \circ U$$

with $p_j : \boxplus_i A_i \to A_j$ the projections for the biproduct $\boxplus_i A_i$. As shown in [4], these general *projectors* $\mathrm{P}_i : A \to A$ are self-adjoint, mutually orthogonal, and their sum is 1_A — we discuss the *sum of morphisms* below. When the spectral decomposition is of type $A \to \boxplus_i I$ the corresponding measurement is *non-degenerated*. We call such a spectral decomposition, which by the defining property of products can be rewritten as $\langle \pi_i \rangle_i : A \to \boxplus_i I$, a *non-degenerated destructive measurement*. For an explicit definition of an *abstract Bell-base measurement*, or any other measurement which allows teleportation, we refer to [4]. *Isolated reversible dynamics* is unitary.

iv. The passage from a non-degenerated non-destructive measurement to a destructive one involves dropping $\psi_i := \pi_i^\dagger : I \to A$. We conceive such a component as a *preparation*. Hence a non-destructive measurement decomposes in $\langle \pi_i \rangle_i$, which gives the measurement's outcome, and ψ_i, which gives the state 'after the collapse' (cf. von Neumann's projection postulate).

Abstract quantum teleportation. The right-hand side of the diagram in Figure 1 *gives a complete description of the teleportation protocol*. The left-hand side expresses the intended behavior (obtaining an identity in each of the four pictures). In [4] we proved correctness, *the diagram commutes* !

Abstract presentations and proofs of correctness of *logic gate teleportation* [15] and *entanglement swapping* [27] can be found in [4].

Immediately after the Bell-base measurement the type is $\left(\boxplus_{i=1}^{i=4} I \right) \boxtimes Q$ where $\boxplus_{i=1}^{i=4} I$ represents the four different measurement outcomes. However, these four pictures only exist 'locally'. After *distributing this information*,

$$\left(\boxplus_{i=1}^{i=4} I \right) \boxtimes Q \xrightarrow{\text{DIST}} \boxplus_{i=1}^{i=4} (I \boxtimes Q) \xrightarrow{\boxplus_{i=1}^{i=4} \lambda_Q^{-1}} \boxplus_{i=1}^{i=4} Q \,,$$

there are four different pictures 'globally'. Hence we can apply the appropriate unitary correction $\beta_i^{-1} : Q \to Q$ in each picture, that is, $\boxplus_{i=1}^{i=4} \beta_i^{-1}$.

The spectrum of a measurement $\langle \mathrm{P}_i \rangle_i$ is the index set $\{i\}_i$, which for example could encode locations in physical space. Since for teleportation we assume to work with spatially located particles, that is, there are no spatial superpositions, the associativity natural isomorphism allows to encode *spatial association* (i.e. proximity) in a qualitative manner.

Scalars, normalization, probabilities and the Born rule. Up to now one might think that the abstract setting is purely qualitative (whatever that means anyway). But it is not!
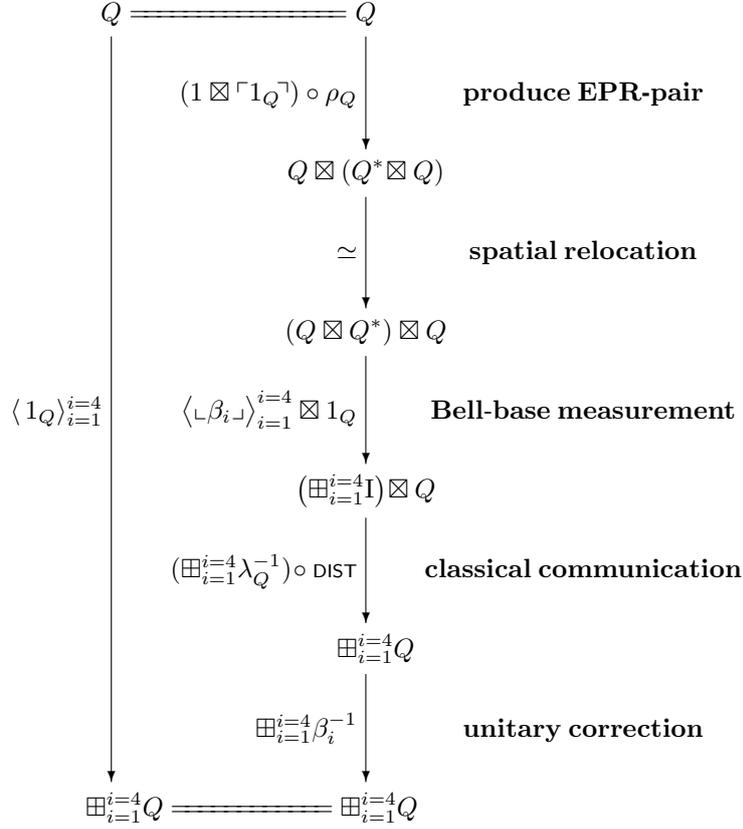
$$Q \xlongequal{\hspace{3cm}} Q$$

