MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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Mock Final Examination
Time allowed: 3 hours

Justify all your answers. There are deliberately more than 3 hours worth of questions here to give you a wider variety of questions. There are also more questions on the later half of the course, and particularly on Ramsey theory than there are likely to be in the final exam.

Compulsory questions

1 *Deduce the finite version of Ramsey's theorem from the infinte version.*

We prove this by contradiction. Suppose that there are k and m such that for every N, there is a 2-colouring of the edges of a K_N without a red K_k or a blue K_m. We will use these to construct a 2-colouring of the edges between natural numbers without a red K_k or a blue K_m. Observe that if we have a 2-colouring of a K_N without a red K_k or a blue K_m, and we restrict to some $n < N$ vertices, we get a 2-colouring of K_n without a red K_k or blue K_m. Given a 2-colouring c of a K_n without a red K_k or blue K_m, we can ask for what N is there a 2-colouring $\overline{c}$ of a K_N without a red K_k or blue K_m, such that c is a restriction of $\overline{c}$. Since there are colourings of K_N without a red K_k or blue K_m, for arbitrarily large N, and there are only finitely many possible colourings of edges of a K_m, one of these colourings must be a restriction of a colouring of a K_N without a red K_k or blue K_m for arbitrarily large N. For the same reason, one of the extensions of this colouring to a colouring of edges of a K_{n+1} must be a restriction of a colouring of a K_n without a red K_k or blue K_m for arbitrarily large N. We will choose a sequence of colourings $c_0, c_1, \ldots$, where each colouring c_i is a colouring of K_i that extends the colouring c_{i-1}, and that is a restriction of colourings of K_N without a red K_k or blue K_m, for arbitrarily large N. Given this sequence, we can put all of these colourings together to get a 2-colouring of edges of the complete graph on $\mathbb{N}$ – colour every edge the colour it is coloured in the colouring where it first appears. This has no red K_k or blue K_m, since any red K_k or blue K_m would have to occur within the first N vertices for some N, and then c_N would have either a red K_k or blue K_m.

2 *A biassed coin with probability 0.3 of getting a head is tossed 50 times. What is the expected number of occurences of the sequence "HTTHH"?*

The first toss in this sequence "HTTHH" can be anything from the 1st to the 46th toss. For each case, the probability that this sequence occurs starting from this toss is $0.3 \times 0.7 \times 0.7 \times 0.3 \times 0.3 = 0.01223$. The expected number of occurrences of this sequence is therefore $46 \times 0.01223 = 0.56258$.

3 *How many non-identical ways are there to colour the sides of a square with 3 colours, where we count rotations as identical, but not reflections.*

First, we count the number of times each colour occurs, and then for each pattern, we can count the number of ways of arranging it.

If three colours are used, they must be used 2,1,1. There are 3 choices for which colour is used twice. Then the two occurences of this colour can be either adjacent or opposite. If they are adjacent, then the choice of which way the other two colours are used makes a difference, so there are 2 ways. If they are opposite, then the two arrangements of the other 2 colours are rotation of each other, so they count as the same. This gives a total of $3 \times 3 = 9$ colourings of the form 2,1,1.

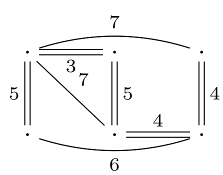
If only two colours are used, they can either be 3,1 or 2,2. If they are 3,1, then only the choice of which colours to use matters, so there are 6 colourings. If they are 2,2, then the pairs can either be opposite or adjacent, so there are 2 possibilities, but when choosing colours, only the colour that is not used matters, so there are 3 choices of colours, making a total of 6 colourings.

Finally, if only one colour is used, all that matters is the choice of colour, so there are 3 possible colourings.

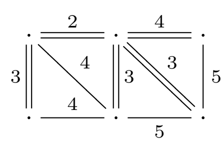
This is a total of $9 + 6 + 6 + 3 = 24$ colourings.

4 *Find minimal spanning trees for the following graphs:*

(a)



(b)



5 *A fair die is rolled repeatedly until a 6 is rolled. What is the expected number of rolls required before a 6 is rolled?*

Let X be the number of rolls until a 6 is rolled. In order for $X = n$, we must have that the first $n-1$ rolls are all not 6, and that the nth roll is a 6. The probability of this is $\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$. Therefore, the expected value of X is

$$\frac{1}{6}\sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n-1}$$

If we let $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, then $f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$, so

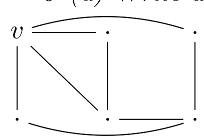
$$\mathbb{E}(X) = \frac{1}{6} f'\left(\frac{5}{6}\right) = 6$$

Alternatively, if we let Y be the number of rolls to get a 6 after the first, then X and Y have the same distribution, and

$$X = \begin{cases} 1 & \text{if the first roll is a 6} \\ Y+1 & \text{otherwise} \end{cases}$$

So $\mathbb{E}(X) = \frac{1}{6} + \frac{5}{6}\mathbb{E}(Y+1) = \frac{1}{6} + \frac{5}{6}(\mathbb{E}(Y)+1) = \frac{1}{6} + \frac{5}{6}(\mathbb{E}(X)+1)$ This gives $\frac{1}{6}\mathbb{E}(X) = 1$, or $\mathbb{E}(X) = 6$.

