MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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Homework Sheet 1
Hints & Model Solutions

Compulsory questions

1 *(a) In a maths exam, there are two sections A and B. There are 5 problems in Section A and 8 problems in Section B; students are required to attempt 4 problems from Section A and 5 problems from Section B. How many different sets of questions satisfy these requirements?*

There are $\binom{5}{4} = 5$ different choices of questions from Section A. For each of these, there are $\binom{8}{5} = 56$ choices of questions from Section B, so in total there are $5 \times 56 = 280$ possible choices of questions.

(b) What if the requirement is a total of 9 questions with at least 4 from Section A?

At least 4 questions must be chosen from Section A, and at most 5 questions can be chosen from Section A. If 4 questions are chosen from Section A, then there are 280 possible choices of questions by part (a). If 5 questions are chosen from Section A, then 4 questions must be chosen from Section B. There is only one way to choose 5 questions from Section A – choose them all, and there are $\binom{8}{4} = 70$ ways to choose 4 questions from Section B, so in total there are $280 + 1 \times 70 = 350$ possible choices of questions.

2 *(a) How many anagrams are there of MATHEMATICS? (Count all anagrams, whether or not they are actual words.)*

The word "mathematics" has a total of 11 letters, 2 'm', 2 'a', 2 't' and 5 letters that are not repeated. The number of ways of arranging these is therefore $\binom{11}{2} \times \binom{9}{2} \times \binom{7}{2} \times 5! = 55 \times 36 \times 21 \times 120 = 4,989,600$ anagrams.

(b) What if we allow multiple word anagrams? (i.e. we divide the letters into different words by putting spaces between them, e.g. MAT HE M ATIC S. The order of words is important, so HE MAT ATIC S M would be a different anagram.)

First, we arrange the letters in one of the 4,989,600 different ways. Now we insert spaces. We can choose to either insert a space, or not insert a space between any two consecutive letters, so there are 10 places we can insert a space, and therefore $2^{10} = 1024$ different ways to divide our anagram into words. Note that given this multiple-word anagram, we can concatenate the words to get back our single word anagram. Therefore, there are $1024 \times 4,989,600 = 5,109,350,400$ possible anagrams.

3 *(a) How many possible 7-digit numbers are there such that every digit is at least as large as its position – so the first digit is at least 1, the second digit is at least 2, etc.?*

There are 9 choices for the first digit, 8 choices for the second digit, $\ldots$, and 3 choices for the last digit. Therefore, there are $9\times8\times7\times6\times5\times4\times3 = 181,440$ possible 7-digit numbers.

(b) How many of these are multiples of 9? Justify your answer. [Hint: recall that a number is a multiple of 9 if and only if the sum of its digits is also a multiple of 9.]

As in part (a), we can choose the last 6 digits in $8\times7\times6\times5\times4\times3 = 20,160$ ways. Once we have chosen the last 6 digits, we can take their sum modulo 9. To get a multiple of 9, the first digit must be congruent to minus this sum modulo 9, but there is exacly one number in the range 1—9 congruent to minus this sum modulo 9, so there is exactly one choice for the first digit, so there are 20,160 7-digit numbers with these properties.

4 *A restaurant menu has 4 choices of starters, 6 choices of main course and 3 choices for dessert.*

(a) How many different 3 course meals can be chosen from this menu?

There are 4 choices for starter. For each of these, there are 6 choices for main course, and for each of these, there are 3 choices for dessert, so in total, there are $4 \times 6 \times 3 = 72$ choices for the whole meal.

(b) How many different 2 course meals can be chosen? (The meals may consist of any two different courses, so e.g. starter + dessert is a two-course meal.)

There are $4 \times 6 = 24$ choices of starter + main course, $4 \times 3 = 12$ choices of starter + dessert, and $6 \times 3 = 18$ choices of main course + dessert, so in total, there are $24 + 12 + 18 = 54$ choices of two-course meal.

5 *A student has 8 identical red socks and 5 identical blue socks. How many matching pairs of socks can the student make from these?*

The student can form $\binom{8}{2} = 28$ pairs of red socks and $\binom{5}{2} = 10$ pairs of blue socks. This is a total of $28 + 10 = 38$ pairs of socks.

Alternatively, the student can form $\binom{13}{2} = 78$ pairs of socks in total. The number of non-matching pairs is $8 \times 5 = 40$, so the number of matching pairs is $78 - 40 = 38$.

Bonus question

6 *Suppose we have a cube made of $n \times n \times n$ smaller cubes ($n \geqslant 2$). We call a line through the cube a set of n of the smaller cubes such that the centres of all the smaller cubes are in a straight line. How many lines are there through the cube?*

Hint: Try describing the lines in terms of one endpoint and the direction. You might find it easiest to count every line twice then divide by 2.