$$(1 \boxtimes \ulcorner 1_Q \urcorner) \circ \rho_Q \Big\downarrow \qquad \textbf{produce EPR-pair}$$

$$Q \boxtimes (Q^* \boxtimes Q)$$

$$\simeq \Big\downarrow \qquad \textbf{spatial relocation}$$

$$(Q \boxtimes Q^*) \boxtimes Q$$

$$\langle 1_Q \rangle_{i=1}^{i=4} \qquad \langle \llcorner \beta_i \lrcorner \rangle_{i=1}^{i=4} \boxtimes 1_Q \Big\downarrow \qquad \textbf{Bell-base measurement}$$

$$\big(\boxplus_{i=1}^{i=4} \mathrm{I} \big) \boxtimes Q$$

$$\big(\boxplus_{i=1}^{i=4} \lambda_Q^{-1} \big) \circ \textsc{dist} \Big\downarrow \qquad \textbf{classical communication}$$

$$\boxplus_{i=1}^{i=4} Q$$

$$\boxplus_{i=1}^{i=4} \beta_i^{-1} \Big\downarrow \qquad \textbf{unitary correction}$$

$$\boxplus_{i=1}^{i=4} Q \xlongequal{\hspace{3cm}} \boxplus_{i=1}^{i=4} Q$$

Figure 1: ABSTRACT QUANTUM TELEPORTATION

The scalars $\mathbf{C}(\mathrm{I}, \mathrm{I})$ of any monoidal category $\mathbf{C}$ have a commutative composition [19], that is, a *multiplication*.

If the biproduct $\mathrm{I} \oplus \mathrm{I}$ exists, we can define a *sum of scalars* $s, s' : \mathrm{I} \to \mathrm{I}$ as $s + s' := \nabla_{\mathrm{I}} \circ (s \boxplus s') \circ \Delta_{\mathrm{I}} : \mathrm{I} \to \mathrm{I}$, and one shows that the above defined multiplication distributes over this sum and that there is a zero $O_{\mathrm{I}} : \mathrm{I} \to \mathrm{I}$. Hence we obtain an *abelian semiring*.[17]

Furthermore, each scalar $s : \mathrm{I} \to \mathrm{I}$ induces a natural transformation

$$s_A : \lambda_A^{-1} \circ (s \otimes 1_A) \circ \lambda_A : A \to A$$

for each object A, which allows us to define *scalar multiplication* as $s \bullet f := f \circ s_A$ for $f : A \to B$, where $f \circ s_A = s_B \circ f$ by naturality, that is, *morphisms preserve scalar multiplication*.

Since we have an inner-product (which, of course, is scalar valued) we can now talk

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[17]That is, a field except that there are no inverses for addition nor for multiplication.

about *normalization* e.g. an element $\psi : \mathrm{I} \to A$ is normalized iff $\psi^\dagger \circ \psi = 1_\mathrm{I}$.[18] Besides the special scalars 1_I and 0_I there are many others, those which satisfy $s^\dagger = s$, those of the form $s^\dagger \circ s$, those which arise from inner-products of normalized elements, and the latter multiplied with their adjoint, in $(\mathbf{FdHilb}, \otimes, \oplus)$ respectively being 1 and 0, the reals $\mathbb{R}$, the positive reals $\mathbb{R}^+$, the unit disc in $\mathbb{C}$ and the unit interval $[0, 1]$.

Consider now the basic protocol of (non-destructively) measuring a state

$$\mathrm{I} \xrightarrow{\;\;\psi\;\;} A \xrightarrow{\;\langle \mathrm{P}_i \rangle_{i=1}^{i=n}\;} \boxplus_{i=1}^{i=n} A \,.$$

If we look at one component of the biproduct, i.e., one picture,

$$\underbrace{\mathrm{I} \xrightarrow{\;\;\psi\;\;} A \xrightarrow{\;\;\pi_i\;\;} \mathrm{I}}_{s_i \in \mathbf{C}(\mathrm{I}, \mathrm{I})} \xrightarrow{\;\;\pi_i^\dagger\;\;} A \,,$$

we discover a special scalar of the 'unit disc type'. One verifies that

$$\mathrm{PROB}(\mathrm{P}_i, \psi) := s_i^\dagger \circ s_i \quad \text{satisfies} \quad \sum_{i=1}^{i=n} \mathrm{PROB}(\mathrm{P}_i, \psi) = 1_\mathrm{I} \,,$$

hence these '$[0, 1]$ type' scalars $\mathrm{PROB}(\mathrm{P}_i, \psi)$ provide an abstract notion of *probability* [4]. Moreover, using our abstract inner-product one verifies that $\mathrm{PROB}(\mathrm{P}_i, \psi) = \langle \psi \mid \mathrm{P}_i \circ \psi \rangle$, that is, we prove the *Born rule*.

