MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
Toby Kenney
Homework Sheet 3
Hints & Model Solutions

Compulsory questions

1 *(a) What is the probability that a randomly dealt 5-card poker hand is a straight? [But not a straight-flush.]*

There are $\binom{52}{5}$ possible poker hands. Of these, the set of ranks in a straight is entirely determined by the rank of the lowest card, which must be between an ace (counting low) and a 10. There are therefore 10 possibilities for the set of ranks in the straight. Once the ranks have been chosen, there are 4 possibilities for the suit of each card. However, in 4 of these possibilities, the suits are all the same, so the hand is actually a straight-flush, so there are a total of $(4^5 - 4) \times 10$ possible straights that are not straight-flushes. Therefore, the probability that a randomly dealt hand is a straight is $\frac{10\times(4^5-4)}{\binom{52}{5}}$.

(b) What is the probability that a randomly chosen 5-card poker hand contains exactly one king?

The number of poker hands that contain exactly one king is $\binom{4}{1}\binom{48}{4}$, so the probability that a random hand has exactly one king is $\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$.

2 *How many numbers between 100 and 1000 inclusive are divisible by at least one of 2, 3 or 5?*

Let A_1 be the set of numbers between 100 and 1000 divisible by 2; let A_2 be the set of numbers between 100 and 1000 divisible by 3; let A_3 be the set of numbers between 100 and 1000 divisible by 5.

Now, $|A_1| = 451$, $|A_2| = 300$, $|A_3| = 181$, $|A_1 \cap A_2| = 150$, $|A_1 \cap A_3| = 91$, $|A_2 \cap A_3| = 60$ and $|A_1 \cap A_2 \cap A_3| = 30$. Therefore, by the inclusion-exclusion principle, $|A_1 \cup A_2 \cup A_3| = 451+300+181-150-91-60+30 = 661$.

3 *Suppose we have 3 dice: one red, one blue, and one green, and we roll all 3, what is the probability that the number on the green die is at least as big as either of the other numbers, and the number on the red die is no bigger than either of the other numbers? i.e.* red $\leqslant$ blue $\leqslant$ green.

The number of rolls with red $\leqslant$ blue $\leqslant$ green is $\binom{8}{3}$ (add one to the blue die and two to the green one; we can get any 3 of the digits 1, 2, 3, 4, 5, 6, 7, 8). Therefore, the probability that the roll has this property is $\frac{\binom{8}{3}}{6^3} = \frac{7}{27}$.

4 *A fair coin is tossed 5 times.*

(a) What is the probability that the sequence HHH occurs somewhere in the 5 tosses?

Let A_1 be the set of sequences of 5 tosses such that the first 3 are all heads; let A_2 be the set of sequences of 5 tosses such that the 2nd, 3rd and 4th are all heads; and let A_3 be the set of sequences of 5 tosses such that the 3rd, 4th and 5th are all heads. Observe that $|A_1| = |A_2| = |A_3| = 4$, and $|A_1 \cap A_2| = |A_2 \cap A_3| = 2$, and $|A_1 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 1$. Thus, by the inclusion-exclusion principle, $|A_1 \cup A_2 \cup A_3| = 4+4+4-2-2-1+1 = 8$. Therefore, the probability that the sequence HHH occurs is $\frac{8}{32} = \frac{1}{4}$.

(b) What is the probability that the sequence THT occurs somewhere in the 5 tosses?

Let A_1 be the set of sequences of 5 tosses such that the first 3 are THT; let A_2 be the set of sequences of 5 tosses such that the 2nd, 3rd and 4th are THT; and let A_3 be the set of sequences of 5 tosses such that the 3rd, 4th and 5th are THT. Observe that $|A_1| = |A_2| = |A_3| = 4$, and $|A_1 \cap A_2| = |A_2 \cap A_3| = 0$, and $|A_1 \cap A_3| = 1$. Thus, by the inclusion-exclusion principle, $|A_1 \cup A_2 \cup A_3| = 4+4+4-0-0-1+0 = 11$. Therefore, the probability that the sequence THT occurs is $\frac{11}{32}$.

5 *n fair dice are rolled.*

(a) What is the probability that the highest number shown is a 5?

The probability that the numbers are all at most 5 is $\left(\frac{5}{6}\right)^n$. The probability that they are all at most 4 is $\left(\frac{4}{6}\right)^n$. The probability that they are all at most 5 but not all at most 4 is therefore $\frac{5^n - 4^n}{6^n}$.

(b) For which value of n is this probability greatest? [Hint: compare probabilities for consecutive values of n, to see if they are increasing or decreasing.]

Let p_n be the probability for n dice. Observe that $\frac{p_{n+1}}{p_n} = \frac{5^{n+1} - 4^{n+1}}{6(5^n - 4^n)}$. We want to find out when this is greater than 1. Observe that $5^{n+1} - 4^{n+1} = 5(5^n - 4^n) + 4^n$. Therefore, $\frac{5^{n+1} - 4^{n+1}}{6(5^n - 4^n)}$ is greater than 1 if and only if $\frac{4^n}{5^n - 4^n} > 1$. This is equivalent to $\frac{5^n - 4^n}{4^n} < 1$, or $\frac{5^n}{4^n} < 2$. However, we can check that this happens if and only if $n \leqslant 3$. Therefore, for $n \leqslant 3$, we have that $p_{n+1} > p_n$, and for $n \geqslant 4$, $p_{n+1} < p_n$. Therefore, the greatest probability occurs when we roll 4 dice. This probability is $\frac{369}{6^4} = \frac{41}{144}$.

6 *In the game Craps, played in casinos, the player first rolls 2 dice and adds them. If the total is 7 or 11, the player wins. If the total is 2, 3 or 12, the player loses. If the total is anything else (4, 5, 6, 8, 9, or 10) this total is recorded and called the point. Now the player continues to roll two dice until the total is either 7 or is equal to the point. If it is 7, the player loses. If it is the point, the player wins.*

(a) Suppose that after the first roll, the point is 5. What is the probability that the player goes on to win? [Assume that the player always eventually rolls either a 7 or the point.]

We reduce the sample space to the set of rolls of the two dice that sum to either 5 or 7. There are 6 rolls that total 7 and 4 rolls that total 5. If anything else is rolled, we simply reroll. Now, of these 10 possibilities in the sample space, the player wins on 4, so the probability that the player goes on to win is 0.4.

Bonus question

(b) What is the probability that the player wins overall?

If the first roll is a 7 or an 11, (a total probability of $\frac{8}{36} = \frac{2}{9}$) the player wins.

If it is 4 or 10 (a total probability of $\frac{6}{36} = \frac{1}{6}$), the player has a $\frac{3}{9} = \frac{1}{3}$ probability of winning, so the probability that the first roll is 4 or 10 and the player wins is $\frac{1}{18}$.

If it is 5 or 9 (a total probability of $\frac{8}{36} = \frac{2}{9}$) the player has a 0.4 probability of winning, so the probability of rolling a 5 or 9 on the first roll then going on to win is $\frac{4}{45}$.

If it is 6 or 8 (a total probability of $\frac{10}{36} = \frac{5}{18}$) the player has a $\frac{5}{11}$ probability of winning, so the probability of rolling a 6 or 8 and going on to win is $\frac{25}{18 \times 11}$.

All other first rolls result in an immediate loss.

The total probability that the player wins is therefore $\frac{2}{9} + \frac{1}{18} + \frac{4}{45} + \frac{25}{18 \times 11} = \frac{220+55+88+125}{5 \times 18 \times 11} = \frac{488}{990} = \frac{244}{495}$