MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
Toby Kenney
Mock Midterm Examination
Time allowed: 1 hour 30 minutes

In order to give examples of questions covering all the topics in the first half of the course, this mock midterm exam is longer than the actual midterm will be. I think the questions here should take you about 2 hours 30 minutes, once you have learned the material in the first half of the course.

1 Use Euclid's algorithm to find the greatest common divisor of the following pairs of numbers. Write down all the steps involved. Use your calculations to find integers a and b such that a times the first number plus b times the second number is their greatest common divisor.

(a) 159 and 265

(b) 237 and 115

2 Which of the following pairs of propositions are logically equivalent? Justify your answers.

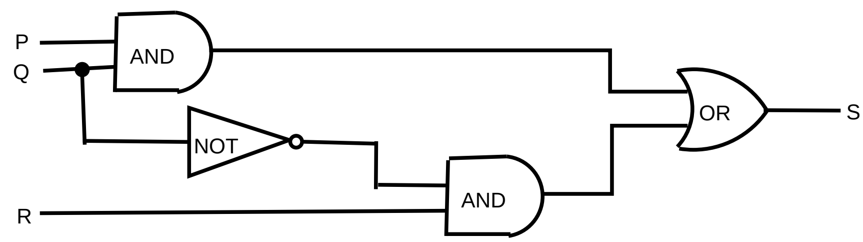
(a) $p \rightarrow (\neg p \vee q)$ and $\neg p \vee q$.

(b) $p \wedge (q \vee r)$ and $(p \wedge q) \vee (q \wedge r)$.

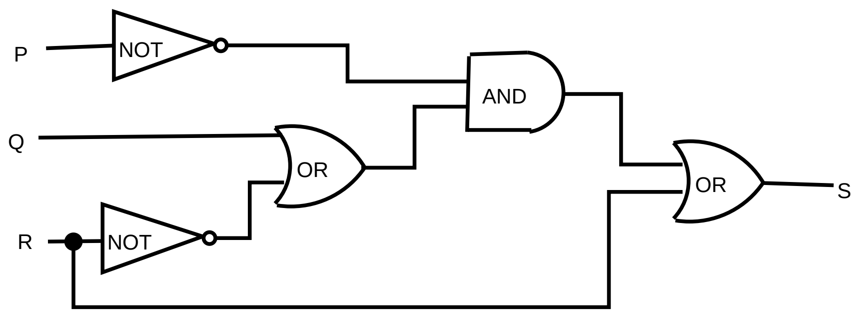
(c) $(p \vee (p \rightarrow q)) \wedge r$ and $(p \vee q) \wedge r$.

3 Find boolean expressions for the following logic circuits.

(a)



(b)



4 Which of the following are true when $A = \{0, 1, 3, 5\}$ and $B = \{1, 2, 4, 6\}$? Justify your answers.

(a) $(\forall x \in A)(x + 1 \in B)$

(b) $(\exists x \in A)(x + 2 \in B)$

(c) $(\forall x \in A)(\exists y \in B)(x + y \text{ is even})$

(d) $(\exists y \in B)(\forall x \in A)(x + y \text{ is even})$

5 Use Venn diagrams to show the following arguments are invalid:

(a)

$$\begin{gathered}(\forall x \in A)(x \in B \vee x \in C)\\ (\forall x \in B)(x \in C)\\ \therefore (\forall x \in A)(x \in B)\end{gathered}$$

(b)

$$\begin{gathered}(\exists x \in A)(x \in B)\\ (\exists x \in B)(x \in C)\\ \therefore (\exists x \in A)(x \in C)\end{gathered}$$

6 Use universal instantiation and rules of inference to show that the following arguments are valid.

(a)

$$\begin{gathered}(\forall x \in A)(x \in B \rightarrow x \in C)\\ y \in A \wedge y \in B\\ \therefore y \in C\end{gathered}$$

(b)

$$\begin{gathered}(\forall x \in A)(x \in B \vee \phi(x))\\ (\forall x \in A)(x \in C \vee \neg\phi(x))\\ y \in A \wedge \neg y \in C\\ \therefore y \in B\end{gathered}$$

7 Prove or disprove the following. You may use results proved in the course or the homework sheets, provided you state them clearly.

(a) $\sqrt[3]{7}$ is rational.

(b) There is a natural number n such that $n^2 + 4n + 16$ is prime.

(c) There is a natural number n such that $n^2 - 169$ is prime.

(d) All integers of the form $n^2 + n + 41$ are prime for $n \in \mathbb{N}$.

(e) $2^{135} + 3^{98} + 5^{32}$ is divisible by 7.

(f) $n^2 + 2 = m^5 + 9$ has no integer solutions [Hint: try modulo 11]

(g) For all natural numbers n, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(h) For all natural numbers n, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

(i) There are infinitely many primes congruent to 2 modulo 3. [Hint: suppose there are only finitely many; take the product of all of them. If this is congruent to 2 modulo 3, then multiply by 2. Add 1 to the resulting product. You may assume that any number that is congruent to 2 modulo 3 is divisible by a prime number congruent to 2 modulo 3.]

(j) For all natural numbers n, $\sum_{i=1}^{n}(i^3 - 3i) = \frac{n^4+2n^3-5n^2-6n+8}{4}$

8 Find $0 \leqslant n < 660$ satisfying all the following congruences:

$$n \equiv 3 \pmod{5} \tag{1}$$
$$n \equiv 5 \pmod{11} \tag{2}$$
$$n \equiv 4 \pmod{12} \tag{3}$$