Mixing classical and quantum uncertainty. This section comprises a *proposal* for the abstract status of density matrices. Having only one page left, we need to be brief. In the von Neumann formalism density matrices are required for two reasons: **i.** to describe part of a larger (compound) system, say *ontic density matrices*, and, **ii.** to describe a system about which we have incomplete knowledge, say *epistemic density matrices*. Hence ontic density matrices arise by considering one component of an element of the name type, $\ulcorner \xi \urcorner : \mathrm{I} \to A_1 \boxtimes A_2$ for $\xi : A_1^* \to A_2$. In order to produce epistemic density matrices, consider the situation of a measurement, but we extract the information concerning the actual outcome from it, that is, we do the converse of distributing classical data,

$$\mathrm{I} \xrightarrow{\;\phi\;} A \xrightarrow{\;\langle \mathrm{P}_i \rangle_i\;} \boxplus_i A \xrightarrow{\;\boxplus_i \lambda_Q\;} \boxplus_i (\mathrm{I} \boxtimes A) \xrightarrow{\;\mathrm{DIST}^{-1}\;} (\boxplus_i \mathrm{I}) \boxtimes A \,.$$

This results again in an element of the name type, $\ulcorner \omega \urcorner : \mathrm{I} \to (\boxplus_i \mathrm{I}) \boxtimes A$ for $\omega : (\boxplus_i \mathrm{I})^* \to A$. Metaphorically one could say that the *classical data is entangled with the quantum data*. Since our formalism allows both to encode classical data and quantum data there is no need for a separate density matrix formalism as it is the case for the von Neumann formalism.

One verifies that the principle of *no signalling faster than light* still holds for the name type in the abstract formalism, that is, operations locally on one component will not alter the other, provided there is no classical data exchange. But there can be a passage from ontic to epistemic e.g.

$$\mathrm{I} \to A_1 \boxtimes A_2 \quad \rightsquigarrow \quad \mathrm{I} \to (\boxplus_i \mathrm{I}) \boxtimes A_2$$

when performing the measurement $\langle \pi_i \rangle_i \boxtimes 1_{A_2} : A_1 \boxtimes A_2 \to (\boxplus_i \mathrm{I}) \boxtimes A_2$. For epistemic density matrices this means that the classical data and the quantum data are truly distinct entities.

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[18]A discussion of normalization of projectors can be found in [5].

Using Lemma 7.6 of [4] one verifies that $\omega_* : \boxplus_i \mathrm{I} \to A^*$ is given by

$$\omega_* = [s_i \bullet (\pi_i^* \circ u_\mathrm{I})]_i$$

where $s_i := \pi_i \circ \phi$ and $u_\mathrm{I} : \mathrm{I} \simeq \mathrm{I}^*$ (a natural transformation which exists by compact closure). Hence ω and hence also $\ulcorner \omega \urcorner$ is determined by a list of orthogonal (pure) states $(\pi_i^\dagger : \mathrm{I} \to A)_i$ and a list of scalars $(s_i^\dagger : \mathrm{I} \to A)_i$ all of the unit disc type — compare this to the orthogonal eigenstates of a standard Hilbert space density matrix and the corresponding eigenvalues which all are of the $[0,1]$ type.

So we can pass from pure states $\phi : \mathrm{I} \to \mathrm{I}$ to density matrices by 'plugging in an ancilla', which either represents classical data (epistemic) or which represents an external part of the system (ontic). The other concepts that can be derived from basic quantum mechanics by 'acting on part of a bigger system' (*non-isolated dynamics*, *generalized measurements*, [16] etc.) can also be defined abstractly, e.g. generalized measurements as

$$\langle f_i \rangle_{i=1}^{i=n} : A \to \boxplus_i A \qquad \text{with} \qquad \sum_{i=1}^{i=n} f_i^\dagger \circ f_i = 1_A \,,$$

while abstract analogous of theorems such as Naimark's can be proven. Of course, many things remain to be verified such as abstract analogous of Gleason's theorem. I might have something to add to this in my talk. $:)$

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