MATH 2113/CSCI 2113, Discrete Structures II
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Homework Sheet 4
Hints & Model solutions

Compulsory questions

1 *On a table there are 129 coins, 128 of which are fair, one of which is two-headed (i.e. it always gives heads when it is tossed.) A coin is selected at random from the table and tossed 6 times. It comes down heads each time. What is the probability that it will come down heads the seventh time it is tossed?*

The probability of getting 6 heads in a row is $\frac{1}{129} + \frac{128}{129} \times \left(\frac{1}{2}\right)^6 = \frac{3}{129}$. The probability of getting 7 heads in a row is $\frac{1}{129} + \frac{128}{129} \times \left(\frac{1}{2}\right)^7 = \frac{2}{129}$, and this implies that we got 6 heads in a row to start with. Therefore, the conditional probability that the seventh toss is a head, given that the first 6 were all heads is $\frac{\frac{2}{129}}{\frac{3}{129}} = \frac{2}{3}$.

2 *A gambler starts with \$3. He proceeds to make a series of \$1 bets, each of which has a 0.5 probability of winning (in which case he gets \$1, and his original \$1 back) until he either has \$10 or \$0. What is the probability of his reaching \$10? [Assume that he definitely reaches either \$10 or \$0.]*

Hint: Consider expected values. Use the fact that $\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n)$.

3 *I have a bowl of spaghetti containing n strands (each of which has two ends). I pick up 2 ends at random and tie them together. I repeat this until there are no loose ends in the bowl. What is the expected number of loops in the bowl?*

Note that each time I tie a pair of loose ends, I decrease the number of loose ends by 2 (or the number of strands that are not loops by 1). Therefore, I perform this operation n times. Each time, I either create a loop if the two ends that I pick are the ends of the same strand, or I don't. I never destroy a loop once it is created. Let Y_i be the number of loops created the ith time I perform this operation. The total number of loops created is then $\sum_{i=1}^{n} Y_i$. Observe that the ith time I perform the operation, there are $n - i + 1$ strands left in the bowl. Once I have picked one end, the probability of picking the other end of the same strand is therefore $\frac{1}{2(n-i+1)-1}$, since only one of the other ends will do. Therefore, $Y_i = 1$ with probability $\frac{1}{2(n-i)+1}$, and 0 otherwise, so $\mathbb{E}(Y_i) = \frac{1}{2(n-i)+1}$. Therefore, the expected number of loops in the bowl after n operations is $\mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathbb{E}(Y_i) = \sum_{i=1}^{n} \frac{1}{2(n-i)+1}$.

[Or $\sum_{j=0}^{n-1} \frac{1}{2j+1}$ by substituting $j = n - i$.]

4 *I have two urns: the first contains 3 blue balls and 5 red balls; the second contains 4 blue balls and 2 red balls. I pick one ball at random out of each urn and get one red ball and one blue ball. What is the probability that the red ball came from the first urn?*

The probability of picking a red ball from the first urn and a blue ball from the second urn is $\frac{5}{8} \times \frac{4}{6} = \frac{5}{12}$. The probability of picking a red ball and a blue ball is this plus the probability of picking a blue ball from the first urn and a red ball from the second urn, which is $\frac{3}{8} \times \frac{2}{6} = \frac{1}{8}$, so the probability of picking a red ball and a blue ball is $\frac{13}{24}$. Therefore, given that I picked a red ball and a blue ball, the probability that the red ball came from the first urn is $\frac{\frac{5}{12}}{\frac{13}{24}} = \frac{10}{13}$.

5 *3 dice are rolled: which of the following sets of events are independant?*

(i) "The numbers are all different." "At least one die is a 6."

The probability that the numbers are all different is $\frac{{}^6P_3}{6^3} = \frac{5}{9}$. The probability that there is at least one 6 is $1 - \left(\frac{5}{6}\right)^3$. The probability that there is at least one six given that the numbers are all different is $\frac{1}{2}$, since there are 3 numbers shown, and any of them could be 6. This is not equal to $1 - \left(\frac{5}{6}\right)^3$, so the events are not independant.

(ii) "There is at least one 6." "There are no 4s."

The probability that there are no 4s is $\left(\frac{5}{6}\right)^3$; the probability that there is at least one 6 is $1 - \left(\frac{5}{6}\right)^3$. The probability that there is at least one 6 and no 4s is $\frac{5^3-4^3}{6^3}$ (there are 5^3 rolls with no 4s; of them 4^3 have no 6s either). This is not equal to $\frac{5^3}{6^3} \times \frac{6^3-5^3}{6^3}$, so the events are not independant.

(iii) "At least 2 dice are even." "The dice are all even or all odd." "The total is 16."

These are not independant – given that at least 2 dice are even and the total is 16, the third die must also be even, so the conditional probability of the second event given the other two is 1, whereas it does not have probability 1.

[The first two events are independant, as are the first and third.]

6 *A and B are two random events. C is the event: Toss a fair coin; either it is H and A occurs, or it is tails and B occurs. [This coin toss is independant of both A and B.] Suppose that C is independant of both A and B. What is the probability of $A \cup B$?*

Let D be the event that the coin toss is a head. C is then the event $(D \cap A) \cup (D^c \cap B)$ (where D^c represents the complement of D). Therefore, the probability of C is $\frac{1}{2}\mathbb{P}(A) + \frac{1}{2}\mathbb{P}(B)$. The event $C \cap A$ is $(D \cap A) \cup (D^c \cap A \cap B)$, so it has probability $\frac{1}{2}\mathbb{P}(A) + \frac{1}{2}\mathbb{P}(A \cap B)$. Similarly, $C \cap B$ has probability

$\frac{1}{2}\mathbb{P}(A \cap B) + \frac{1}{2}\mathbb{P}(B)$. If we let $x = \mathbb{P}(A)$, $y = \mathbb{P}(B)$, and $z = \mathbb{P}(A \cap B)$, then the conditions C is independant of A and B become

$$\frac{x+z}{2} = \mathbb{P}(C \cap A) = \mathbb{P}(A)\mathbb{P}(C) = x\left(\frac{x+y}{2}\right)$$

and

$$\frac{y+z}{2} = \mathbb{P}(C \cap B) = \mathbb{P}(B)\mathbb{P}(C) = y\left(\frac{x+y}{2}\right)$$

Subtracting the second equation from the first and multiplying by 2, we get:

$$x - y = (x-y)\frac{x+y}{2}$$

This gives either $\frac{x+y}{2} = 1$ or $x - y = 0$. In the first case, A and B are both certain, so we have that $\mathbb{P}(A \cup B) = 1$. In the second, we have that $x = y$, so we get $x^2 = \frac{x+z}{2}$. However, we know that $x \leqslant 1$, and since $z \geqslant 0$, $x^2 \geqslant \frac{x}{2}$, meaning $x \geqslant \frac{1}{2}$ or $x = 0$. If $x = 0$ then $\mathbb{P}(A \cup B) = 0$. Otherwise, by inclusion-exclusion, $\mathbb{P}(A \cup B) = x + y - z = 2x - (2x^2 - x) = 3x - 2x^2$. However, for $\frac{1}{2} \leqslant x \leqslant 1$, $3x - 2x^2 \geqslant 1$, so we get that $\mathbb{P}(A \cup B) = 1$ in this case.

Therefore, in general, we have 2 solutions – $\mathbb{P}(A \cup B) = 0$ and $\mathbb{P}(A \cup B) = 1$.