MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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Midterm Examination

Calculators not permitted. Answers may be left in reasonably simplified forms – e.g. binomial coefficients, factorials, etc. Justify all your answers.

Section A – Wednesday 5th March 12:35-1:20PM

1 How many subsets of $\{1, 2, \ldots, 19\}$ contain twice as many odd numbers as even numbers?

2 250 students sit a multiple choice exam with n questions, each of which has 4 choices. The students know that no three consecutive questions all have the same answer, so no student submits a set of solutions with 3 consecutive answers the same.

(a) Let a_n be the number of possible solutions to the exam. Find a recurrence relation satisfied by the a_n.

(b) If no two students hand in identical sets of solutions, what is the smallest possible value of n? [The answer is small enough that you do not have to find a general solution to the recurrence from (a).]

3 How many numbers between 1008 and 2008 inclusive are multiples of 2, 3, or 7?

4 I have 3 urns. The first contains 3 blue balls, 2 red balls and 1 yellow ball. The second contains 7 blue balls and 1 red ball. The third contains 1 blue ball and 4 red balls.

(a) I pick an urn uniformly at random, and pick a ball from it uniformly at random. It is blue. What is the probability that it came from the first urn?

(b) I pick another ball from the same urn without replacement. What is the probability that it is yellow?

Section B – Friday 7th March 12:35-1:20PM

1 A gambler starts with \$12. He makes a series of bets of \$1 on the roll of a fair die. If the roll is 6, he gets his original dollar back, and a further \$4. If it is anything else, he loses his \$1.

(a) Let X_n be the amount the gambler gains on the nth bet that he makes. What is $\mathbb{E}(X_n)$?

(b) What is the expected amount of money that the gambler has after n rolls?

(c) The gambler continues playing for 12 rolls. Let p be the probability that he ends up with at least \$20. Show that $p \leqslant \frac{1}{2}$. Explain your reasoning carefully.

2 Define a recurrence by $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$.

(a) Show that the generating function for the sequence a_n is

$$A(x) = \frac{1}{(1-2x)(1-3x)}$$

(b) Find a general formula for a_n.

3 How many 4-digit numbers are there with the digits in increasing order? [Not necessarily strictly increasing order, so 1147 and 3556 are OK. Leading zeros are not permitted.]

4 n fair dice are rolled. What is the probability that the highest number rolled is a 4?

Section C – Thursday 4th March. Makeup exam.

1 I have 3 fair dice: a red die whose faces are numbered 3,3,3,3,3,6; a green die whose faces are numbered 2,2,2,5,5,5; and a blue die whose faces are numbered 1,4,4,4,4,4.

Two players, A and B each pick a die and roll it. Whoever gets the higher number wins.

(a) What is the probability that Player A wins if:

(i) Player A chooses the red die and player B chooses the green die?

(ii) Player A chooses the green die and player B chooses the blue die?

(iii) Player A chooses the blue die and player B chooses the red die?

(b) Is it better to choose the die first or second?

2 I toss 5 (independant) fair coins. Which of the following sets of events are independant?

(a) (i) The first two are both heads (ii) the third, fourth and fifth are all the same (iii) There are an even number of heads in the 5 tosses.

(b) (i) There are at least 3 heads (ii) There are at least 3 tails.

3 Define a recurrence by $a_n = 2a_{n-1} + 7n + 3$, $a_0 = 0$.

(a) Show that the generating function for the sequence a_n is

$$A(x) = \frac{10x - 3x^2}{(1-x)^2(1-2x)}$$

(b) Find a general formula for a_n.

4 How many solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ with x_1, x_2, x_3, x_4, x_5 all natural numbers $\{0, 1, 2, \ldots\}$.