6 (a) Write down the adjacency matrix for the graph:



The matrix is:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

(b) How many walks of length 8 are there starting and ending at v?

$$\begin{aligned}
A^2 &= \begin{pmatrix} 4 & 1 & 0 & 0 & 1 & 3 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 & 3 & 0 \\ 3 & 1 & 0 & 0 & 0 & 3 \end{pmatrix} \\
A^4 &= \begin{pmatrix} 27 & 10 & 3 & 3 & 8 & 22 \\ 10 & 9 & 8 & 8 & 8 & 8 \\ 3 & 8 & 13 & 13 & 15 & 1 \\ 3 & 8 & 13 & 13 & 15 & 1 \\ 8 & 8 & 15 & 15 & 19 & 4 \\ 22 & 8 & 1 & 1 & 4 & 19 \end{pmatrix} \\
A^8 &= \begin{pmatrix} 1395 & 648 & 381 & 381 & 626 & 800 \\ 648 & 437 & 438 & 438 & 576 & 492 \\ 381 & 438 & 637 & 637 & 767 & 235 \\ 381 & 438 & 637 & 637 & 767 & 235 \\ 626 & 576 & 767 & 767 & 955 & 422 \\ 800 & 492 & 235 & 235 & 422 & 927 \end{pmatrix}
\end{aligned}$$

So there are 1395 such walks.

(c) Does the graph have an Euler Circuit?

It does not have an Euler circuit because it has a vertex of degree 3, which is odd.

(d) Does it have a Hamiltonian cycle?

It does not have a Hamiltonian cycle. If it does, then we can find a Hamiltonian cycle starting and ending at v. Let w be the bottom right vertex. The two vertices of degree 2 are both adjacent to v and w, so in a Hamiltonian cycle, their neighbours would have to be v and w. This means that w must occur as both the third and the third-last vertex in the Hamiltonian cycle. However, there are 7 vertices in the cycle, so the third and third-last vertices should be different.

7 *A fair die is rolled, and the result is used to select a number of coins to toss – e.g. if a 3 is rolled, we toss 3 coins. What is the probability that the die roll was a 4 given that there were exactly 2 heads among the coins tossed?*

The probability that we roll n then get exactly 2 heads is $\frac{1}{6}\left(\frac{1}{2}\right)^n\binom{n}{2}$. Therefore, the total probability that we get exactly 2 heads is

$$\frac{1}{6}\left(\frac{1}{4}+\frac{3}{8}+\frac{6}{16}+\frac{10}{32}+\frac{15}{64}\right)=\frac{1}{6}\times\frac{16+24+24+20+15}{64}$$

The probability that we roll a 4 and get exactly 2 heads is $\frac{1}{6}\times\frac{6}{16}=\frac{1}{6}\times\frac{24}{64}$. The probability that we rolled a 4 given that we got 2 heads is

$$\frac{\frac{1}{6}\times\frac{24}{64}}{\frac{1}{6}\times\frac{16+24+24+20+15}{64}}=\frac{24}{16+24+24+20+15}=\frac{24}{99}=\frac{8}{33}$$

8 *Show that any 2-colouring of a K_6 actually has 2 monochromatic triangles. [Hint: we know it must have one. Let it be v_1, v_2, v_3, and w.l.o.g., let it be red. Consider cases:*

1. There are no red edges wv_i where w is not one of the v_i.

2. There is a w not one of the v_i with exactly one red edge wv_i, and the two edges from w to vertices outside the triangle are both red.

3. There is a w not one of the v_i with exactly one red edge wv_i, and one of the two edges from w to a vertex outside the triangle is blue.

4. There are red edges wv_i and wv_j for w not one of the v_i.]

As in the hint, let v_1, v_2, v_3 be a red triangle. Let the other 3 vertices be w_1, w_2, w_3. Suppose there are no red edges w_iv_j, then all edges v_1w_i are blue; if any edge w_iw_j is blue, then v_1, w_i, w_j forms a second monochromatic triangle, while if no edge w_iw_j is blue, then they are all red, and w_1, w_2, w_3 form a red triangle.

Now suppose there is exactly one red edge from w_1 (w.l.o.g.) to some v_i, and the edges w_1w_2 and w_1w_3 are also red. Now again, if any of the edges

w_2v_i, w_2w_3 or w_3v_i is red, then we get a red triangle. Otherwise, we get a blue triangle.

Next suppose there is exactly one red edge from w_1 (w.l.o.g) to v_1 (w.l.o.g), and the edges w_1w_2 (w.l.o.g.), w_1v_2 and w_1v_3 are all blue. Now a blue edge between any two of w_2, v_2, v_3 will give a blue triangle, while if the edges between them are all red, then they form a red triangle.

Finally, suppose two of the edges w_jv_i for some fixed j are red, then they form a red triangle with the edge between the v.

