MATH 1115, Mathematics for Commerce
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Midterm Examination
Model Solutions

Each multiple choice question is worth one mark, other questions are worth two marks. Include all your working for the other questions, but for multiple-choice questions, it is sufficient to just indicate the letter. [Make it clear which letter you mean as your answer. Ambiguous answers will be marked wrong.]

1. *In a 25-year mortgage for \$300,000 with interest 9% compounded monthly, the monthly repayments are given by:*

$$\begin{aligned} P &= 300000 \\ i &= \frac{0.09}{12} = 0.0075 \\ n &= 12 \times 25 = 300 \end{aligned}$$

   (A) $\frac{300000 \times 1.09}{25}$

   (B) $\frac{300000 \times 1.09}{1-1.09^{-25}}$

   (C) $300000\frac{1-1.0075^{-300}}{0.0075}$

   (D) $\boldsymbol{\frac{300000 \times 0.0075}{1-1.0075^{-300}}}$

   (E) $300000\frac{1.0075^{300}-1}{0.0075}$

2. *For a \$200,000 mortgage with interest 6% compounded monthly, to be repaid with monthly repayments over 20 years, the monthly repayments are \$1432.87, except for the final repayment, which is \$1429.23. The outstanding balance after 5 years is given by:*

   There are two possible ways to calculate this, the first is to calculate the future value of the loan so far and the second is to calculate the discounted value of the payments still to be made.

   For the first way, the future value of the loan after 5 years is 200000×1.005^{60}. The future value of the payments made so far is $1432.87\frac{1.005^{60}-1}{0.005}$. Therefore the outstanding value is the difference between these, which is (B).

   The second way is more difficult because the final payment is irregular. The present value of the regular payments of \$1432.87 is $1432.87\frac{1-1.005^{-179}}{0.005}$ (there are 180 payments left, and one of them is irregular). The total outstanding value is therefore $1432.87\frac{1-1.005^{-179}}{0.005} + 1429.23 \times 1.005^{-180}$. This answer is not given as an option.

(A) $1432.87\frac{1-1.005^{-60}}{0.005}$

(B) $\mathbf{200000 \times 1.005^{60} - 1432.87\frac{1.005^{60}-1}{0.005}}$

(C) $1429.23\frac{1-1.005^{-180}}{0.005}$

(D) $200000 - 1432.87\frac{1.005^{60}-1}{0.005}$

(E) $200000 - 1432.87\frac{1-1.005^{-60}}{0.005}$

3. *For the system of equations:*

$$\begin{array}{rrrcr} x & -y & +3z & = & 3 \\ 3x & -y & -2z & = & -1 \\ 2x & & -z & = & 4 \end{array}$$

the solution includes:

Write the augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 3 & -1 & -2 & -1 \\ 2 & 0 & -1 & 4 \end{array}\right)$$

We subtract 3 times the first row from the second row and subtract 2 times the first row from the third row to get:

$$\left(\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 2 & -11 & -10 \\ 0 & 2 & -7 & -2 \end{array}\right)$$

We divide the second row by 2, then add the second row to the first row, and subtract 2 times the second row from the third row.

$$\left(\begin{array}{rrr|r} 1 & 0 & -2.5 & -2 \\ 0 & 1 & -5.5 & -5 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

Finally we divide the last row by 4, and add 2.5 times the last row to the first row and add 5.5 times the last row to the second row to get

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The solution is $x = 3$, $y = 6$, $z = 2$.

We check:

$$\begin{array}{rrrrr} 3 & -6 & +3\times 2 & = & 3 \\ 3\times 3 & -6 & -2\times 2 & = & -1 \\ 2\times 3 & & -2 & = & 4 \end{array}$$

(A) $\mathbf{x = 3}$

(B) $y = 5$

(C) $z = 7$

(D) There is no solution.

(E) There are infinitely many solutions.

