MATH 1115, Mathematics for Commerce
WINTER 2011
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Homework Sheet 2
Model Solutions

Each multiple choice question is worth one mark, other questions are worth two marks.

1. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & -5 \end{pmatrix}$, then $A^{\mathrm{T}} + 2B$ is:

   (A) $\begin{pmatrix} 3 & 8 \\ 7 & 12 \\ 5 & -5 \end{pmatrix}$

   (B) $\begin{pmatrix} 14 & 20 & -14 \\ 30 & 44 & -22 \\ 28 & 36 & -56 \end{pmatrix}$

   (C) $\begin{pmatrix} 3 & 7 & 5 \\ 8 & 12 & -5 \end{pmatrix}$

   (D) $\begin{pmatrix} 1 & 7 & 7 \\ 11 & 12 & -8 \end{pmatrix}$

   (E) undefined.

2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & -5 \end{pmatrix}$ then the following matrices are defined:

   (A) AB, BA but not AB + BA.

   (B) AB but not BA.

   (C) BA but not AB

   (D) AB, BA and $AB + BA$

   (E) neither AB nor BA

3. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$ then AB is

   (A) $\begin{pmatrix} 1 & 4 \\ 9 & -20 \end{pmatrix}$

   (B) $\begin{pmatrix} 7 & 15 \\ -8 & -14 \end{pmatrix}$

   (C) $\begin{pmatrix} 7 & -8 \\ 15 & -14 \end{pmatrix}$

(D) $\begin{pmatrix} 7 & 10 \\ -12 & -14 \end{pmatrix}$

(E) undefined.

4. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 1 \end{pmatrix}$ then A^{-1} is

(A) $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & 1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 4 \\ -1 & -3 \\ 1 & 1 \end{pmatrix}$

(D) defined, but not equal to (A), (B) or (C).

(E) undefined.

5. For the system of equations:

$$\begin{array}{rrrrr} x & +2y & -z & = & 4 \\ 2x & -y & & = & 3 \\ x & +y & -z & = & -1 \end{array}$$

the solution includes:

(A) $x = 3$

(B) $\mathbf{y = 5}$

(C) $z = 7$

(D) There is no solution.

(E) There are infinitely many solutions.

6. The matrix $A = \begin{pmatrix} 18 & 5 & 8 & 5 & 1 & 0 & 1 \\ 5 & 20 & -12 & -4 & -3 & 0 & -1 \\ 8 & -12 & 16 & 9 & 4 & 1 & 1 \\ 5 & -4 & 9 & 13 & 6 & 4 & -2 \\ 1 & -3 & 4 & 6 & 3 & 2 & -1 \\ 0 & 0 & 1 & 4 & 2 & 2 & -1 \\ 1 & -1 & 1 & -2 & -1 & -1 & 1 \end{pmatrix}$ is invertible

with inverse $A^{-1} = \begin{pmatrix} 1 & -2 & -3 & 4 & -6 & 1 & 3 \\ -2 & 5 & 8 & -12 & 17 & -2 & -10 \\ -3 & 8 & 14 & -21 & 27 & -2 & -20 \\ 4 & -12 & -21 & 34 & -45 & 3 & 31 \\ -6 & 17 & 27 & -45 & 67 & -8 & -35 \\ 1 & -2 & -2 & 3 & -8 & 4 & 1 \\ 3 & -10 & -20 & 31 & -35 & 1 & 36 \end{pmatrix}$.

Which of the following is part of the solution to the system of equations

$$\begin{array}{rrrrrrrcr}
18a & +5b & +8c & +5d & +e & & +g & = & 2 \\
5a & +20b & -12c & -4d & -3e & & -g & = & 6 \\
8a & -12b & +16c & +9d & +4e & +f & +g & = & 3 \\
5a & -4b & +9c & +13d & +6e & +4f & -2g & = & 2 \\
a & -3b & +4c & +6d & +3e & +2f & -g & = & -1 \\
 & & c & +4d & +2e & +2f & -g & = & -2 \\
a & -b & +c & -2d & -e & -f & +g & = & 1
\end{array}$$

(A) $a = 5$

(B) $b = -6$

(C) $c = 4$

(D) d = 11

(E) $e = 10$

7. In an economy with three sectors, and whose Leontief matrix is $A = \begin{pmatrix} 0.2 & 0.3 & 0.6 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0.4 & 0.2 \end{pmatrix}$ the amount of production needed in each sector to satisfy external demands $\begin{pmatrix} 30 \\ 15 \\ 25 \end{pmatrix}$ is:

(A) (150 100 100)

(B) (25.5 18 14)

(C) (55.5 33 39)

(D) (30 15 25)

(E) (43 35.5 52.5)

8. *Solve the system of equations*

$$\begin{array}{rrrrcr}
a & +2b & +3c & +d & = & 7 \\
2a & +3b & -c & +2d & = & 12 \\
a & -b & -15c & +d & = & 1 \\
3a & & -33c & +3d & = & 9
\end{array}$$

We write the augmented coefficient matrix.