9 *(a) Show that whenever we 4-colour the edges of a K_{66}, we always get a monochromatic triangle.*

We know that whenever we 3-colour the edges of a K_17, we always get a monochromatic triangle. Pick a vertex of the K_52. It has 65 neighbours, so there must be some colour (w.l.o.g. red) such that it has at least 17 neighbours of that colour. If any of the edges between its red neighbours is also red, then we get a red triangle. If not, then we have a 3-colouring of a K_{17}, which we know must yield a monochromatic triangle.

(b) What is the expected number of monochromatic triangles?

Any triangle has probability $\frac{1}{16}$ of being monochromatic – once a colour is chosen for the first edge, there is a $\frac{1}{4}$ probability of each other edge being the same colour. There are $\binom{66}{3}$ triangles, so the expected number of monochromatic triangles is $\frac{\binom{66}{3}}{16}$ $(= 2860)$.

10 *How many 4-digit numbers counting numbers with leading zeros, contain at least one of the digits 1 and 2.*

There are 10^4 4-digit numbers in total. Of these, 8^4 use only the digits $0, 3, 4, 5, 6, 7, 8, 9$, so there are $10^4 - 8^4 = 5904$ 4-digit numbers that contain at least one of the digits 1 and 2.

11 *If we 2-colour a K_n, must there be a monochromatic spanning tree?*

A graph has a spanning tree if and only if it is connected, so this question is asking whether we can 2-colour a K_n in such a way that neither the graph on the red edges nor the graph on the blue edges is connected.

Suppose the red edges do not form a connected graph. Then we can partition the vertices as the disjoint union of two non-empty sets A and B, such that all red edges are either between two vertices in A or two vertices in B. This means that any edge from a vertex in A to a vertex in B is blue. This forces the blue graph to be connected, since if we let v be a vertex in A and w a vertex in B, then given any two vertices x and y. If $x \in A$ and $y \in B$ or vice versa, then the edge xy is blue. If they are both in B, then xv and yv are both blue, so there is a path of length 2. If they are both in A, then xw and yw are both blue, so there is a path of length 2.

(b) What if we 3-colour it?

If $n > 2$, then we can 3-colour K_n without a monochromatic spanning tree – Pick two vertices v and w, colour the edge vw green, colour all other edges from v red, and all other edges from w blue. Colour the other edges in any way. There is now no red or blue path from v to w, since there is no red edge to v and no blue edge to w. If u is some other vertex, then there is no green path from v to u, since there are no green edges from v or w to any other vertex.

12 *3 fair dice are rolled. One is red; one is green and one is blue.*

(a) What is the probability that red $\leqslant$ green $\leqslant$ blue*.*

The number of rolls with red $\leqslant$ green $\leqslant$ blue is $\binom{8}{3}$ – either add one to the green die and 2 to the blue die to get 3 different numbers from 1 to 8, or view

The probability that red $\leqslant$ green $\leqslant$ blue is therefore $\frac{\binom{8}{3}}{216} = \frac{56}{216} = \frac{7}{27}$

(b) What is the probability that blue $<$ green *given that* red $\leqslant$ green*?*

The probability that red $\leqslant$ green is $\frac{\binom{7}{2}}{36} = \frac{7}{12}$. The probability that green $\leqslant$ blue given that red $\leqslant$ green is therefore $\frac{\frac{7}{27}}{\frac{7}{12}} = \frac{12}{27} = \frac{4}{9}$. Therefore, the probability that blue $<$ green given that red $\leqslant$ green is $\frac{5}{9}$.

13 *Show that at least 4 trees are required to cover all edges of K_8?*

A tree on some of the vertices of a K_8 has at most 8 vertices, so it has at most 7 edges. K_8 has $\binom{8}{2} = 28$ edges. Therefore, at least 4 trees are required to cover all the edges.

14 *Two players play a game: player A rolls a fair die and scores the result of the roll. Player B tosses 6 fair coins and records the number of heads. Player B wins if his score is greater than or equal to A's score. What is the probability that B wins.*

Let H be the number of heads that B rolls. The probability that B wins if $X = n$ is $\frac{n}{6}$. Therefore, the probability that B wins is

$$\frac{1}{6}\sum_{i=1}^{6} i\mathbb{P}(X = i) = \frac{1}{6}\mathbb{E}(X)$$

Note that X is just the sum of the number of heads rolled with each coin, so $\mathbb{E}(X) = 6 \times \frac{1}{2} = 3$. Therefore, $\mathbb{P}(B \text{ wins}) = \frac{3}{6} = \frac{1}{2}$.

15 *Let p_n be the probability of tossing a fair coin n times without getting 4 consecutive heads. Show that $p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2} + \frac{1}{8}p_{n-3} + \frac{1}{16}p_{n-4}$.*

To roll n times without getting 4 consecutive heads, consider all the rolls since the last tail. There are 4 possibilities: "T", "TH", "THH" and

"THHH". The first of these has probability $\frac{1}{2}p_{n-1}$, the second has probability $\frac{1}{4}p_{n-2}$, the third has probability $\frac{1}{8}p_{n-3}$, and the fourth has probability $\frac{1}{16}p_{n-4}$. They are mutually exclusive, so

$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2} + \frac{1}{8}p_{n-3} + \frac{1}{16}p_{n-4}$$