4. *In an economy with three sectors, and whose Leontief matrix is* $A = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$ *the amount of production needed in each sector to satisfy external demands* $\begin{pmatrix} 30 \\ 0 \\ 28 \end{pmatrix}$ *is:*

We need to solve

$$\begin{pmatrix} 0.6 & -0.3 & -0.1 \\ -0.3 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{pmatrix} X = \begin{pmatrix} 30 \\ 0 \\ 28 \end{pmatrix}$$

We can solve this by row reduction as in the last example (though there is an easier way — see below). Write the augmented matrix:

$$\left(\begin{array}{rrr|r} 0.6 & -0.3 & -0.1 & 30 \\ -0.3 & 0.9 & -0.2 & 0 \\ -0.1 & -0.3 & 0.8 & 28 \end{array}\right)$$

We multiply the first row by $\frac{10}{6}$, add 0.3 times the first row (or half the original first row) to the second row and add 0.1 times the first row to the third row to get:

$$\left(\begin{array}{lcr|c} 1 & -\frac{1}{2} & -\frac{1}{6} & 50 \\ 0 & 0.75 & -0.25 & 15 \\ 0 & -0.35 & 0.8 - \frac{0.1}{6} & 33 \end{array}\right)$$

We multiply the second row by $\frac{4}{3}$, then add $\frac{1}{2}$ times the second row to the first row, and add 0.35 times the second row from the third row.

$$\left(\begin{array}{cccc|c} 1 & 0 & & -\frac{1}{3} & 60 \\ 0 & 1 & & -\frac{1}{3} & 20 \\ 0 & 0 & 0.8-\frac{0.1}{6} & -\frac{0.7}{6} & 40 \end{array}\right)$$

The entry in the bottom right is equal to $\frac{4.8-0.1-0.7}{6} = \frac{4}{6} = \frac{2}{3}$. Multiplying the last row by 1.5, and adding $\frac{1}{3}$ times the last row to the first and second rows, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 80 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 60 \end{array}\right)$$

In this multiple choice format, the easier way is to just test each possible solution. That is, multiply $I - A$ by each answer, and see if one of the answers is $\begin{pmatrix} 30 \\ 0 \\ 28 \end{pmatrix}$.

(A) (150 100 100)

(B) (80 40 60)

(C) (15.2 14.6 19.4)

(D) (-10 130 -50)

(E) It is not possible to satisfy these demands.

5. *A business makes 4 kinds of product. These products require 3 different kinds of components. The number of each type of component required to make each product is represented by the table*

	Component A	*Component B*	*Component C*
Product 1	*1*	*2*	*0*
Product 2	*3*	*0*	*1*
Product 3	*2*	*2*	*1*
Product 4	*1*	*3*	*1*

These 3 components are made from 3 different kinds of raw materials. The number of units of each raw material needed to make each component is given by the table

	Raw material X	*Raw material Y*	*Raw material Z*
Component A	*0*	*1*	*4*
Component B	*3*	*5*	*2*
Component C	*2*	*4*	*0*

The cost per unit for each raw material is given by the table

Raw material X	*60*
Raw material Y	*20*
Raw material Z	*10*

The company receives an order for 20 of product 1 and 5 of product 3. The total cost of the raw materials needed to satisfy this order is:

This is obtained as the matrix product

$$\begin{pmatrix} 20 & 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 4 \\ 3 & 5 & 2 \\ 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} 60 \\ 20 \\ 10 \end{pmatrix}$$

This can be evaluated in any order. Perhaps the easiest way to evaluate it is by multiplying the two at the left, then the two at the right, to get

$$\begin{pmatrix} 30 & 50 & 5 \end{pmatrix} \begin{pmatrix} 60 \\ 300 \\ 200 \end{pmatrix}$$

Then multiply these together to get $30 \times 60 + 50 \times 300 + 5 \times 200 = 1800 + 15000 + 1000 = 17800$.

[The most time-consuming way is to multiply the matrices in the middle first.]

(A) 13,800

(B) 15,800

(C) 17,800

(D) Not equal to (A), (B) or (C) (but it can be calculated from the above data).

(E) Impossible to calculate from the above data.

6. *The first row of the inverse of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & -2 \\ -1 & -1 & -10 \end{pmatrix}$$

is:

We try to row reduce the augmented matrix:

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & -2 & 0 & 1 & 0 \\ -1 & -1 & -10 & 0 & 0 & 1 \end{array}\right)$$

We subtract the first row from the second row and add the first row to the third row.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -6 & -1 & 1 & 0 \\ 0 & 1 & -6 & 1 & 0 & 1 \end{array}\right)$$

Now we see that the second and third row are equal, so that when we subtract the second row from the first, we get a row of 0s. This means the matrix is not invertible.

(A) (1 -1 1)

(B) (2 -1 1)

(C) $(1\ \frac{1}{2}\ \frac{1}{4})$

(D) (1 1 0)

(E) The matrix is not invertible.