$$\left(\begin{array}{rrrr|r} 1 & 2 & 3 & 1 & 7 \\ 2 & 3 & -1 & 2 & 12 \\ 1 & -1 & -15 & 1 & 1 \\ 3 & 0 & -33 & 3 & 9 \end{array}\right)$$

Subtracting 2 times Row 1 from Row 2, subtracting Row 1 from Row 3, and subtracting 3 times Row 1 from Row 4 gives:

$$\left(\begin{array}{rrrr|r} 1 & 2 & 3 & 1 & 7 \\ 0 & -1 & -7 & 0 & -2 \\ 0 & -3 & -18 & 0 & -6 \\ 0 & -6 & -42 & 0 & -12 \end{array}\right)$$

Multiplying Row 2 by -1, subtracting 2 times the new Row 2 from Row 1, adding 3 times the new Row 2 to Row 3, and adding 6 times the new Row 2 to Row 4 gives:

$$\left(\begin{array}{rrrr|r} 1 & 0 & -11 & 1 & 5 \\ 0 & 1 & 7 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Dividing Row 3 by 3, adding 11 times the new Row 3 to Row 1, and subtracting 7 times the new Row 3 from Row 2 gives:

$$\left(\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is in reduced form, and we can read off the parametric solution:

$a = 3 - x$, $b = 2$, $c = 0$, $d = x$ for any value of x.

9. *Given the following table, calculate the number of units of each sector that must be produced in order to meet the external demand of 20 units of sector A, 40 units of sector B and 35 units of sector C.*

	Sector A	*Sector B*	*Sector C*	*Other outputs*	*Total*
Sector A	*100*	*350*	*200*	*500*	*1150*
Sector B	*300*	*700*	*150*	*300*	*1450*
Sector C	*50*	*200*	*100*	*400*	*750*
Other costs	*700*	*200*	*300*		
Total	*1150*	*1450*	*750*		

We obtain the Leontief matrix A by dividing the entries in each column by the total for that column.

This gives

$$A = \begin{pmatrix} \frac{100}{1150} & \frac{350}{1450} & \frac{200}{750} \\ \frac{300}{1150} & \frac{700}{1450} & \frac{150}{750} \\ \frac{50}{1150} & \frac{200}{1450} & \frac{100}{750} \end{pmatrix} = \begin{pmatrix} \frac{2}{23} & \frac{7}{29} & \frac{4}{15} \\ \frac{6}{23} & \frac{14}{29} & \frac{3}{15} \\ \frac{1}{23} & \frac{4}{29} & \frac{2}{15} \end{pmatrix}$$

We need to solve $(I - A)X = D$, where the demand vector D is

$$D = \begin{pmatrix} 20 \\ 40 \\ 35 \end{pmatrix}$$

We write out the augmented matrix

$$\left(\begin{array}{ccc|c} \frac{21}{23} & -\frac{7}{29} & -\frac{4}{15} & 20 \\ -\frac{6}{23} & \frac{15}{29} & -\frac{3}{15} & 40 \\ -\frac{1}{23} & -\frac{4}{29} & \frac{13}{15} & 35 \end{array}\right)$$

Adding $\frac{2}{7}$ times Row 1 to Row 2 and $\frac{1}{21}$ times Row 1 to Row 3, we get.

$$\left(\begin{array}{ccc|c} \frac{21}{23} & -\frac{7}{29} & -\frac{4}{15} & 20 \\ 0 & \frac{13}{29} & -\frac{29}{15\times 7} & \frac{320}{7} \\ 0 & -\frac{13}{29\times 3} & \frac{269}{15\times 21} & \frac{755}{21} \end{array}\right)$$

Adding $\frac{1}{3}$ times Row 2 to Row 3 and $\frac{7}{13}$ times Row 2 to Row 1, then gives

$$\left(\begin{array}{ccc|c} \frac{21}{23} & 0 & -\frac{81}{15\times 13} & \frac{580}{13} \\ 0 & \frac{13}{29} & -\frac{29}{15\times 7} & \frac{320}{7} \\ 0 & 0 & \frac{240}{15\times 21} & \frac{1075}{21} \end{array}\right)$$

Adding $\frac{29}{80}$ times Row 3 to Row 2 and $\frac{21\times 81}{13\times 240}$ times Row 3 to Row 1 gives:

$$\left(\begin{array}{ccc|c} \frac{21}{23} & 0 & 0 & \frac{580}{13} + \frac{1075\times 81}{13\times 240} \\ 0 & \frac{13}{29} & 0 & \frac{320}{7} + \frac{1075\times 29}{80\times 21} \\ 0 & 0 & \frac{240}{15\times 21} & \frac{1075}{21} \end{array}\right)$$

We can now multiply each row by an appropriate number to get an identity matrix, and then read the solution

$\left(\frac{23\times(580\times 80+1075\times 27)}{13\times 21\times 80} \quad \frac{29\times(320\times 80\times 3+1075\times 29)}{13\times 80\times 21} \quad \frac{15\times 1075}{240}\right)$

[or (79.43 143.37 67.19) to 2 decimal places.]