7. *The maximum value of* $8x + 6y$ *subject to the constraints*

$$\begin{aligned} x + y &\geqslant 5 \\ -x + y &\geqslant -2 \\ x + 2y &\leqslant 8 \\ 3x + y &\leqslant 6 \end{aligned}$$

is

These inequalities are inconstent. One way to see this is to draw a diagram. Another is to add 0.4 times the third inequality and 0.2 times the fourth to get $x + y \leqslant 4.4$, which is inconsistent with the first inequality.

(A) Attained at $x = 0.5, y = 4.5$

(B) 37

(C) Attained at $x = 2, y = 3$

(D) 44

(E) Undefined. (That is, there is no maximum value).

8. *While applying the simplex method to solve a linear programming problem to maximise* Z*, you obtain the following simplex tableau.*

x_1	x_2	x_3	s_1	s_2	s_3	s_4	Z	
1	0	-2	0	0	3	0	0	5
0	3	-1	1	0	4	0	0	3
0	2	2	0	1	-1	0	0	8
0	-3	2	0	0	0	1	0	7
0	1	2	0	0	3	0	1	15

Increasing which of the non-basic variables in this tableau would increase the value of Z?

Note that all entries in the last row are non-negative, so increasing any non-basic variable would decrease Z.

(A) x_1, x_2 or x_3

(B) x_2, x_3 or s_3

(C) Just x_3

(D) x_2 or x_3

(E) None of them — the current point is the one which maximises Z.

9. *While applying the simplex method to solve a linear programming problem to maximise Z, you obtain the following simplex tableau.*

x_1	x_2	x_3	s_1	s_2	s_3	s_4	Z	
1	3	-2	0	0	3	0	0	5
0	1	1	1	0	1	0	0	3
0	2	-2	0	1	-2	0	0	8
0	-6	1	0	0	3	1	0	8
0	-1	4	0	0	2	0	1	10

You decide that the entering variable should be x_2. The value of x_2 in the next BFS, and the departing variable are:

The quotients corresponding to the 4 rows are: $\frac{5}{3}$, $\frac{3}{1} = 3$, $\frac{8}{2} = 4$, and $\frac{8}{-6} = -\frac{4}{3}$. The smallest positive quotient is $\frac{5}{3}$, which is in row 1. The basic variable with entry 1 in this row is x_1. Therefore x_1 is the departing variable.

	New x_2 value	Departing variable
(A)	$\mathbf{\frac{5}{3}}$	$\mathbf{x_1}$
(B)	$\frac{5}{3}$	s_1
(C)	3	s_2
(D)	$\frac{4}{3}$	s_4
(E)	$\frac{4}{3}$	x_1

10. *If 4 cards are drawn at random (without replacement) from a standard deck of 52 cards, the probability that there are at least two of the same suit is: [A standard deck contains 4 suits each of which has 13 cards.]*

If there are not at least two of the same suit, then all 4 cards must be different suits. This means that there must be exactly one card of each suit. The number of ways this can happen is 13^4, since there are 13 choices in each suit. The total number of possibilities is ${}_{52}c_4$.

(A) $4 \times \frac{{}_{13}C_2 \times {}_{48}C_2 + {}_{13}C_3 \times {}_{48} + {}_{13}C_4}{{}_{52}C_4}$

(B) $1 - \frac{4!}{4^4}$

(C) $\frac{{}_{13}C_2 \times {}_{48}C_2}{{}_{52}C_4}$

(D) $\mathbf{1 - \frac{13^4}{{}_{52}C_4}}$

(E) $\frac{(13!)^4}{52!}$

11. *The only anagram of the word "midterm" is "trimmed". If the letters in the word "midterm" are arranged at random, the probability that the result is a word is:*

The letter "m" is repeated, so the total number of possible rearrangements is $\frac{7!}{2}$. Two of these are words. The probability of a word is therefore $\frac{2}{\left(\frac{7!}{2}\right)} = \frac{4}{7!}$

(A) $\frac{7!}{7^7}$

(B) $\mathbf{\frac{4}{7!}}$

(C) $\frac{2}{{}_7C_2}$

(D) $\frac{1}{7!}$

(E) $\frac{2}{7}$

12. *A bank makes loans to three customers. They estimate that the first customer has probability 0.03 of defaulting (not paying the money back), the second has probability 0.02, and the third has probability 0.01. Assuming that these events are independant, the probability that at least two of the loans are paid back is:*

Let F be the event that the first customer repays, let S be the event that the second customer repays, and let T be the event that the third customer repays. We are told

$$P(F) = 1 - 0.03 = 0.97$$
$$P(S) = 1 - 0.02 = 0.98$$
$$P(T) = 1 - 0.01 = 0.99$$

The event described is $(F \cap S \cap T) \cup (F \cap S \cap T^c) \cup (F \cap S^c \cap T) \cup (F^c \cap S \cap T)$. This is a union of mutually exclusive events, which are intersections of independent events, so the overall probability is $P(F)P(S)P(T) + P(F)P(S)P(T^c) + P(F)P(S^c)P(T) + P(F^c)P(S)P(T)$.

(A) $\mathbf{0.97 \times 0.98 \times 0.01 + 0.97 \times 0.99 \times 0.02 + 0.98 \times 0.99 \times 0.03 + 0.97 \times 0.98 \times 0.99}$

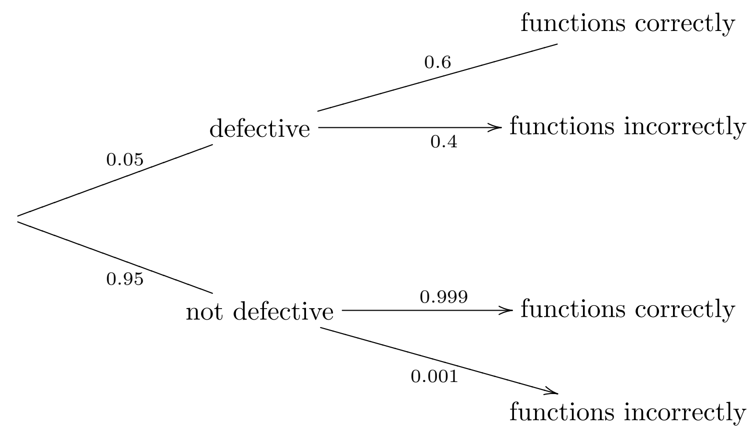
(B) $0.97 \times 0.98 + 0.97 \times 0.99 + 0.98 \times 0.99$

(C) $0.01 + 0.02 + 0.03$

(D) $1 - (0.01 + 0.02 + 0.03)$

(E) $1 - 0.01 \times 0.02 \times 0.03$

13. *A company is demonstrating a new product, which functions correctly 99.9% of the time. However, 5% of the products in the company's warehouse are defective, and only function correctly 60% of the time. If the company selects a product at random from its warehouse for the demonstration, and does not test it beforehand, the probability that the product functions correctly at the demonstration is:*



(A) $\mathbf{0.95 \times 0.999 + 0.05 \times 0.6}$

(B) $(0.95 + 0.999) \times (0.05 + 0.6)$

(C) $0.95 \times 0.6 + 0.05 \times 0.999$

(D) $0.95 \times 0.999 \times 0.05 \times 0.6$

(E) $0.95 \times 0.999 - 0.05 \times 0.6$

14. *A company manufactures two different types of product, A and B. Each product requires the following resouces:*

Product	*A*	*B*
Raw material X	*6*	*1*
Raw material Y	*2*	*4*
Employee hours	*1*	*2*
Profit	*20*	*15*

The company has a total of 200 units of raw material X, 150 units of raw material Y, and a total of 100 employee hours available for production of these products. The manager wants to determine how many of each type of product should be produced in order to maximise profit.

Write out the linear programming problem that she should solve in order to calculate the maximum profit. [You do not need to actually calculate the maximum profit, just write down the problem that needs to be solved.]

Let a be the total amount of product A produced, and let b be the total amount of product B produced.

Maximise $20a + 15b$, subject to:

$$\begin{array}{rcrcr} 6a & + & b & \leqslant & 200 \\ 2a & + & 4b & \leqslant & 150 \\ a & + & 2b & \leqslant & 100 \\ a & & & \geqslant & 0 \\ & & b & \geqslant & 0 \end{array}$$

[The third inequality is not strictly necessary, as it follows from the second.]

15. *An investor estimates that the probability that a particular stock ABC will increase in value by 10% in the coming year is 0.4, and that the probability that another stock XYZ will increase by 10% is 0.2. If the probability that at least one of them increases by 10% is 0.5, what is the probability that they both increase by 10%?*

    Let A be the event that stock ABC increases in value by 10%. Let X be the probability that stock XYZ increases in value by 10%. We are told that $P(A) = 0.4$, $P(X) = 0.2$, and $P(A \cup X) = 0.5$. The general formula for probability of a union gives:

$$\begin{array}{rrcrcr} P(A \cup X) = & P(A) & + & P(X) & - & P(A \cap X) \\ 0.5 = & 0.4 & + & 0.2 & - & P(A \cap X) \\ P(A \cap X) = & 0.1 & & & & \end{array}$$